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Blow up for Initial-Boundary Value Problem of Wave Equation with a Nonlinear Memory in 1-D*

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Abstract The present paper is devoted to studying the initial-boundary value problem of a 1-D wave equation with a nonlinear memory:

$$u_{tt} - u_{xx} = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |u(s)|^p \mathrm{d}s.$$

The blow up result will be established when $p > 1$ and $0 < \gamma < 1$, no matter how small the initial data are, by introducing two test functions and a new functional.

Keywords Blow up, Wave equation, Nonlinear memory, Initial-boundary value problem
2010 MR Subject Classification 35L05, 35L70

1 Introduction

In this paper we consider the initial-boundary value problem of a wave equation with a nonlinear memory as following:

$$\begin{cases} u_{tt} - u_{xx} = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} |u(s)|^p \mathrm{d}s, \quad t > 0, \ x \in (0,\infty), \\ u(0,x) = \varepsilon u_0(x), \quad u_t(0,x) = \varepsilon u_1(x), \quad x \in (0,\infty), \\ u|_{x=0} = 0, \end{cases} \tag{1.1}$$

where $0 < \gamma < 1$, $p > 1$, Γ denotes the Euler gamma function and ε is a positive small parameter. We assume that the data are compact supported and satisfy

$$\begin{aligned} &u_0 \in C^2(0,\infty), \quad u_1 \in C^1(0,\infty), \\ &\text{supp } u_i \subset \{x \mid 0 < x < R\}, \quad i = 0, 1, \end{aligned} \tag{1.2}$$

where $R > 0$ is a constant. We also assume that the initial data (u_0, u_1) and the Dirichlet boundary condition satisfy the following compatibility condition at the origin:

$$u_0(0) = u_0''(0) = u_1(0) = 0. \tag{1.3}$$

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Noting that the limit

$$\lim_{\gamma\to 1}\frac{1}{\Gamma(1-\gamma)}s_+^{-\gamma}=\delta(s)$$

holds in the distribution sense. The first equation of system (1.1) can be considered as the approximation of the classical one-dimensional semilinear wave equation

$$u_{tt}-u_{xx}=|u|^p. \tag{1.4}$$

It is well known that the above equation in higher dimensions is connected with the Strauss' conjecture, which is a Cauchy problem with small data of the following:

$$\begin{cases} u_{tt}-\Delta u=u_{tt}-\sum_{i=1}^{n}u_{x_ix_i}=|u|^p, & t>0,\ x\in\mathbb{R}^n,\\ u(0,x)=\varepsilon f(x),\quad u_t(0,x)=\varepsilon g(x), & x\in\mathbb{R}^n. \end{cases} \tag{1.5}$$

The first work for system (1.5) is due to John [9], in which he considered the case $n=3$ and regular data with compact support and obtained two main results: (I) When $1<p<1+\sqrt{2}$ and the data (f,g) are nonnegative, the solution blows up in a finite time; (II) when $p>1+\sqrt{2}$, the solution exists globally in time if the data are small enough. After that, Strauss [18] conjectured that for each $n\geq 2$, there exists a critical power $p_c(n)$, which is the positive root of the quadratic equation

$$(n-1)p^2-(n+1)p-2=0,$$

such that this critical power divides $(1,\infty)$ into two subintervals. If $p\in(1,p_c(n)]$, then solutions with nonnegative data blow up in a finite time. When $p\in(p_c(n),\infty)$, solutions with small initial data will exist globally in time. Since then, this problem has become the focus of interest of many authors and was finally well solved recently. Glassey [4–5] showed the blow up result for $1<p<p_c(2)$ and global existence for $p>1+p_c(2)$. Sideris [16] established the blow up result for $1<p<p_c(n)$ and $n\geq 4$, the proof of which was quite sophisticated and was simplified by Rammaha [14] and Jiao and Zhou [8]. Schaeffer [15] showed the blow up results for a critical exponent $p=p_c(n)$ and $n=2,3$. The global existence for $p>p_c(4)$ was proved by Zhou [22]. Lindblad and Sogge [13] established the global existence for all dimensions under the assumption of radial symmetry and for $n\leq 8$ without the radial symmetric assumption. Not much later, the case of $\frac{n+3}{n-1}\geq p>p_c(n)$ and $n\geq 4$ was solved by Georgiev, Lindblad and Sogge [3]. Tataru [19] also gave a simple proof for the latter case. The remaining part of Strauss' conjecture is the blow up for $p=p_c(n)$ and $n\geq 4$, which was solved by Yordanov and Zhang [21] and Zhou [23] independently. Recently, the first author and Zhou [10] gave an elementary proof of Strauss' conjecture and obtained furthermore the sharp lifespan estimate from below for the case $1<p<p_c(n)$ and $n\geq 4$.

For the Strauss' conjecture of the initial-boundary value problem in an exterior domain, Zhou and Han [24] established the blow up result for $1<p<p_c(n)$ and $n\geq 3$. Li and Wang

[12] showed the non-global existence for $1 < p < p_c(2)$. The first author and Zhou [11] obtained the blow up result and the upper bound of the lifespan estimate for $p = p_c(3)$. In the case of $p > p_c(n)$, it is known that the solutions exist globally in time from the works of Du et al [1] for $n = 4$, Hidano et al [7] for $n = 3, 4$ and Smith, Sogge and Wang [17] for $n = 2$. Recently, Han [6] studied the initial-boundary value problem of (1.4) in 1-D and obtained the blow up result for $p > 1$.

For the Cauchy problem of a wave equation with a nonlinear memory, Fino, Kirane and Georgiev [2] introduced the left-handed and right-handed Riemann-Liouville fractional derivatives and showed the blow up result in a finite time when the exponents γ and p satisfy some conditions. In this paper, we will study the initial-boundary value problem (1.1) with small initial data satisfying (1.2). We expect that the problem admits a similar result as that of the classical semilinear wave equation for $p > 1$ and $0 < \gamma < 1$. For this purpose, we will use the test function method, which was originally due to the work of Yordanov and Zhang [20–21].

Set

$$\phi_0(x) = x,$$
$$\phi(t,x) = \mathrm{e}^{-t}\phi_1(x) = \mathrm{e}^{-t}(\mathrm{e}^{x} - \mathrm{e}^{-x}),$$

while the weighted average functional

$$F_0(t) = \int_0^\infty u(t,x)\phi_0(x)\mathrm{d}x,$$

where u solves the problem (1.1). The necessity for introducing the two test functions lies in that $\phi_0(x)$ satisfies the null Dirichlet boundary condition at $x = 0$, which is helpful when applying integration by parts, and $\phi(t,x)$ is used to improve the lower bound of $F_0(t)$. Comparing with the initial-boundary value problem of semilinear wave equations studied in [6], we can not use the classical Kato-type lemma (see Lemma 4.1) directly because of the nonlinear memory term. However, following the idea in [2], we employ an iteration method to prove the auxiliary functional

$$I(t) = \int_b^t (t-s)^\alpha F_0(s)\mathrm{d}s$$

with some constant $\alpha > 0$ satisfying the conditions of the Kato-type lemma and then we obtain the desired blow up result.

We can get the local existence of the initial-boundary value problem (1.1) as following.

Theorem 1.1 *Assuming that the data* (u_0, u_1) *satisfy* (1.2) *and the compatibility condition* (1.3). *Then there exists* $T > 0$ *and a unique solution which solves problem* (1.1) *and belongs to*

$$C^2([0,T] \times (0,\infty)). \tag{1.6}$$

Proof The first step to prove the local existence theorem is to extend u_0 and u_1 as odd functions on $-\infty < x < \infty$:

$$\widetilde{u}_0(x) = \begin{cases} u_0(x), & x \geq 0, \\ -u_0(x), & x < 0, \end{cases} \qquad \widetilde{u}_1(x) = \begin{cases} u_1(x), & x \geq 0, \\ -u_1(x), & x < 0. \end{cases} \tag{1.7}$$

Using the boundary condition, we can get that the solution of problem (1.1) can be obtained by restricting the solution of the following Cauchy problem:

$$u_{tt} - u_{xx} = \begin{cases} \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} |u(s)|^p \mathrm{d}s, & t > 0,\ x \geq 0, \\ -\dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} |u(s)|^p \mathrm{d}s, & t > 0,\ x < 0. \end{cases} \tag{1.8}$$

The local existence of solutions to the Cauchy problem (1.8) with initial data (1.7) and $1 < p < \infty$ has been established in [2] by the contraction mapping method and we omit the details here. Then, we can get the conclusion of Theorem 1.1.

Furthermore, we can also consider the blow up result of the initial-boundary value problem of system (1.1) under the positive assumptions on the initial data. We can get the following main result.

Theorem 1.2 *Let $0 < \gamma < 1$ and $1 < p < \infty$. Under the assumptions that the initial data (u_0, u_1) satisfy* (1.2)–(1.3), *we further assume*

$$u_i(x) \geq 0, \quad u_i(x) \not\equiv 0, \quad i = 0, 1. \tag{1.9}$$

Then the solution to the initial-boundary value problem of system (1.1) *in the space* (1.6) *must blow up in a finite time.*

This paper is organized as follows: In Section 2 we will introduce two test functions and give some estimates about them. In Section 3, an iteration argument for $F_0(t)$ is shown. The blow up for the function $I(t)$ is given in Section 4.

2 Preliminaries

In this section we will introduce two test functions and give some basic estimates which play an important role in the proof of our main result. Denote the following functions:

$$\phi_0(x) = x,$$
$$\phi_1(x) = \mathrm{e}^x - \mathrm{e}^{-x}.$$

Then, we have

$$\begin{cases} \dfrac{\mathrm{d}^2\phi_0(x)}{\mathrm{d}x^2} = 0, & x \geq 0, \\ \phi_0(x)|_{x=0} = 0 \end{cases} \tag{2.1}$$

and

$$\begin{cases} \dfrac{\mathrm{d}^2\phi_1(x)}{\mathrm{d}x^2} = \phi_1(x), & x \geq 0, \\ \phi_1(x)|_{x=0} = 0. \end{cases} \tag{2.2}$$

Set

$$\phi(t,x) = \mathrm{e}^{-t}\phi_1(x),$$

then it holds that

$$\phi_t(t,x) = -\phi(t,x), \quad \phi_{tt}(t,x) = \phi(t,x), \quad \phi_{xx}(t,x) = \phi(t,x). \tag{2.3}$$

Lemma 2.1 *Let $\phi(t,x)$ be as above. Then, for any $p > 1$ and $t > 0$, we have*

$$\int_0^{t+R} |\phi(t,x)|^{\frac{p}{p-1}} \mathrm{d}x \le C. \tag{2.4}$$

Here and in the following, C denotes a positive constant which may change from line to line and R is as in (1.2).

Proof It can be done by a direct computation:

$$\begin{aligned}\int_0^{t+R} |\phi(t,x)|^{\frac{p}{p-1}} \mathrm{d}x &= \mathrm{e}^{-\frac{pt}{p-1}} \int_0^{t+R} (\mathrm{e}^x - \mathrm{e}^{-x})^{\frac{p}{p-1}} \mathrm{d}x \\ &\le \mathrm{e}^{-\frac{pt}{p-1}} \int_0^{t+R} \mathrm{e}^{\frac{px}{p-1}} \mathrm{d}x \\ &\le C.\end{aligned}$$

Lemma 2.2 *Let $\phi(t,x)$ be as above. Then for $p > 1$, we have*

$$\int_0^{t+R} x^{-\frac{1}{p-1}} |\phi(t,x)|^{\frac{p}{p-1}} \mathrm{d}x \le C. \tag{2.5}$$

Proof We divide the left-hand side of the above integral into two parts

$$\begin{aligned}&\int_0^{t+R} x^{-\frac{1}{p-1}} |\phi(t,x)|^{\frac{p}{p-1}} \mathrm{d}x \\ &= \int_0^{R} x^{-\frac{1}{p-1}} |\phi(t,x)|^{\frac{p}{p-1}} \mathrm{d}x + \int_R^{t+R} x^{-\frac{1}{p-1}} |\phi(t,x)|^{\frac{p}{p-1}} \mathrm{d}x \\ &\triangleq \mathrm{I}_1 + \mathrm{I}_2.\end{aligned} \tag{2.6}$$

Firstly, we will estimate the term I_1. For $x \in [0, R]$, it is easy to get

$$\phi_1(x) = \phi_1(x) - \phi_1(0) \le \max_{x\in[0,R]} \phi_1'(x)|x-0| \le Cx. \tag{2.7}$$

Then

$$\begin{aligned}\mathrm{I}_1 &= \int_0^{R} x^{-\frac{1}{p-1}} |\phi(t,x)|^{\frac{p}{p-1}} \mathrm{d}x \\ &= \mathrm{e}^{-\frac{pt}{p-1}} \int_0^{R} x^{-\frac{1}{p-1}} \phi_1(x)^{\frac{p}{p-1}} \mathrm{d}x \\ &\le C\mathrm{e}^{-\frac{pt}{p-1}} \int_0^{R} x^{-\frac{1}{p-1}} x^{\frac{p}{p-1}} \mathrm{d}x\end{aligned}$$

$$\leq C. \tag{2.8}$$

For the term I_2, using (2.4) in Lemma 2.1, we have

$$\begin{aligned}
\mathrm{I}_2 &= \int_R^{t+R} x^{-\frac{1}{p-1}} |\phi(t,x)|^{\frac{p}{p-1}} \mathrm{d}x \\
&\leq R^{-\frac{p}{p-1}} \int_R^{t+R} \phi(t,x)^{\frac{p}{p-1}} \mathrm{d}x \\
&\leq C \int_0^{t+R} \phi(t,x)^{\frac{p}{p-1}} \mathrm{d}x \\
&\leq C.
\end{aligned} \tag{2.9}$$

3 Iteration Argument for $F_0(t)$

For proving our main result, in this section we will give the following two functionals:

$$\begin{aligned}
F_0(t) &= \int_0^\infty u(t,x) x \mathrm{d}x, \\
F_1(t) &= \int_0^\infty u(t,x)\phi(t,x) \mathrm{d}x,
\end{aligned}$$

where u is the local solution of problem (1.1). It is easy to see that $F_0(t)$ and $F_1(t)$ are twice continuously differentiable with respect to t. First we claim that

$$F_1(t) \geq C, \quad t > 0 \tag{3.1}$$

for some positive constant C.

In the following, we will give the proof of (3.1). From (2.3), it is easy to get

$$\begin{aligned}
F_1'(t) &= \int_0^\infty u_t \phi \mathrm{d}x + \int_0^\infty u \phi_t \mathrm{d}x \\
&= \int_0^\infty u_t \phi \mathrm{d}x - \int_0^\infty u \phi \mathrm{d}x
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
F_1''(t) &= \int_0^\infty u_{tt} \phi \mathrm{d}x + 2\int_0^\infty u_t \phi_t \mathrm{d}x + \int_0^\infty u \phi_{tt} \mathrm{d}x \\
&= \int_0^\infty u_{tt} \phi \mathrm{d}x - 2\int_0^\infty u_t \phi \mathrm{d}x + \int_0^\infty u \phi \mathrm{d}x.
\end{aligned} \tag{3.3}$$

Then

$$\begin{aligned}
F_1''(t) + 2F_1'(t) &= \int_0^\infty u_{tt} \phi \mathrm{d}x - \int_0^\infty u \phi \mathrm{d}x \\
&= \int_0^\infty u_{tt} \phi \mathrm{d}x - \int_0^\infty u \phi_{xx} \mathrm{d}x \\
&= \int_0^\infty u_{tt} \phi \mathrm{d}x - \int_0^\infty u_{xx} \phi \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty (u_{tt} - u_{xx})\phi \mathrm{d}x \\
&= \frac{1}{\Gamma(1-\gamma)} \int_0^\infty \int_0^t (t-s)^{-\gamma} |u(s,x)|^p \phi \mathrm{d}s \mathrm{d}x \\
&\geq 0. \qquad (3.4)
\end{aligned}
$$

Integrating (3.4) over $[0, t]$, we can get

$$
\begin{aligned}
F_1'(t) + 2F_1(t) &\geq F_1'(0) + 2F_1(0) \\
&= \varepsilon \int_0^\infty (u_0 + u_1)\phi_1(x) \mathrm{d}x. \qquad (3.5)
\end{aligned}
$$

Multiplying both sides of (3.5) with e^{2t} and integrating it over $[0, t]$, we have

$$
\begin{aligned}
F_1(t) &\geq \varepsilon \mathrm{e}^{-2t} \int_0^\infty u_0(x)\phi_1(x) \mathrm{d}x + \frac{\varepsilon}{2}(1 - \mathrm{e}^{-2t}) \int_0^\infty (u_0 + u_1)\phi_1(x) \mathrm{d}x \\
&\geq C. \qquad (3.6)
\end{aligned}
$$

Then, we finish the proof of the estimate (3.1).

From the equation of system (1.1), we have

$$
\begin{aligned}
F_0''(t) &= \int_0^\infty u_{tt} x \mathrm{d}x \\
&= \int_0^\infty \Big(u_{xx} + \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |u(s,x)|^p \mathrm{d}s \Big) x \mathrm{d}x \\
&= \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} \int_0^\infty x |u(s,x)|^p \mathrm{d}x \mathrm{d}s, \qquad (3.7)
\end{aligned}
$$

where we use the boundary condition. By Hölder inequality and the finite speed of the propagation property, we can get

$$
\begin{aligned}
&\Big| \int_0^\infty u(t,x)\phi(t,x) \mathrm{d}x \Big|^p \\
&= \Big| \int_0^{t+R} u(t,x) x^{\frac{1}{p}} x^{-\frac{1}{p}} \phi(t,x) \mathrm{d}x \Big|^p \\
&\leq \int_0^\infty |u(t,x)|^p x \mathrm{d}x \Big(\int_0^{t+R} x^{-\frac{1}{p-1}} \phi^{\frac{p}{p-1}}(t,x) \mathrm{d}x \Big)^{p-1} \\
&\leq C \int_0^\infty |u(t,x)|^p x \mathrm{d}x, \qquad (3.8)
\end{aligned}
$$

where we use the estimate (2.5) in Lemma 2.2. This in turn implies

$$
\int_0^\infty |u(s,x)|^p x \mathrm{d}x \geq C|F_1(s)|^p \geq C. \qquad (3.9)
$$

Plugging (3.9) into (3.7), one gets

$$
F_0''(t) \geq C \int_0^t (t-s)^{-\gamma} \mathrm{d}s = Ct^{1-\gamma}. \qquad (3.10)
$$

Integrating (3.10) over $[0,t]$ twice, we obtain

$$F_0(t) \geq F_0(0) + F_0'(0)t + Ct^{3-\gamma}, \tag{3.11}$$

which means

$$F_0(t) \geq Ct^{3-\gamma}, \tag{3.12}$$

due to the fact that $F_0(0) \geq 0,\ F_0'(0) \geq 0$ and $0 < \gamma < 1$.

On the other hand, by Hölder inequality, we have

$$\begin{aligned} |F_0(s)|^p &= \Big|\int_0^\infty u(s,x)x\mathrm{d}x\Big|^p \\ &\leq \int_0^\infty |u(s)|^p x\mathrm{d}x\Big(\int_0^{s+R} x\mathrm{d}x\Big)^{p-1} \\ &\leq C(s+R)^{2(p-1)}\int_0^\infty |u(s)|^p x\mathrm{d}x, \end{aligned} \tag{3.13}$$

which yields

$$\int_0^\infty |u(s)|^p x\mathrm{d}x \geq C(s+R)^{-2(p-1)}|F_0(s)|^p. \tag{3.14}$$

By combining (3.7), (3.12) and (3.14), one gets

$$\begin{aligned} F_0''(t) &\geq C\int_0^t (t-s)^{-\gamma}(s+R)^{-2(p-1)}s^{p(3-\gamma)}\mathrm{d}s \\ &\geq C\int_{\frac{t}{2}}^t (t-s)^{-\gamma}(s+R)^{2+p-p\gamma}\mathrm{d}s \\ &\geq C(t+R)^{3+p-p\gamma-\gamma}. \end{aligned} \tag{3.15}$$

Integrating it over $[0,t]$ twice, we conclude that for t large enough $(t > a_1 > 0)$,

$$\begin{aligned} F_0(t) &\geq C(t+R)^{5+p-p\gamma-\gamma} \\ &\triangleq C(t+R)^{p_1}. \end{aligned} \tag{3.16}$$

Lastly, using the iteration argument, we will prove the desired estimate. From (3.7) and (3.14), it follows that

$$F_0''(t) \geq C\int_0^t (t-s)^{-\gamma}(s+R)^{2-2p}|F_0(s)|^p\mathrm{d}s. \tag{3.17}$$

Then plugging (3.16) into (3.17) yields that for t large enough $\big(a_1 \leq \frac{t}{2}\big)$,

$$\begin{aligned} F_0''(t) &\geq C\int_{a_1}^t (t-s)^{-\gamma}(s+R)^{2-2p+pp_1}\mathrm{d}s \\ &\geq C\int_{\frac{t}{2}}^t (t-s)^{-\gamma}(s+R)^{2-2p+pp_1}\mathrm{d}s \end{aligned}$$

$$\geq C(t+R)^{pp_1-2p+3-\gamma}, \tag{3.18}$$

which in turn gives that for t large enough ($t > a_2$),

$$\begin{aligned} F_0(t) &\geq C(t+R)^{5+pp_1-2p-\gamma} \\ &\triangleq C(t+R)^{p_2}. \end{aligned} \tag{3.19}$$

If we do the same iteration k times, we then get that for t large enough

$$F_0(t) \geq C(t+R)^{p_{k+1}} \tag{3.20}$$

with

$$\begin{aligned} p_{k+1} &= pp_k + 2 - 2p + 1 - \gamma + 2 \\ &= pp_k + 5 - 2p - \gamma. \end{aligned} \tag{3.21}$$

Hence by the knowledge of geometric series, we have

$$p_{k+1} = \frac{p^2(1-\gamma)+2p}{p-1}p^k - \frac{5-2p-\gamma}{p-1}, \tag{3.22}$$

which implies that due to the fact $p > 1$ and $0 < \gamma < 1$,

$$\lim_{k\to\infty} p_{k+1} = \infty. \tag{3.23}$$

In conclusion, for any $n > 0$, if t is large enough ($t > b > 0$), then we have

$$F_0(t) \geq C_n(t+R)^n \tag{3.24}$$

and

$$\begin{aligned} F_0''(t) &\geq C\int_0^t (t-s)^{-\gamma}(s+R)^{2-2p}|F_0(s)|^p \mathrm{d}s \\ &\geq C\int_b^t (t-s)^{-\gamma}(s+R)^{2-2p}|F_0(s)|^p \mathrm{d}s \\ &\geq C\int_b^t (t-s)^{-\gamma}|F_0(s)|^q \mathrm{d}s, \end{aligned} \tag{3.25}$$

where $C_n > 0$ is a constant depending on n and $1 < q < p$.

4 Proof of Theorem 1.2

In this section, using the above estimates obtained in Section 3, we will give the proof of Theorem 1.2. Set

$$I(t) = \int_b^t (t-s)^\alpha F_0(s)\mathrm{d}s$$

with

$$\alpha > \frac{1-\gamma}{q-1} - 1 > 0. \tag{4.1}$$

By Hölder inequality and the condition (4.1), we have

$$
\begin{aligned}
I(t) &\le \Big(\int_b^t (t-s)^{1-\gamma+\alpha}|F_0(s)|^q\Big)^{\frac{1}{q}}\Big(\int_b^t (t-s)^{\alpha-\frac{1-\gamma}{q-1}}\mathrm{d}s\Big)^{\frac{q-1}{q}}\\
&\le C(t+R)^{\frac{(\alpha+1)(q-1)}{q}-\frac{1-\gamma}{q}}\Big(\int_b^t (t-s)^{1-\gamma+\alpha}|F_0(s)|^q\mathrm{d}s\Big)^{\frac{1}{q}}, \qquad (4.2)
\end{aligned}
$$

which yields

$$
\int_b^t (t-s)^{1-\gamma+\alpha}|F_0(s)|^q\mathrm{d}s \ge C(t+R)^{1-\gamma-(\alpha+1)(q-1)}I(t)^q. \qquad (4.3)
$$

We compute $I'(t)$ and use the integration by parts

$$
\begin{aligned}
I'(t) &= \int_b^t \frac{\mathrm{d}}{\mathrm{d}t}(t-s)^\alpha F_0(s)\mathrm{d}s\\
&= -\int_b^t \frac{\mathrm{d}}{\mathrm{d}s}(t-s)^\alpha F_0(s)\mathrm{d}s\\
&= F_0(b)(t-b)^\alpha + \int_b^t (t-s)^\alpha F_0'(s)\mathrm{d}s. \qquad (4.4)
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
I''(t) &= \alpha F_0(b)(t-b)^{\alpha-1} + F_0'(b)(t-s)^\alpha + \int_b^t (t-s)^\alpha F_0''(s)\mathrm{d}s\\
&\ge \int_b^t (t-s)^\alpha F_0''(s)\mathrm{d}s, \qquad (4.5)
\end{aligned}
$$

where the fact $F_0(t)\ge 0$ and $F_0'(t)\ge 0$ has been used.

Combining (3.25) and (4.5), we can get

$$
\begin{aligned}
I''(t) &\ge C\int_b^t (t-s)^\alpha \int_b^s (s-\tau)^{-\gamma}|F_0(\tau)|^q\mathrm{d}\tau\mathrm{d}s\\
&= C\int_b^t |F_0(\tau)|^q \int_\tau^t (t-s)^\alpha (s-\tau)^{-\gamma}\mathrm{d}s\mathrm{d}\tau. \qquad (4.6)
\end{aligned}
$$

Since $(s-\tau)^{-\gamma}\ge (t-\tau)^{-\gamma}$ for $0<\gamma<1$, it holds that

$$
\begin{aligned}
I''(t) &\ge C\int_b^t |F_0(\tau)|^q (t-\tau)^{-\gamma}\int_\tau^t (t-s)^\alpha\mathrm{d}s\mathrm{d}\tau\\
&\ge C\int_b^t |F_0(\tau)|^q (t-\tau)^{1+\alpha-\gamma}\mathrm{d}\tau\\
&\ge C(t+R)^{1-\gamma-(\alpha+1)(q-1)}I(t)^q, \qquad (4.7)
\end{aligned}
$$

where we use the inequality (4.3).

On the other hand, for any $n>0$, we have, for large t $\big(\frac{t}{2}>b\big)$,

$$
I(t) = \int_b^t (t-s)^\alpha F_0(s)\mathrm{d}s
$$

$$\begin{aligned}&\geq C_n\int_b^t (t-s)^\alpha (t+R)^n \mathrm{d}s\\ &\geq C_n\int_{\frac{t}{2}}^t (t-s)^\alpha (t+R)^n \mathrm{d}s\\ &\geq C_n (t+R)^n. \end{aligned}\tag{4.8}$$

By Lemma 4.1 stated below, we conclude that $I(t)$ will blow up in a finite time and hence we finish the proof of Theorem 1.2.

Lemma 4.1 (see [16, Lemma 4]) *Suppose that $F(t)\in C^2[a,b)$ and for $a\leq t<b$,*

$$\begin{aligned}&F(t)\geq C_0(k+t)^i,\\ &F''(t)\geq C_1(k+t)^{-q}F(t)^p,\end{aligned}$$

where $C_0, C_1, k>0$. If $p>1$, $i\geq 1$ and $(p-1)i>q-2$, then b must be finite.

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