Chin. Ann. Math.
38B(3), 2017, 741–758
DOI: 10.1007/s11401-017-1093-6

Chinese Annals of Mathematics, Series B

Blowup Criteria for Full Compressible Navier-Stokes Equations with Vacuum State*

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Abstract This paper deals with the global strong solution to the three-dimensional (3D) full compressible Navier-Stokes systems with vacuum. The authors provide a sufficient condition which requires that the Sobolev norm of the temperature and some norm of the divergence of the velocity are bounded, for the global regularity of strong solution to the 3D compressible Navier-Stokes equations. This result indicates that the divergence of velocity fields plays a dominant role in the blowup mechanism for the full compressible Navier-Stokes equations in three dimensions.

Keywords Compressible Navier-Stokes equations, Heat-conduction, Blowup criterion, Divergence of velocity
2000 MR Subject Classification 35Q35, 35B65, 35Q30, 76N10

1 Introduction

The motion of compressible viscous fluid with heat-conduction in $\mathbb{R}^3$ is governed by the following full compressible Navier-Stokes equations

$$\begin{cases}\rho_t+\operatorname{div}(\rho u)=0,\\ (\rho u)_t+\operatorname{div}(\rho u\otimes u)-\mu\Delta u-(\mu+\lambda)\nabla\operatorname{div}u+\nabla P=0,\\ c_\upsilon\left[(\rho\theta)_t+\operatorname{div}(\rho u\theta)\right]-\kappa\Delta\theta+P\operatorname{div}u=2\mu|\mathfrak{D}(u)|^2+\lambda(\operatorname{div}u)^2\end{cases}\tag{1.1}$$

and the initial conditions

$$(\rho,u,\theta)(x,0)=(\rho_0,u_0,\theta_0)(x),\quad x\in\mathbb{R}^3.\tag{1.2}$$

Here we denote by ρ, u and θ the unknown density, velocity fields, temperature of the fluid, respectively. $P=R\rho\theta$ $(R>0)$ is the pressure of the fluid. $\mathfrak{D}(\mathfrak{u})$ is the deformation tensor, which is described as

$$\mathfrak{D}(\mathfrak{u})=\frac{1}{2}(\nabla u+(\nabla u)^{\mathrm{tr}}).$$

The shear viscosity coefficient μ and bulk viscosity coefficient λ satisfy the physical restrictions

$$\mu>0,\quad 2\mu+3\lambda\ge 0.$$

The constant c_υ is the heat capacity, κ (>0) is the heat conductivity.

Manuscript received May 21, 2015. Revised September 2, 2015.
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*This work was supported by the Sichuan Youth Science and Technology Foundation (No. 2014JQ0003).

The following boundary conditions are considered in this paper, for some constant $\widetilde{\rho} \geq 0$,

$$(\rho, u, \theta)(x, t) \to (\widetilde{\rho}, 0, 0) \quad \text{as } |x| \to \infty \tag{1.3}$$

in weak sense.

In the absence of vacuum, the global existence of classical solutions to the system (1.1) has been established by Mastumura and Nishida in [21], when the initial data is close to an non-vacuum equilibrium in some Sobolev spaces H^s. Later, in [13], Hoff obtained the global existence of weak solutions for discontinuous initial data when initial density and temperature are strictly positive.

In case of that the initial vacuum is allowed, this problem becomes much more complicated to the system (1.1). Feireisl [10] first proved the global existence of the variational weak solutions to the full compressible Navier-Stokes equations in dimension $N \geq 2$. Especially, Lions [20] proved the global existence of weak solution to isentropic compressible Navier-Stokes system. However, the global existence or finite-time blowup of strong solution is still an open problem, and only local existence results have been obtained for sufficiently regular data with some compatibility conditions. For details, in [4] Cho and Kim showed the local existence of the strong solution to 3D compressible Navier-Stokes equations (see also in [11–12, 19–20] for isentropic flows). Recently, the global existence and uniqueness of classical solutions to the Cauchy problem in three spatial dimensions with smooth initial data with small energy is obtained by Huang, Li and Xin [17].

Meanwhile, the regularity and uniqueness of weak solution to 3D compressible Navier-Stokes equations with large data remains open. In the significant work [27], Xin showed that the classical solutions will blow up in finite time when initial density has compact support. Therefore, we would not expect higher regularity of Lions' weak solutions in general. In additional, in [28] Xin and Yan showed that any classical solutions of viscous compressible fluids without heat-conduction will blow up in finite time, as long as the initial data has an isolated mass group.

Hence, it is natural to study the blowup mechanism and the possible singularity of the smooth solutions. We would like to mention the two well-known blowup criteria, Beale-Kato-Majda criterion in [1] for incompressible inviscid flows and Serrin-type in [23] for incompressible viscous flows. Namely, if $T^* < \infty$ is the maximal time for the existence of a strong (or classical) solution, then

$$\lim_{T \to T^*} \|\nabla u\|_{L^1(0,T;L^\infty)} = \infty \quad \text{(Beale-Kato-Majda type)}$$

and

$$\lim_{T \to T^*} \|u\|_{L^s(0,T;L^r)} = \infty \quad \text{(Serrin type)},$$

where

$$\frac{2}{s} + \frac{3}{r} \leq 1, \quad 3 < r \leq \infty. \tag{1.4}$$

A natural problem is that whether the blowup criteria are valid for 3D compressible Navier-Stokes equations. Firstly, Huang, Li and Xin [16] have shown that maximum norm of the deformation tensor of velocity gradients controls the possible breakdown of smooth (strong) solutions to compressible isentropic Navier-Stokes system. We would like to mention the references [15–16, 24] and references therein for the blowup criteria to the 3D barotropic viscous flows. Moreover, for 3D full Navier-Stokes equations, Fan et al. in [9] established the following blowup criterion

$$\lim_{T\to T^*}(\|\theta\|_{L^\infty(0,T;L^\infty)}+\|\nabla u\|_{L^1(0,T;L^\infty)})=\infty, \tag{1.5}$$

with the additional condition (1.6)

$$7\mu>\lambda \tag{1.6}$$

and positive initial density. Recently, a BKM criterion

$$\lim_{T\to T^*}(\|\rho,\rho^{-1}\|_{L^\infty(0,T;L^\infty)}+\|\theta\|_{L^\infty(0,T;L^\infty)})=\infty$$

was established in [25] for 3D compressible Navier-Stokes equations with heat-conduction with the stringent condition (1.6).

Furthermore, Huang and Li in [14] removed the condition (1.6) and established the following blowup criterion

$$\limsup_{T\to T^*}(\|\theta\|_{L^2(0,T;L^\infty)}+\|\mathfrak{D}(u)\|_{L^1(0,T;L^\infty)})=\infty. \tag{1.7}$$

The motivations in this paper come from the following two facts. The one is that for compressible viscous flows with heat-conduction in dimension two, a blowup criterion only involving the divergence of velocity fields has been established by Wang in [26]. In three-dimension, by the blowup criterion (1.7), we don't know whether the velocity gradient tensor plays an essential role in the blowup mechanism. Inspired by the results (see [26]) in two dimension, we strongly expect to show the diagonal elements of the velocity gradient tensor plays a leading role in the possible singularity of solution instead of the velocity gradient tensor itself. The second fact is that for 3D incompressible viscous flows, Cao and Titi in [3] established the blowup criterion involving one entry of the velocity gradient tensor, which implies that only one entry of the velocity gradient tensor $\frac{\partial u^i}{\partial x_j}$ can guarantee the global regularity of 3D incompressible viscous flows. Hence, the major purpose of this paper is to establish a blowup criterion for 3D full compressible Navier-Stokes equations (1.1), in terms of the temperature θ and $\operatorname{div} u$. In additional, this result improved the previous blowup criteria in [9, 14, 25], substituting ∇u (or $\mathfrak{D}(u)$) by $\operatorname{div} u$.

Throughout this paper, we adopt the following simplified notation

$$\int f\mathrm{d}x=\int_{\mathbb{R}^3}f\mathrm{d}x,$$

and the simplified ones for standard homogeneous and inhomogeneous Sobolev spaces

$$\begin{cases} L^r=L^r(\mathbb{R}^3),\quad W^{k,r}=W^{k,r}(\mathbb{R}^3),\quad H^k=W^{k,2}(\mathbb{R}^3),\\ D^1=\{u\in L^6\mid \|\nabla u\|_{L^2}<\infty\},\quad D^{k,r}=\{u\in L^1_{\mathrm{loc}}\mid \|\nabla^k u\|_{L^r}<\infty\},\end{cases}$$

where $1\le r\le\infty$ and k is a positive integer.

Next, we give the definition of strong solutions as follows.

Definition 1.1 (Strong Solutions) *For $\widetilde{\rho} \geqslant 0$, (ρ, u, θ) is called a strong solution to* (1.1) *in $\mathbb{R}^3 \times (0,T)$, provided that for some $r_0 \in (3,6]$,*

$$\begin{cases} \rho \geq 0, \quad \rho - \widetilde{\rho} \in C([0,T]; W^{1,r_0}), \quad \rho_t \in C([0,T]; L^{r_0}), \\ (u,\theta) \in C([0,T]; D^1 \cap D^{2,2}) \cap L^2(0,T; D^{2,r_0}), \\ (u_t, \theta_t) \in L^2(0,T; D^1), \quad (\rho^{\frac{1}{2}} u_t, \rho^{\frac{1}{2}} \theta_t) \in C(0,T; L^2), \end{cases}$$

(ρ, u, θ) satisfies both (1.1) *almost everywhere in $\mathbb{R}^3 \times (0,T)$ and initial condition* (1.2) *almost everywhere in $\mathbb{R}^3$.*

The main results in this paper are stated as follows.

Theorem 1.1 *Suppose that the initial data (ρ_0, u_0, θ_0) satisfy*

$$\rho_0, \theta_0 \geq 0, \quad \rho_0 \in H^1 \cap W^{1,q_0}, \quad (u_0, \theta_0) \in D^1 \cap D^{2,2}, \quad \rho_0 \theta_0^2 \in L^1 \tag{1.8}$$

for $q_0 \in (3,6]$ and compatibility conditions

$$-\mu \Delta u_0 - (\mu + \lambda) \nabla \mathrm{div}\, u_0 + R \nabla(\rho_0 \theta_0) = \rho_0^{\frac{1}{2}} g_1, \tag{1.9}$$

$$\kappa \Delta \theta_0 + 2\mu |\mathfrak{D}(u_0)|^2 + \lambda (\mathrm{div}\, u_0)^2 = \rho_0^{\frac{1}{2}} g_2 \tag{1.10}$$

with $(g_1, g_2) \in L^2$, and (ρ, u, θ) is the strong solution of the initial boundary value problem (1.1)–(1.2) *together with* (1.3). *If $T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T \to T^*} (\|\mathrm{div}\, u\|_{L^2(0,T;L^\infty)} + \|\theta\|_{L^\alpha(0,T;L^\beta)}) = \infty \tag{1.11}$$

with $1 \leq \alpha \leq 2$, $\beta \geq 4$, $\frac{3}{\alpha} + \frac{2}{\beta} \geq 2$ and $\frac{1}{\alpha} + \frac{2}{\beta} \leq 1$.

We would like to give some comments on our results.

Remark 1.1 Theorem 1.1 shows that $\mathrm{div}\, u$ plays an important role in the mechanism of blowup for the 3D compressible viscous flows. If we compare with the 3D incompressible viscous flows, when the density and the temperature remain constants, the Leray-Hopf weak solution is the unique strong solution, provided that the pressure possesses some nice regularity (see [2] for global regularity criterion for the pressure). For the heat-conduct compressible Navier-Stokes equations, the pressure is determined by the density and temperature. In the proof of our results, some estimates for the pressure can be obtained, as long as some regularity assumptions on the $\mathrm{div}\, u$ and θ are given apriorily. Due to these facts, our results seem to be natural and reasonable.

Remark 1.2 Recently, Huang et al. in [18] provided a Serrin-type blowup criterion for the Cauchy problem of system (1.1), roughly speaking, that if $T^* < \infty$ is the maximal time for the existence of a strong solutions, then

$$\lim_{T \to T^*} (\|\mathrm{div}\, u\|_{L^1(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r)}) = \infty, \tag{1.12}$$

where s and r are also restricted by (1.4). Similarly, the result implies that the divergence of velocity plays a key role in the blowup mechanism instead of the velocity gradient tensor.

Remark 1.3 If $\theta \equiv \theta_0 \equiv 0$, Theorem 1.1 directly yields the following blowup criterion for the three-dimensional compressible Navier-Stokes equations, more precisely, if $T^* < \infty$ is the maximal time for the existence of a strong solution, then

$$\lim_{T \to T^*} \|\mathrm{div}\, u\|_{L^2(0,T;L^\infty)} = \infty,$$

which is consistent with the corresponding result in [5]. Some similar blowup criteria for the isentropic compressible magnetohydrodynamic flows in two dimensions and liquid-gas two-phase flow model have been established in the recent papers [6–8].

The remain of this paper is organized as follows. In Section 2, we will recall some elementary facts and inequalities. The proof of Theorem 1.1 will be given in Section 3.

2 Preliminaries

In this section, we first give some results for the existence of the local strong solution, which have shown in [4] for the initial-boundary value problem (1.1)–(1.3).

Lemma 2.1 *Assume that the initial data* (ρ_0, u_0, θ_0) *satisfy* (1.8)–(1.10), *then there exists a positive constant* T_0 *and a unique strong solution* (ρ, u, θ) *to the problem* (1.1), (1.2) *together with* (1.3) *on* $\mathbb{R}^3 \times (0, T_0]$.

Next, we recall some important inequalities, which will play an important role in the following arguments (see [10, 22] for the details).

Lemma 2.2 *For* $q \in (1, \infty)$ *and* $r \in (3, \infty)$, *there exists a positive constant* C, *such that for any* $f \in H^1$, $g \in L^q \cap D^{1,r}$, *we have*

$$\|f\|_{L^6} \le C\|\nabla f\|_{L^2}, \tag{2.1}$$

$$\|g\|_{L^\infty} \le C\|g\|_{L^q} + C\|\nabla g\|_{L^r}, \tag{2.2}$$

where C *depends only on* q, r.

It thus follows from the momentum equations that we have the following elliptic system

$$\Delta F = \mathrm{div}(\rho \dot{u}), \quad \mu \Delta \omega = \mathrm{curl}(\rho \dot{u}), \tag{2.3}$$

where

$$F = (2\mu + \lambda)\mathrm{div}\, u - P, \quad \dot{f} = f_t + u \cdot \nabla f, \quad \omega = \mathrm{curl}\, u \tag{2.4}$$

are the effective viscous flux, the material derivative of f, and the vorticity of the velocity fields, respectively.

It follows from the standard L^p-estimate for the elliptic system (2.3) that we have the following lemmas (see [17] for details).

Lemma 2.3 *Let* (ρ, u, θ) *be a solution of* (1.1), *then there exists a general positive constant* C *depending only on* λ *and* μ *such that for any* $p \in [2, 6]$,

$$\|F\|_{L^2} + \|\omega\|_{L^2} \le C(\|\nabla u\|_{L^2} + \|P\|_{L^2}), \tag{2.5}$$

$$\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \le C\|\rho \dot{u}\|_{L^p}, \tag{2.6}$$

$$\|\nabla u\|_{L^p} \le C\|\nabla u\|_{L^2}^{\frac{6-p}{2p}} (\|\rho \dot{u}\|_{L^2} + \|P\|_{L^6})^{\frac{3p-6}{2p}}. \tag{2.7}$$

In order to estimate $\|\nabla u\|_{L^\infty}$, we introduce the following BKM-type inequality, which can be found in [15].

Lemma 2.4 *For $q \in (3,\infty)$, suppose $\nabla u \in L^2 \cap D^{1,q}$. There is a constant C depending on q, such that*

$$\|\nabla u\|_{L^\infty} \le C(\|\mathrm{div}\, u\|_{L^\infty} + \|\mathrm{curl}\, u\|_{L^\infty}) \log(\mathrm{e} + \|\nabla^2 u\|_{L^q}) + C(1 + \|\nabla u\|_{L^2}). \tag{2.8}$$

3 Proof of the Main Results

In this section, we will show Theorem 1.1 by the contradiction arguments. We assume the contrary to the results of Theorem 1.1, namely, there exists a bounded positive constant M, such that

$$\lim_{T\to T^*} (\|\mathrm{div}\, u\|_{L^2(0,T;L^\infty)} + \|\theta\|_{L^\alpha(0,T;L^\beta)}) \le M \tag{3.1}$$

with $1 \le \alpha \le 2,\ \beta \ge 4,\ \frac{3}{\alpha} + \frac{2}{\beta} \ge 2$ and $\frac{1}{\alpha} + \frac{2}{\beta} \le 1$. Without loss of generality, we assume $\widetilde{\rho} = 0$ in the following.

The upper bound estimate of the density ρ is standard, which comes from the estimate (3.1) and the continuity equation immediately (see [15–16] for details).

Lemma 3.1 *Suppose that*

$$\|\mathrm{div}\, u\|_{L^1(0,T;L^\infty)} \le M, \quad 0 \le T < T^*. \tag{3.2}$$

Then

$$\sup_{t\in[0,T]} \|\rho\|_{L^\infty} \le C, \quad 0 \le T < T^*. \tag{3.3}$$

Throughout this paper, C, C_i denote some generic constants depending only on M, μ, λ, R, κ, c_v, T^* and the initial data.

Next, we will give the energy inequality as follows.

Lemma 3.2 *Under the assumption* (3.1)*, it holds that for $0 \le T < T^*$,*

$$\sup_{t\in[0,T]} \int (\rho\theta + \rho|u|^2)\mathrm{d}x + \int_0^T \int |\nabla u|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{3.4}$$

Proof Applying standard maximum principle to the temperature equation in (1.1) together with $\theta_0 \ge 0$ (see [9–10]) yields that

$$\inf_{\mathbb{R}^3\times[0,T]} \theta(x,t) \ge 0.$$

Denote the specific energy as $E \triangleq c_v\theta + \frac{1}{2}|u|^2$ and it follows from (1.1) that

$$(\rho E)_t = \Delta\Big(\kappa\theta + \frac{1}{2}\mu|u|^2\Big) + \mathrm{div}\,(\mu u\cdot\nabla u + \lambda u \mathrm{div}\, u - \rho E u - P u). \tag{3.5}$$

Integrating (3.5) over $\mathbb{R}^3 \times [0,T]$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \Big(c_v\rho\theta + \frac{1}{2}\rho|u|^2\Big)\mathrm{d}x = 0. \tag{3.6}$$

Next, multiplying the momentum equations by u and integrating the resulting equation in $\mathbb{R}^3$, yield that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^2\mathrm{d}x + \mu\|\nabla u\|_{L^2}^2 + (\mu+\lambda)\|\mathrm{div}\, u\|_{L^2}^2 \le C\|\mathrm{div}\, u\|_{L^\infty}\int \rho\theta\mathrm{d}x. \tag{3.7}$$

Moreover, adding (3.6) to (3.7) implies the estimate (3.4) by Gronwall's inequality and (3.2). This completes the proof of Lemma 3.2.

The following lemma gives the key estimate of $L^\infty(0,T;L^2)$-norm of ∇u.

Lemma 3.3 *Under the assumption* (3.1), *the estimate*

$$\sup_{t\in[0,T]}\int(\rho\theta^2+|\nabla u|^2)\mathrm{d}x + \int_0^T\int(\rho|\dot{u}|^2+|\nabla\theta|^2)\mathrm{d}x\mathrm{d}t \le C \tag{3.8}$$

holds for $0\le T<T^*$.

Proof Multiplying the third equation in (1.1) by θ and integrating the resulting equation over $\mathbb{R}^3$ give that

$$\begin{aligned} c_\upsilon\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\theta^2\mathrm{d}x + 2\kappa\|\nabla\theta\|_{L^2}^2 &= -2\int P\theta\mathrm{div}\, u\mathrm{d}x + 4\mu\int|\mathfrak{D}(u)|^2\theta\mathrm{d}x + 2\lambda\int(\mathrm{div}\, u)^2\theta\mathrm{d}x \\ &\le C\|\mathrm{div}\, u\|_{L^\infty}\int\rho\theta^2\mathrm{d}x + C\int|\nabla u|^2\theta\mathrm{d}x. \end{aligned} \tag{3.9}$$

In order to estimate the last term on the right-hand side of (3.9), multiply the momentum equation by $u\theta$ and integrate the resulting equation over $\mathbb{R}^3$ to obtain

$$\begin{aligned} \mu\int|\nabla u|^2\theta\mathrm{d}x &\le \int|\rho\dot{u}\cdot u\theta|\mathrm{d}x + \Big|\int\nabla P\cdot u\theta\mathrm{d}x\Big| + C\int|u||\nabla u||\nabla\theta|\mathrm{d}x \\ &\le \int|\rho\dot{u}\cdot u\theta|\mathrm{d}x + C\int\rho\theta^2|\mathrm{div}\, u|\mathrm{d}x + C\int\rho\theta|u||\nabla\theta|\mathrm{d}x + C\int|u||\nabla u||\nabla\theta|\mathrm{d}x \\ &\le \delta\int\rho|\dot{u}|^2\mathrm{d}x + \varepsilon\|\nabla\theta\|_{L^2}^2 + C\|\mathrm{div}\, u\|_{L^\infty}\int\rho\theta^2\mathrm{d}x + C\int\rho u^2\theta^2\mathrm{d}x \\ &\quad + C\int|u|^2|\nabla u|^2\mathrm{d}x. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), after choosing ε suitably small, yields that

$$\begin{aligned} &c_\upsilon\frac{\mathrm{d}}{\mathrm{d}t}\int\rho\theta^2\mathrm{d}x + \kappa\|\nabla\theta\|_{L^2}^2 \\ &\le C\|\mathrm{div}\, u\|_{L^\infty}\int\rho\theta^2\mathrm{d}x + C\delta\int\rho|\dot{u}|^2\mathrm{d}x + C\int\rho u^2\theta^2\mathrm{d}x + C\int|u|^2|\nabla u|^2\mathrm{d}x. \end{aligned} \tag{3.11}$$

On the other hand, multiplying the second equation in (1.1) by u_t and integrating equation over $\mathbb{R}^3$ yield

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}\, u)^2)\mathrm{d}x + \int\rho|\dot{u}|^2\mathrm{d}x \\ &= \int\rho\dot{u}(u\cdot\nabla)u\mathrm{d}x + \int P\mathrm{div}\, u_t\mathrm{d}x, \end{aligned}$$

then it follows from Young's inequality that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\mu|\nabla u|^2+(\mu+\lambda)(\operatorname{div}u)^2)\mathrm{d}x+\int\rho|\dot u|^2\mathrm{d}x\\
&\le\frac14\int\rho|\dot u|^2\mathrm{d}x+C\int|u|^2|\nabla u|^2\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int P\operatorname{div}u\mathrm{d}x-\int P_t\operatorname{div}u\mathrm{d}x\\
&=\frac14\int\rho|\dot u|^2\mathrm{d}x+C\int|u|^2|\nabla u|^2\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int P\operatorname{div}u\mathrm{d}x\\
&\quad-\frac{1}{2(2\mu+\lambda)}\frac{\mathrm{d}}{\mathrm{d}t}\int P^2\mathrm{d}x-\frac{1}{2\mu+\lambda}\int P_tF\mathrm{d}x,
\end{aligned}\tag{3.12}
$$

where we have used the definition of the effective viscous flux F in (2.4).

For the last term in (3.12), it follows from the third equation in (1.1) and (2.4) that

$$
\begin{aligned}
\Big|\int P_tF\mathrm{d}x\Big|&=R\Big|\int(\rho\theta)_tF\mathrm{d}x\Big|\\
&=\frac{R}{c_v}\Big|\int(-c_v\operatorname{div}(\rho u\theta)+\kappa\Delta\theta-P\operatorname{div}u+2\mu|\mathfrak{D}(u)|^2+\lambda(\operatorname{div}u)^2)F\mathrm{d}x\Big|\\
&=\frac{R}{c_v}\Big|\int(c_v\rho u\theta-\kappa\nabla\theta)\cdot\nabla F+(-P\operatorname{div}u+2\mu|\mathfrak{D}u|^2\\
&\quad+\lambda(\operatorname{div}u)^2)((2\mu+\lambda)\operatorname{div}u-P)\mathrm{d}x\Big|.
\end{aligned}
$$

According to the estimates (2.6), (3.3) and Young's inequality, we have

$$
\begin{aligned}
\Big|\int P_tF\mathrm{d}x\Big|&\le\varepsilon\|\nabla F\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^2+C\int\rho^2u^2\theta^2\mathrm{d}x\\
&\quad+C\int|\nabla u|^2\theta\mathrm{d}x+C\|\operatorname{div}u\|_{L^\infty}\Big(\|\nabla u\|_{L^2}^2+\int\rho\theta^2\mathrm{d}x\Big).
\end{aligned}
$$

Choosing ε suitably small and together with (2.6) and (3.12) yields that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\frac{\mu}{2}|\nabla u|^2+\frac{\mu+\lambda}{2}(\operatorname{div}u)^2-P\operatorname{div}u+\frac{1}{2(2\mu+\lambda)}P^2\Big)\mathrm{d}x+\frac12\int\rho|\dot u|^2\mathrm{d}x\\
&\le C\int|u|^2|\nabla u|^2\mathrm{d}x+C_0\Big(\|\nabla\theta\|_{L^2}^2+\int|\nabla u|^2\theta\mathrm{d}x\Big)+C\int\rho u^2\theta^2\mathrm{d}x\\
&\quad+C\|\operatorname{div}u\|_{L^\infty}\Big(\|\nabla u\|_{L^2}^2+\int\rho\theta^2\mathrm{d}x\Big).
\end{aligned}\tag{3.13}
$$

Taking a constant $C_1>0$ with

$$\kappa C_1\ge C_0+1$$

and

$$(\mu+\lambda)(\operatorname{div}u)^2-2P\operatorname{div}u+(2C_1c_v-1)\rho\theta^2\ge0,\tag{3.14}$$

adding (3.11) multiplied by C_1 to (3.13) and (3.10), after choosing ε, δ suitably small, one has

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\mu|\nabla u|^2+(\mu+\lambda)(\operatorname{div}u)^2+2C_1c_v\rho\theta^2-2P\operatorname{div}u+\frac{1}{2\mu+\lambda}P^2\Big)\mathrm{d}x\\
&\quad+\frac12\int\rho|\dot u|^2\mathrm{d}x+\|\nabla\theta\|_{L^2}^2\\
&\le C_2\int|u|^2|\nabla u|^2\mathrm{d}x+C\int\rho|u|^2\theta^2\mathrm{d}x+C\|\operatorname{div}u\|_{L^\infty}\Big(\|\nabla u\|_{L^2}^2+\int\rho\theta^2\mathrm{d}x\Big).
\end{aligned}\tag{3.15}
$$

Note that we can choose constant C_1 sufficiently large such that the inequality (3.14) holds.

In the following, it suffices to estimate the key terms of $\int |u|^2|\nabla u|^2\mathrm{d}x$.

In fact, multiplying the momentum equation by $4|u|^2u$ and integrating the resulting equation over $\mathbb{R}^3$ lead to

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^4\mathrm{d}x + 4\int |u|^2[\mu|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + 2\mu|\nabla|u||^2]\mathrm{d}x\\
&= -4(\mu+\lambda)\int u\cdot\nabla|u|^2\operatorname{div} u\mathrm{d}x + 4\int \operatorname{div}(|u|^2u)P\mathrm{d}x\\
&\le 4(\mu+\lambda)\int |u|^2|\nabla u||\operatorname{div} u|\mathrm{d}x + 4R\int (|u|^2\operatorname{div} u + 2u\cdot\nabla u\cdot u)\rho\theta\mathrm{d}x\\
&\le 4(\mu+\lambda)\int |u|^2|\nabla u||\operatorname{div} u|\mathrm{d}x + C\int |u|^2|\nabla u|\rho\theta\mathrm{d}x\\
&\le \eta\int |u|^2|\nabla u|^2\mathrm{d}x + C\int |u|^2(\operatorname{div} u)^2\mathrm{d}x + C\int \rho u^2\theta^2\mathrm{d}x,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^4\mathrm{d}x + (4\mu-\eta)\int |u|^2|\nabla u|^2\mathrm{d}x + 4\int |u^2|[(\mu+\lambda)(\operatorname{div} u)^2 + 2\mu|\nabla|u||^2]\mathrm{d}x\\
&\le C\int |u|^2(\operatorname{div} u)^2\mathrm{d}x + C\int \rho u^2\theta^2\mathrm{d}x. \qquad (3.16)
\end{aligned}
$$

Then choosing the constant η suitably small such that $4\mu-\eta>0$, and adding (3.16) multiplied by $\frac{C_2}{4\mu-\eta}$ to (3.15), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \Big(\frac{C_2\rho|u|^4}{4\mu-\eta} + \mu|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + 2C_1c_v\rho\theta^2 - 2P\operatorname{div} u + \frac{1}{2\mu+\lambda}P^2\Big)\mathrm{d}x\\
&+\frac{1}{2}\int \rho|\dot{u}|^2\mathrm{d}x + \|\nabla\theta\|_{L^2}^2\\
&\le C\int |u|^2(\operatorname{div} u)^2\mathrm{d}x + C\int \rho u^2\theta^2\mathrm{d}x + C\|\operatorname{div} u\|_{L^\infty}\Big(\|\nabla u\|_{L^2}^2 + \int \rho\theta^2\mathrm{d}x\Big). \qquad (3.17)
\end{aligned}
$$

Next, we estimate the term of $\int |u|^2(\operatorname{div} u)^2\mathrm{d}x$ and $\int \rho u^2\theta^2\mathrm{d}x$, respectively.

It follows from Hölder inequality and (2.1) that

$$
\begin{aligned}
\int |u|^2|\operatorname{div} u|^2\mathrm{d}x &\le \|u\|_{L^6}^2\|\operatorname{div} u\|_{L^3}^2\\
&\le C\|\nabla u\|_{L^2}^2\|\operatorname{div} u\|_{L^3}^2\\
&\le C\|\nabla u\|_{L^2}^2(\|\operatorname{div} u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^\infty}^2). \qquad (3.18)
\end{aligned}
$$

Finally, Hölder's inequality and Young's inequality yield that

$$
\begin{aligned}
\int \rho u^2\theta^2\mathrm{d}x &= \int \rho^{\frac12}u^2\theta^\alpha\rho^{\frac12}\theta^{2-\alpha}\mathrm{d}x\\
&\le \|\rho^{\frac14}u\|_{L^4}^2\|\theta\|_{L^\beta}^\alpha\|\rho^{\frac{1}{2(2-\alpha)}}\theta\|_{L^{\frac{2\beta(2-\alpha)}{\beta-2\alpha}}}^{2-\alpha}\\
&\le C\|\theta\|_{L^\beta}^\alpha\big(\|\rho^{\frac14}u\|_{L^4}^4 + \|\rho^{\frac{1}{2(2-\alpha)}}\theta\|_{L^{\frac{2\beta(2-\alpha)}{\beta-2\alpha}}}^{4-2\alpha}\big),
\end{aligned}
$$

where $1\le\alpha<2,\ \beta\ge 4,\ \frac{3}{\alpha}+\frac{2}{\beta}\ge 2$ and $\frac{1}{\alpha}+\frac{2}{\beta}\le 1$, which implies that

$$\begin{aligned}\int\rho u^2\theta^2\mathrm{d}x&\le C\|\theta\|_{L^\beta}^\alpha(\|\rho^{\frac14}u\|_{L^4}^4+\|\rho\theta\|_{L^1}^2+\|\sqrt{\rho}\theta\|_{L^2}^2)\\&\le C\|\theta\|_{L^\beta}^\alpha\Big(1+\int\rho u^4\mathrm{d}x+\int\rho\theta^2\mathrm{d}x\Big),\end{aligned}\tag{3.19}$$

due to the interpolation inequality and (3.4).

Especially, when $\alpha=2$, we have the following estimate:

$$\int\rho u^2\theta^2\mathrm{d}x\le C\|\theta\|_{L^4}^2\|\rho^{\frac14}u\|_{L^4}^2.\tag{3.20}$$

In summary, we can choose $1\le\alpha\le 2,\ \beta\ge 4,\ \frac{3}{\alpha}+\frac{2}{\beta}\ge 2$ and $\frac{1}{\alpha}+\frac{2}{\beta}\le 1$.

Combining (3.17)–(3.20), it follows form Gronwall's inequality and the estimates (3.1), (3.4) that

$$\begin{aligned}&\sup_{t\in[0,T]}\int(\mu|\nabla u|^2+(\mu+\lambda)(\operatorname{div}u)^2+2C_1c_v\rho\theta^2-2P\operatorname{div}u)\mathrm{d}x\\&+\int_0^T\int(\rho|\dot{u}|^2+|\nabla\theta|^2)\mathrm{d}x\mathrm{d}t\le C.\end{aligned}\tag{3.21}$$

Finally, thanks to the condition (3.14), we obtain the estimate (3.4) and complete the proof of Lemma 3.3.

Lemma 3.4 *Suppose that the condition* (3.1) *holds. We obtain that*

$$\sup_{t\in[0,T]}\int(\rho|\dot{u}|^2+|\nabla\theta|^2)\mathrm{d}x+\int_0^T\int(\rho\dot{\theta}^2+|\nabla\dot{u}|^2)\mathrm{d}x\mathrm{d}t\le C,\quad 0\le T<T^*.\tag{3.22}$$

Proof Applying the operator $\dot{u}^j[\partial_t+\operatorname{div}(u\cdot)]$ to the j-th equation of the momentum equations ($j=1,2,3$) and integrating the resulting equations over $\mathbb{R}^3$, we obtain after integration by parts that

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\dot{u}|^2\mathrm{d}x&=-\int\dot{u}^j[\partial_jP_t+\operatorname{div}(u\partial_jP)]\mathrm{d}x+\mu\int\dot{u}^j[\Delta u_t^j+\operatorname{div}(u\Delta u^j)]\mathrm{d}x\\&\quad+(\mu+\lambda)\int\dot{u}^j[\partial_j\operatorname{div}u_t+\operatorname{div}(u\partial_j\operatorname{div}u)]\mathrm{d}x\\&=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.\end{aligned}\tag{3.23}$$

It follows from integration by parts and the continuity equation (1.1) that

$$\begin{aligned}\mathrm{I}_1&=-\int\dot{u}^j[\partial_jP_t+\operatorname{div}(u\partial_jP)]\mathrm{d}x\\&=R\int\partial_j\dot{u}^j[-\operatorname{div}(\rho u)\theta+\rho\theta_t]\mathrm{d}x+\int\partial_k\dot{u}^j\partial_jPu^k\mathrm{d}x\\&=R\int\partial_j\dot{u}^j(-\nabla\rho\cdot u\theta-\rho\operatorname{div}u\theta+\rho\dot{\theta}-\rho u\cdot\nabla\theta)\mathrm{d}x+\int\partial_k\dot{u}^j\partial_jPu^k\mathrm{d}x\\&=R\int\partial_j\dot{u}^j(-\rho\theta\operatorname{div}u+\rho\dot{\theta})\mathrm{d}x-R\int\partial_j\dot{u}^j(\theta\partial_k\rho u^k+\rho u^k\partial_k\theta)\mathrm{d}x\\&\quad-\int P(\partial_k\dot{u}^j\partial_ju^k+\partial_k\partial_j\dot{u}^ju^k)\mathrm{d}x\\&=R\int\partial_j\dot{u}^j(-\rho\theta\operatorname{div}u+\rho\dot{\theta})\mathrm{d}x+\int P\operatorname{div}u\operatorname{div}\dot{u}\mathrm{d}x-\int P\partial_k\dot{u}^j\partial_ju^k\mathrm{d}x.\end{aligned}$$

By Young's inequality, one has

$$
\begin{aligned}
|\mathrm{I}_1| &\le \frac{\mu}{8}\|\nabla\dot{u}\|_{L^2}^2 + C\int\rho\dot{\theta}^2\mathrm{d}x + C\int\rho^2\theta^2|\nabla u|^2\mathrm{d}x \\
&\le \frac{\mu}{8}\|\nabla\dot{u}\|_{L^2}^2 + C\int\rho\dot{\theta}^2\mathrm{d}x + C\|\rho\theta\|_{L^2}^{\frac12}\|\theta\|_{L^6}^{\frac32}\|\nabla u\|_{L^4}^2 \\
&\le \frac{\mu}{8}\|\nabla\dot{u}\|_{L^2}^2 + C\int\rho\dot{\theta}^2\mathrm{d}x + C\|\nabla u\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C,
\end{aligned}
\tag{3.24}
$$

where we have used the estimates (2.1), (3.3)–(3.4) and (3.8).

Furthermore, for the second term I_2, integrating by parts leads to

$$
\begin{aligned}
\mathrm{I}_2 &= \mu\int\dot{u}^j[\Delta u_t^j + \mathrm{div}(u\Delta u^j)]\mathrm{d}x \\
&= -\mu\int(\partial_i\dot{u}^j\partial_i u_t^j + \Delta u^j u\cdot\nabla\dot{u}^j)\mathrm{d}x \\
&= -\mu\int(|\nabla\dot{u}|^2 - \partial_i\dot{u}^j u^k\partial_k\partial_i u^j - \partial_i\dot{u}^j\partial_i u^k\partial_k u^j + \partial_i\partial_i u^j u^k\partial_k\dot{u}^j)\mathrm{d}x \\
&= -\mu\int(|\nabla\dot{u}|^2 + \partial_i\dot{u}^j\partial_i u^j\,\mathrm{div}\,u - \partial_i\dot{u}^j\partial_i u^k\partial_k u^j - \partial_k\dot{u}^j\partial_i u^k\partial_i u^j)\mathrm{d}x.
\end{aligned}
$$

Hence, it follows from Young's inequality that

$$
\mathrm{I}_2 \le -\frac{5\mu}{8}\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4. \tag{3.25}
$$

Similarly,

$$
\mathrm{I}_3 \le -\frac{5}{8}(\mu+\lambda)\|\mathrm{div}\,\dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 \le C\|\nabla u\|_{L^4}^4. \tag{3.26}
$$

Substituting (3.24)–(3.26) into (3.23), and using the estimate (2.7) with $p = 4$ and (3.8) yield that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\dot{u}|^2\mathrm{d}x + \mu\|\nabla\dot{u}\|_{L^2}^2 &\le C\int\rho\dot{\theta}^2\mathrm{d}x + C\|\nabla u\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C \\
&\le C\int\rho\dot{\theta}^2\mathrm{d}x + C\|\nabla u\|_{L^2}(\|\rho^{\frac12}\dot{u}\|_{L^2}^3 + \|\theta\|_{L^6}^3) + C\|\nabla\theta\|_{L^2}^4 + C \\
&\le C\int\rho\dot{\theta}^2\mathrm{d}x + C\|\rho^{\frac12}\dot{u}\|_{L^2}^4 + C\|\nabla\theta\|_{L^2}^4 + C.
\end{aligned}
\tag{3.27}
$$

On the other hand, multiplying the third equation in (1.1) by $\dot{\theta}$ and integrating the resulting equation over $\mathbb{R}^3$ give that

$$
\begin{aligned}
&c_v\int\rho\dot{\theta}^2\mathrm{d}x + \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\theta|^2\mathrm{d}x \\
&= -\kappa\int\nabla\theta\cdot\nabla(u\cdot\nabla\theta)\mathrm{d}x + 2\mu\int|\mathfrak{D}(u)|^2\dot{\theta}\mathrm{d}x + \lambda\int(\mathrm{div}\,u)^2\dot{\theta}\mathrm{d}x - R\int\rho\theta\dot{\theta}\mathrm{div}\,u\mathrm{d}x \\
&= \sum_{i=1}^4\mathrm{J}_i.
\end{aligned}
\tag{3.28}
$$

For the first term J_1, integrating by parts and applying Young's inequality, Gagliardo-Nirenberg inequality and (2.1) give that

$$\begin{aligned}|\mathrm{J}_1| &= \kappa\Big|\int \nabla\theta\cdot\nabla(u\cdot\nabla\theta)\mathrm{d}x\Big|\\ &\le C\int|\nabla u||\nabla\theta|^2\mathrm{d}x + C\int |u||\nabla^2\theta||\nabla\theta|\mathrm{d}x\\ &\le C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac12}\|\nabla\theta\|_{L^6}^{\frac32} + C\|\nabla^2\theta\|_{L^2}\|\nabla\theta\|_{L^3}\|u\|_{L^6}\\ &\le C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac12}\|\nabla\theta\|_{L^6}^{\frac32} + C\|\nabla^2\theta\|_{L^2}^{\frac32}\|\nabla\theta\|_{L^2}^{\frac12}\|u\|_{L^6}\\ &\le C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac12}\|\nabla^2\theta\|_{L^2}^{\frac32}\\ &\le \varepsilon\|\nabla^2\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2.\end{aligned}\tag{3.29}$$

By the standard L^2-estimate of the third equation in (1.1) and Hölder inequality, one has

$$\begin{aligned}\|\nabla^2\theta\|_{L^2}^2 &\le C\|\rho\dot\theta\|_{L^2}^2 + C\int\rho^2\theta^2|\nabla u|^2\mathrm{d}x + C\|\nabla u\|_{L^4}^4\\ &\le C\|\rho\dot\theta\|_{L^2}^2 + C\|\rho\theta\|_{L^2}^{\frac12}\|\theta\|_{L^6}^{\frac32}\|\nabla u\|_{L^4}^2 + C\|\nabla u\|_{L^4}^4\\ &\le C\|\rho\dot\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^{\frac32}\|\nabla u\|_{L^4}^2 + C\|\nabla u\|_{L^4}^4\\ &\le C\|\rho\dot\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^4 + C\|\nabla u\|_{L^4}^4 + C.\end{aligned}\tag{3.30}$$

In fact, we have used the interpolation equality here.

Then, substituting (3.30) into (3.29) yields that

$$|\mathrm{J}_1| \le \varepsilon\|\rho\dot\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^4 + C\|\nabla u\|_{L^4}^4 + C.\tag{3.31}$$

For the second term in the right hand side of (3.28), a series of direct computation yields that

$$\begin{aligned}\mathrm{J}_2 =&\ 2\mu\int|\mathfrak{D}(u)|^2\dot\theta\mathrm{d}x\\ =&\ 2\mu\int|\mathfrak{D}(u)|^2\theta_t\mathrm{d}x + 2\mu\int|\mathfrak{D}(u)|^2u\cdot\nabla\theta\mathrm{d}x\\ =&\ 2\mu\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathfrak{D}(u)|^2\theta\mathrm{d}x - 4\mu\int\theta\mathfrak{D}(u):\mathfrak{D}(u_t)\mathrm{d}x + 2\mu\int|\mathfrak{D}(u)|^2u\cdot\nabla\theta\mathrm{d}x\\ =&\ 2\mu\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathfrak{D}(u)|^2\theta\mathrm{d}x - \mu\int\theta(\partial_iu^j+\partial_ju^i)(\partial_iu_t^j+\partial_ju_t^i)\mathrm{d}x + 2\mu\int|\mathfrak{D}(u)|^2u\cdot\nabla\theta\mathrm{d}x\\ =&\ 2\mu\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathfrak{D}(u)|^2\theta\mathrm{d}x - \mu\int\theta(\partial_iu^j+\partial_ju^i)(\partial_i\dot u^j+\partial_j\dot u^i-\partial_i(u\cdot\nabla u^j)-\partial_j(u\cdot\nabla u^i))\mathrm{d}x\\ &+2\mu\int|\mathfrak{D}(u)|^2u\cdot\nabla\theta\mathrm{d}x\\ =&\ 2\mu\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathfrak{D}(u)|^2\theta\mathrm{d}x - 4\mu\int\theta\mathfrak{D}(u):\mathfrak{D}(\dot u)\mathrm{d}x + \mu\int\theta(\partial_iu^j+\partial_ju^i)(\partial_iu^k\partial_ku^j\\ &+\partial_ju^k\partial_ku^i)\mathrm{d}x + 4\mu\int\theta\mathfrak{D}(u):(u\cdot\nabla)\mathfrak{D}(u)\mathrm{d}x + 2\mu\int|\mathfrak{D}(u)|^2u\cdot\nabla\theta\mathrm{d}x\\ =&\ 2\mu\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathfrak{D}(u)|^2\theta\mathrm{d}x - 4\mu\int\theta\mathfrak{D}(u):\mathfrak{D}(\dot u)\mathrm{d}x + \mu\int\theta(\partial_iu^j+\partial_ju^i)(\partial_iu^k\partial_ku^j\\ &+\partial_ju^k\partial_ku^i)\mathrm{d}x - 2\mu\int\theta|\mathfrak{D}(u)|^2\mathrm{div}\,u\mathrm{d}x.\end{aligned}$$

Then, by the elementary inequalities and the interpolation inequality, we have

$$\begin{aligned}
\mathrm{J}_2 &\le 2\mu\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathfrak{D}(u)|^2\theta\mathrm{d}x + C\int\theta|\nabla u||\nabla\dot{u}|\mathrm{d}x + C\int\theta|\nabla u|^3\mathrm{d}x\\
&\le 2\mu\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathfrak{D}(u)|^2\theta\mathrm{d}x + C\|\theta\|_{L^6}\|\nabla u\|_{L^3}(\|\nabla\dot{u}\|_{L^2} + \|\nabla u\|_{L^4}^2)\\
&\le 2\mu\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathfrak{D}(u)|^2\theta\mathrm{d}x + C\|\nabla\theta\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{3}}\|\nabla u\|_{L^4}^{\frac{2}{3}}(\|\nabla\dot{u}\|_{L^2} + \|\nabla u\|_{L^4}^2)\\
&\le 2\mu\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathfrak{D}(u)|^2\theta\mathrm{d}x + \frac{\eta}{2}\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C.
\end{aligned}\tag{3.32}$$

Similarly to the arguments to J_2, we obtain that

$$\mathrm{J}_3 \le \lambda\frac{\mathrm{d}}{\mathrm{d}t}\int|\operatorname{div} u|^2\theta\mathrm{d}x + \frac{\eta}{2}\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C.\tag{3.33}$$

Finally, it follows from (2.1) and the basic inequalities that

$$\begin{aligned}
|\mathrm{J}_4| &= R\Big|\int\rho\theta\dot{\theta}\operatorname{div} u\mathrm{d}x\Big|\\
&\le C\|\rho^{\frac{1}{2}}\dot{\theta}\|_{L^2}\|\rho^{\frac{1}{2}}\theta\|_{L^4}\|\nabla u\|_{L^4}\\
&\le C\|\rho^{\frac{1}{2}}\dot{\theta}\|_{L^2}\|\rho^{\frac{1}{2}}\theta\|_{L^2}^{\frac{1}{4}}\|\rho^{\frac{1}{2}}\theta\|_{L^6}^{\frac{3}{4}}\|\nabla u\|_{L^4}\\
&\le \varepsilon\int\rho\dot{\theta}^2\mathrm{d}x + C\|\nabla u\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C.
\end{aligned}\tag{3.34}$$

Substituting (3.31)–(3.34) into (3.28), and choosing ε suitably small give that for any $\eta\in(0,1]$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\frac{\kappa}{2}|\nabla\theta|^2 - \theta(2\mu|\mathfrak{D}(u)|^2 + \lambda(\operatorname{div} u)^2)\Big)\mathrm{d}x + \frac{c_v}{2}\int\rho\dot{\theta}^2\mathrm{d}x\\
&\le \eta\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^4 + C\|\nabla u\|_{L^4}^4 + C.
\end{aligned}\tag{3.35}$$

On the other hand, it follows from (2.7) that

$$\begin{aligned}
\|\nabla u\|_{L^4} &\le C\|\nabla u\|_{L^2}^{\frac{1}{4}}(\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2} + \|\theta\|_{L^6})^{\frac{3}{4}}\\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{4}}(\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2} + \|\nabla\theta\|_{L^2})^{\frac{3}{4}}.
\end{aligned}\tag{3.36}$$

Then, substituting (3.36) into (3.35) yields that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\frac{\kappa}{2}|\nabla\theta|^2 - \theta(2\mu|\mathfrak{D}(u)|^2 + \lambda(\operatorname{div} u)^2)\Big)\mathrm{d}x + \frac{c_v}{2}\int\rho\dot{\theta}^2\mathrm{d}x\\
&\le \eta\|\nabla\dot{u}\|_{L^2}^2 + C\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2}^4 + C\|\nabla\theta\|_{L^2}^4 + C.
\end{aligned}\tag{3.37}$$

Hence, choosing η suitably small and adding (3.27) multiplied by $C_3 = \frac{2\eta}{\mu}$ to (3.37), we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(C_3\rho|\dot{u}|^2 + \frac{\kappa}{2}|\nabla\theta|^2 - \theta(2\mu|\mathfrak{D}(u)|^2 + \lambda(\operatorname{div} u)^2)\Big)\mathrm{d}x + \eta\|\nabla\dot{u}\|_{L^2}^2 + \frac{c_v}{4}\int\rho\dot{\theta}^2\mathrm{d}x\\
&\le C\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2}^4 + C\|\nabla\theta\|_{L^2}^4 + C.
\end{aligned}\tag{3.38}$$

If follows from Gronwall's inequality and (3.8) that

$$\sup_{t\in[0,T]}\Big(\int C_3\rho|\dot u|^2+\frac{\kappa}{2}|\nabla\theta|^2-\theta(2\mu|\mathfrak{D}(u)|^2+\lambda(\operatorname{div}u)^2)\mathrm{d}x\Big)$$
$$+\eta\int_0^T\int|\nabla\dot u|^2\mathrm{d}x\mathrm{d}t+\frac{c_v}{4}\int_0^T\int\rho\dot\theta^2\mathrm{d}x\mathrm{d}t\le C. \tag{3.39}$$

Finally, note that by (2.7) and the elementary inequalities, one has

$$\begin{aligned}\int\theta(2\mu|\mathfrak{D}(u)|^2+\lambda(\operatorname{div}u)^2)\mathrm{d}x&\le C\|\theta\|_{L^6}\|\nabla u\|^2_{L^{\frac{12}{5}}}\\&\le C\|\nabla\theta\|_{L^2}\|\nabla u\|^{\frac43}_{L^2}\|\nabla u\|^{\frac23}_{L^4}\\&\le\frac{\kappa}{8}\|\nabla\theta\|^2_{L^2}+C\|\nabla u\|^{\frac43}_{L^4}\\&\le\frac{\kappa}{8}\|\nabla\theta\|^2_{L^2}+C\|\nabla u\|^{\frac13}_{L^2}(\|\rho^{\frac12}\dot u\|_{L^2}+\|\theta\|_{L^6})\\&\le\frac{\kappa}{8}\|\nabla\theta\|^2_{L^2}+C\|\rho^{\frac12}\dot u\|_{L^2}+C\|\nabla\theta\|_{L^2}\\&\le\frac{\kappa}{4}\|\nabla\theta\|^2_{L^2}+\frac{C_3}{2}\|\rho^{\frac12}\dot u\|^2_{L^2}+C.\end{aligned}$$

Substituting it into (3.39) yields (3.22) and this completes the proof of Lemma 3.4.

Lemma 3.5 *Suppose that the conditions* (3.1) *holds. We have*

$$\sup_{t\in[0,T]}\int\rho\dot\theta^2\mathrm{d}x+\int_0^T\int|\nabla\dot\theta|^2\mathrm{d}x\mathrm{d}t\le C,\quad 0\le T<T^*. \tag{3.40}$$

Proof First, it follows from the estimates (2.7), (3.8), (3.22) and (3.30) that we have the following fact

$$\begin{aligned}&\sup_{t\in[0,T]}(\|\theta\|_{L^6}+\|\nabla u\|_{L^6})+\int_0^T\|\nabla^2\theta\|^2_{L^2}\mathrm{d}t\\&\le\sup_{t\in[0,T]}(C\|\nabla\theta\|_{L^2}+C\|\rho\dot u\|_{L^2})+C\int_0^T(\|\rho\dot\theta\|^2_{L^2}+\|\nabla\theta\|^4_{L^2}+\|\nabla u\|^4_{L^4}+1)\mathrm{d}t\\&\le C.\end{aligned} \tag{3.41}$$

Furthermore, applying the operator $\partial_t+\operatorname{div}(u\cdot)$ to the third equation in (1.1) and a series of direct computations give that

$$\begin{aligned}c_v(\rho\partial_t\dot\theta+\rho u\cdot\nabla\dot\theta)=\ &\kappa\triangle\dot\theta+\kappa(\triangle\theta\operatorname{div}u-\partial_i(\partial_iu\cdot\nabla\theta)-\partial_iu\cdot\nabla\partial_i\theta)\\&+(2\mu|\mathfrak{D}(u)|^2+\lambda(\operatorname{div}u)^2)\operatorname{div}u+P\partial_ku^l\partial_lu^k\\&-R\rho\dot\theta\operatorname{div}u-P\operatorname{div}\dot u+2\lambda(\operatorname{div}\dot u-\partial_ku^l\partial_lu^k)\operatorname{div}u\\&+\mu(\partial_iu^j+\partial_ju^i)(\partial_i\dot u^j+\partial_j\dot u^i-\partial_iu^k\partial_ku^j-\partial_ju^k\partial_ku^i).\end{aligned} \tag{3.42}$$

Then, multiplying (3.42) by $\dot\theta$, after integration by parts and using (3.8), (3.22) and (3.41) yield that

$$
\begin{aligned}
&\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\dot\theta^2\mathrm{d}x+\kappa\|\nabla\dot\theta\|_{L^2}^2\\
&\le C\int|\nabla u|(|\nabla^2\theta||\dot\theta|+|\nabla\theta||\nabla\dot\theta|)\mathrm{d}x+C\int|\nabla u|^2|\dot\theta|(|\nabla u|+\theta)\mathrm{d}x\\
&\quad+C\int\rho|\dot\theta|^2|\nabla u|\mathrm{d}x+C\int\rho\theta|\dot\theta||\nabla\dot u|\mathrm{d}x+C\int|\nabla u||\dot\theta||\nabla\dot u|\mathrm{d}x\\
&\le C\|\nabla u\|_{L^3}\|\nabla\theta\|_{H^1}(\|\dot\theta\|_{L^6}+\|\nabla\dot\theta\|_{L^2})+C\|\nabla u\|_{L^3}^2\|\dot\theta\|_{L^6}(\|\nabla u\|_{L^6}+\|\theta\|_{L^6})\\
&\quad+C\|\nabla u\|_{L^3}\|\rho\dot\theta\|_{L^2}\|\dot\theta\|_{L^6}+C\|\rho^{\frac12}\theta\|_{L^2}^{\frac12}\|\theta\|_{L^6}^{\frac12}\|\nabla\dot u\|_{L^2}\|\dot\theta\|_{L^6}+C\|\nabla u\|_{L^3}\|\nabla\dot u\|_{L^2}\|\dot\theta\|_{L^6}\\
&\le\frac{\kappa}{2}\|\nabla\dot\theta\|_{L^2}^2+C\|\nabla^2\theta\|_{L^2}^2+C\|\rho^{\frac12}\dot\theta\|_{L^2}^2+C\|\nabla\dot u\|_{L^2}^2+C.
\end{aligned}
$$

Applying Gronwall's inequality, (3.22) and (3.41) directly gives (3.40).

Finally, the following lemma gives the higher order estimates of the solutions.

Lemma 3.6 *Suppose that the condition* (3.1) *holds. We have*

$$
\sup_{t\in[0,T]}(\|\nabla u\|_{H^1}+\|\nabla\theta\|_{H^1}+\|\rho\|_{H^1\cap W^{1,q_0}})\le C,\quad 0\le T<T^*. \tag{3.43}
$$

Proof First, combining the known estimates (3.8), (3.22) and (3.40) and the inequalities (2.1)–(2.2), we have

$$
\sup_{t\in[0,T]}\|\nabla\theta\|_{H^1}\le C \tag{3.44}
$$

and

$$
\sup_{t\in[0,T]}\|\theta\|_{L^\infty}\le C. \tag{3.45}
$$

Thus from (2.1)–(2.2), (2.5)–(2.6), we have

$$
\begin{aligned}
\|\mathrm{div}\,u\|_{L^\infty}^2+\|\omega\|_{L^\infty}^2&\le C(\|F\|_{L^\infty}^2+\|P\|_{L^\infty}^2)+\|\omega\|_{L^\infty}^2\\
&\le C(\|F\|_{L^2}^2+\|\nabla F\|_{L^6}^2+\|\omega\|_{L^2}^2+\|\nabla\omega\|_{L^6}^2+\|\theta\|_{L^\infty}^2)\\
&\le C\|\nabla\dot u\|_{L^2}^2+C.
\end{aligned}\tag{3.46}
$$

Next, for $2\le p\le q_0$ $(3<q_0\le6)$, $|\nabla\rho|^p$ satisfies

$$
\begin{aligned}
&(|\nabla\rho|^p)_t+\mathrm{div}(|\nabla\rho|^pu)+(p-1)|\nabla\rho|^p\mathrm{div}\,u\\
&+p|\nabla\rho|^{p-2}(\nabla\rho)^{\mathrm{tr}}\nabla u(\nabla\rho)+p\rho|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\mathrm{div}\,u=0.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^p}&\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p}+C\|\nabla^2u\|_{L^p}\\
&\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p}+C\|\nabla\dot u\|_{L^2}+C,
\end{aligned}\tag{3.47}
$$

where we have used the following facts

$$
\begin{aligned}
\|\nabla^2u\|_{L^p}&\le C(\|\rho\dot u\|_{L^p}+\|\nabla(\rho\theta)\|_{L^p})\\
&\le C(\|\rho\dot u\|_{L^2}+\|\rho\dot u\|_{L^6}+\|\theta\|_{L^\infty}\|\nabla\rho\|_{L^p})+C\\
&\le C(1+\|\nabla\dot u\|_{L^2}+\|\nabla\rho\|_{L^p})
\end{aligned}\tag{3.48}
$$

due to (3.22), (3.44)–(3.45) and interpolation inequality.

Thus following from (2.8), (3.8) and (3.48), we have

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le C + C(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty})\ln(\mathrm{e} + \|\nabla^2 u\|_{L^{q_0}})\\ &\le C + C(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty})\ln(\mathrm{e} + \|\nabla\rho\|_{L^{q_0}})\\ &\quad + C(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty})\ln(\mathrm{e} + \|\nabla\dot u\|_{L^2}).\end{aligned} \tag{3.49}$$

Let $p = q_0$, substituting (3.49) into (3.47), and using (3.46), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\ln(\mathrm{e} + \|\nabla\rho\|_{L^{q_0}})\\ &\le C(1 + \|\operatorname{div} u\|_{L^\infty} + \|\operatorname{curl} u\|_{L^\infty} + \|\nabla\dot u\|_{L^2})\ln(\mathrm{e} + \|\nabla\dot u\|_{L^2})\ln(\mathrm{e} + \|\nabla\rho\|_{L^{q_0}})\\ &\le C(1 + \|\nabla\dot u\|_{L^2}^2)\ln(\mathrm{e} + \|\nabla\rho\|_{L^{q_0}}).\end{aligned}$$

This together with Gronwall's inequality and (3.22) gives that

$$\sup_{t\in[0,T]}\|\nabla\rho\|_{L^{q_0}} \le C. \tag{3.50}$$

Combining (3.22), (3.46) and (3.49), we have

$$\int_0^T \|\nabla u\|_{L^\infty}\mathrm{d}t \le C. \tag{3.51}$$

Then, taking $p = 2$ in (3.47), we obtain

$$\sup_{t\in[0,T]}\|\nabla\rho\|_{L^2} \le C, \tag{3.52}$$

due to (3.22) and (3.51).

Moreover, letting $p = 2$ in (3.48) and together with (3.45), (3.52) and (3.22) yields

$$\|\nabla^2 u\|_{L^2} \le C(\|\rho\dot u\|_{L^2} + \|\nabla\rho\|_{L^2} + \|\nabla\theta\|_{L^2}) \le C. \tag{3.53}$$

Hence, the estimates (3.44), (3.50), (3.52)–(3.53) imply (3.43) and we complete the proof of Lemma 3.6.

With the aid of the a priori estimates established above, we will complete the proof of Theorem 1.1.

In fact, the generic constants C in Lemmas 3.1–3.6 remain uniformly bounded for all $T < T^*$, so we can extend the strong solution of (ρ, u, θ) beyond $t > T^*$. Furthermore, the functions $(\rho, u, \theta)(x, T^*) \triangleq \lim_{t\to T^*}(\rho, u, \theta)(x, t)$ satisfy the conditions imposed on the initial data at the time $t = T^*$. In additional, we have $(\rho\dot u, \rho\dot\theta) \in C([0, T]; L^2)$, which implies

$$(\rho\dot u, \rho\dot\theta)(x, T^*) = \lim_{t\to T^*}(\rho\dot u, \rho\dot\theta)(x, t) \in L^2.$$

The compatibility conditions are given as follows:

$$\begin{aligned}&(-\mu\Delta u - (\mu+\lambda)\nabla\operatorname{div} u + R\nabla(\rho\theta))|_{t=T^*} = \rho^{\frac12}(x, T^*)g_1(x, T^*),\\ &(\kappa\Delta\theta + 2\mu|\mathfrak{D}(u)|^2 + \lambda(\operatorname{div} u)^2)|_{t=T^*} = \rho^{\frac12}(x, T^*)g_2(x, T^*),\end{aligned}$$

where

$$g_1(x) \triangleq \begin{cases} \rho^{-\frac{1}{2}}(x,T^*)(\rho\dot{u})(x,T^*), & \text{for } x \in \{x \mid \rho(x,T^*) > 0\}, \\ 0, & \text{for } x \in \{x \mid \rho(x,T^*) = 0\} \end{cases}$$

and

$$g_2(x) \triangleq \begin{cases} \rho^{-\frac{1}{2}}(x,T^*)[\rho\dot{\theta} + R\rho\theta \operatorname{div} u](x,T^*), & \text{for } \ x \in \{x \mid \rho(x,T^*) > 0\}, \\ 0, & \text{for } \ x \in \{x \mid \rho(x,T^*) = 0\}. \end{cases}$$

It is clear that $g^1,\ g^2 \in L^2$ due to the estimates (3.22), (3.40) and (3.43). Thus, (ρ,u,θ) (x,T^*) satisfy compatibility conditions (1.9) and (1.10). Therefore, the local strong solution beyond T^* can be extended by taking $(\rho,u,\theta)(x,T^*)$ as the initial data and Lemma 2.1, which contradicts to the assumption on T^*. This completes the proof of Theorem 1.1.

Acknowledgement The authors would like to thank the referees for careful reading and valuable suggestions to update our paper.

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