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A Survey on Mixed Spin P-Fields*

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Abstract The mixed spin P-fields (MSP for short) theory sets up a geometric platform to relate Gromov-Witten invariants of the quintic three-fold and Fan-Jarvis-Ruan-Witten invariants of the quintic polynomial in five variables. It starts with Wittens vision and the P-fields treatment of GW invariants and FJRW invariants. Then it briefly discusses the master space technique and its application to the set-up of the MSP moduli. Some key results in MSP theory are explained and some examples are provided.

Keywords Mixed Spin P-fields, GW invariants, FJRW invariants, P-fields, Cosection localization
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1 Gromov-Witten Invariants of Quintics

The counting of genus g curves of degree d on the quintic Calabi-Yau three-folds

$$Q = \{W_5 := x_1^5 + \cdots + x_5^5 = 0\} \subset \mathbb{P}^4$$

is a challenging problem in enumerative geometry. Since the seminal paper of Candelas, de la Ossa-Green-Parkes [3], a modified version has been intensively studied in string theory as well as algebraic geometry via the stable maps of Kontsevich and virtual cycles theory developed by Li-Tian [23] and Behrend-Fantachi [1].

For $d, g \in \mathbb{Z}$, the moduli space of stable maps from genus g nodal curves to Q of degree d is

$$\overline{M}_g(X,d) = \{[f : C \to X] \mid C \text{ nodal}, \, g(C) = g, \, f_*[C] = d, \, \mathrm{Aut}(f) < \infty\}.$$

The Gromov-Witten invariants are defined as

$$N_{g,d} \colon = \int_{[\overline{M}_g(X,d)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

One of the main unsolved problems in Gromov-Witten theory is to determine

$$F_g(q) \colon = \sum_d N_{g,d} q^d.$$

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From the Super-String theory side, in 1991 Candelas et al. found a closed formula for the genus zero series $F_0(q)$ using T-duality and mirror symmetry (see [3]). In 1993, Bershadsky-Cecotti-Ooguri-Vafa developed the Kodaira-Spencer theory and determined the genus one series $F_1(q)$ (see [2]). For the higher genus theory, in 2009 Huang-Klemm-Quackenbush determined $F_g(q)$ for g up to 51 (see [19]).

From the mathematical side, Kontsevich utilized torus localization to calculate the genus zero GW-invariants $N_{0,d}$. Givental [16], Lian-Liu-Yau [25] determined the genus zero case $F_0(q)$. Later on, more people worked on this topic. The genus one case $F_1(q)$ was solved in 2000's. The second named author and Zinger in [24] obtained a formula $N^{\rm red}_{1,d} = N_{1,d} - \frac{1}{12}N_{0,d}$ where $N^{\rm red}_{1,d}$ is a series of certain reduced GW-invariants. Using this formula and $\mathbb{C}^*$-localization, Zinger in [30] succeeded in determining $F_1(q)$. Gathmann [15] provided an algorithm for $N_{1,d}$ using relative GW invariants. For higher genus case, Maulik-Pandharipande found an algorithm (see [27, Section 3.2]) using the algebraic version of the degeneration formula (see [21] and analogous degeneration formula [22]) and used it for some theoretical applications. Despite the progress, a lot of questions on higher genus GW invariants of quintic Calabi-Yau three-folds remain open.

It remains a central problem in Gromov-Witten theory to develop new techniques to calculate all genus GW-invariants of quintic Calabi-Yau three-folds.

2 Witten's Vision and FJRW Invariants

2.1 Witten's vision

The same quintic polynomial $W_5 = x_1^5 + \cdots + x_5^5$ can also give a map $\mathbb{C}^5 \to \mathbb{C}$. The corresponding physical theory is the Landau-Ginzburg theory. In [29], Witten studied phase transitions involving GW theory on the quintic Q and the LG-model for W_5. Mathematically, the set-up is as follows. Let $\mathbb{C}^*$ act on

$$\mathbb{C}^6 = \mathbb{C}^5 \times \mathbb{C} = \{(x_1, \cdots, x_5, p)\}$$

with weights $(1, \cdots, 1, -5)$. Then the map $p \cdot W_5 \colon \mathbb{C}^6 \to \mathbb{C}$ is $\mathbb{C}^*$-equivariant. The quotient $[\mathbb{C}^6/\mathbb{C}^*]$ has two GIT quotients:

$$((\mathbb{C}^5 - \{\vec{0}\}) \times \mathbb{C})/\mathbb{C}^* = K_{\mathbb{P}^4}$$

and

$$(\mathbb{C}^5 \times (\mathbb{C} - 0))/\mathbb{C}^* = [\mathbb{C}^5/\mathbb{Z}_5].$$

Here $[\mathbb{C}^5/\mathbb{Z}_5]$ represents the quotient stack. The field theory valued in $K_{\mathbb{P}^4}$ is the GW theory of the quintic Q and the field theory valued in $[\mathbb{C}^5/\mathbb{Z}_5]$ leads to Witten's spin class. The latter was generalized to quasi-homogeneous polynomials by Fan-Jarvis-Ruan [12–13], and is called FJRW theory. Witten's vision is that these two theories are related via a phase transition.

2.2 P-fields treatment of GW and FJRW

The notion of P-fields was introduced by Guffin and Sharpe in [18] for genus zero LG-theory of $(K_{\mathbb{P}^4}, W_5)$. Mathematically, the first and second named authors developed the theory of P-fields for all genus GW invariants.

We start with LG-theory for $K_{\mathbb{P}^4}$. A field taking values in $((\mathbb{C}^5 - \{\vec{0}\}) \times \mathbb{C})/\mathbb{C}^*$ is

$$\xi = (\mathcal{C}, \mathcal{L}, \varphi_1, \cdots, \varphi_5, \rho),$$

where $\mathcal{C}$ is a complete nodal curve, $\mathcal{L}$ is an invertible sheaf on $\mathcal{C}$, $\varphi_i \in H^0(\mathcal{C}, \mathcal{L})$ and $\rho \in H^0(\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}})$. Since the weights of the action $\mathbb{C}^*$ on $\mathbb{C}^5$ and $\mathbb{C}$ are $(1, \cdots, 1)$ and -5 respectively, the naive choice is to take φ_i as a section of $\mathcal{L}$ and ρ as that of $\mathcal{L}^{\vee 5}$, as the equivalence $(\mathrm{id}_{\mathcal{C}}, c\,\mathrm{id}_{\mathcal{L}}) : (\mathcal{C}, \mathcal{L}) \to (\mathcal{C}, \mathcal{L})$, where $c \in \mathbb{C}^*$, represents taking quotient by $\mathbb{C}^*$. Instead, as suggested in [18], we let φ_i be a section of $\mathcal{L}$ and ρ be a section of $\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}}$. As we will see, this choice is crucial to the introduction of the cosection in (2.1). Since the origin $\vec{0}$ is removed from $\mathbb{C}^5$, $(\varphi_1, \cdots, \varphi_5)$ must be nowhere zero.

Finally we say ξ is stable if $\mathrm{Aut}(\xi)$ is finite.

In fact, what we have obtained so far is a stable map to $\mathbb{P}^4$ with a P-field. The moduli space of such objects is

$$\overline{M}_g(\mathbb{P}^4, d)^p = \{[f, \mathcal{C}, \rho] \mid [f, \mathcal{C}] \in \overline{M}_g(\mathbb{P}^4, d),\ \rho \in H^0(\mathcal{C}, f^*\mathcal{O}(5) \otimes \omega_{\mathcal{C}})\}.$$

Note that the data $([f, \mathcal{C}], \rho)$ is equivalent to the data $(\mathcal{C}, \mathcal{L}, \varphi_1, \cdots, \varphi_5, \rho)$ since the map f is equivalent to the line bundle $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^4}(1)$ with a nowhere-zero section $(\varphi_1, \cdots, \varphi_5)$ of $\mathcal{L}^{\oplus 5}$.

The first and second named authors constructed the GW invariants of stable maps with P-fields as follows. The moduli stack $\overline{M}_g(\mathbb{P}^4, d)^p$, relative to the stack $\mathcal{D} = \{(\mathcal{C}, \mathcal{L})\}$, has a perfect obstruction theory. At $\xi = (\mathcal{C}, \mathcal{L}, \varphi_i, \rho)$, the obstruction sheaf $\mathcal{O}b$ restricted to ξ is

$$\mathcal{O}b|_\xi = H^1(\mathcal{L})^{\oplus 5} \oplus H^1(\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}}).$$

We define a cosection (a sheaf homomorphism)

$$\sigma \colon \mathcal{O}b \to \mathcal{O}_{\overline{M}_g(\mathbb{P}^4, d)^p}$$

as follows: For

$$(\dot\varphi_1, \cdots, \dot\varphi_5, \dot\rho) \in H^1(\mathcal{L})^{\oplus 5} \oplus H^1(\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}}) = \mathcal{O}b|_\xi,$$

we define

$$\sigma|_\xi(\dot\varphi_1, \cdots, \dot\varphi_5, \dot\rho) = \dot\rho \sum_{i=1}^5 \varphi_i^5 + \rho \sum_{i=1}^5 5\varphi_i^4 \dot\varphi_i. \tag{2.1}$$

Since $\dot\rho \in H^1(\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}})$ and $\varphi_i \in H^0(\mathcal{L})$, the two terms in the RHS of (2.1) lie in $H^1(\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}} \otimes \mathcal{L}^{\otimes 5})$, which is canonically isomorphic to $H^1(\omega_{\mathcal{C}}) \equiv \mathbb{C}$. Thus the cosection σ takes values in $\mathcal{O}_{\overline{M}_g(\mathbb{P}^4, d)^p}$. This reveals the technical reason for introducing the twisting $\omega_{\mathcal{C}}$ to the line bundle $\mathcal{L}^{\vee 5}$.[1]

The degeneracy locus $D(\sigma)$ of the cosection σ consists of ξ such that $\sigma|_\xi$ is zero, i.e., $\sigma|_\xi(\dot\varphi_1, \cdots, \dot\varphi_5, \dot\rho) = 0$ for all $\dot\varphi_i$ and $\dot\rho$. Using that the section $(\varphi_1, \cdots, \varphi_5)$ is nowhere zero, one easily sees that (as sets)

$$D(\sigma) = \Big\{\xi \in \overline{M}_g(\mathbb{P}^4, d)^p \mid \rho = 0 \text{ and } \sum_{i=1}^5 \varphi_i^5 = 0\Big\} = \overline{M}_g(Q, d) \subset \overline{M}_g(\mathbb{P}^4, d).$$

[1] In physics literature, taking ρ to be a section of $\mathcal{L}^{\vee 5}$ twisted by $\omega_{\mathcal{C}}$ is called twisted by gravity.

Remark 2.1 We remark that, by taking the first-order variation of the expression $p\cdot(x_1^5+\cdots+x_5^5)$, we obtain $\delta\rho\sum\varphi_i^5+\rho\sum 5\varphi_i^4\delta\varphi_i$. Substituting $\delta\rho$ by $\dot\rho$, etc., we obtain the expression of the cosection σ.

As ρ are sections of line bundles, the moduli space $\overline{M}_g(\mathbb{P}^4,d)^p$ is not proper (except when $g=0$), which makes taking degrees of zero-cycles in $\overline{M}_g(\mathbb{P}^4,d)^p$ meaningless. To define its analogous GW invariants, we apply the theory of cosection localized virtual cycles, developed by Kiem and the second named author [20], based on that the degeneracy locus $D(\sigma)$ is the moduli space of stable maps to the quintic Q and thus proper.

Theorem 2.1 (see [5]) *Applying the theory of cosection localized virtual cycles, one obtains a cycle*

$$[\overline{M}_g(\mathbb{P}^4,d)^p]^{\mathrm{vir}}_{\mathrm{loc}}\in A_*D(\sigma)=A_*\overline{M}_g(Q,d).$$

Furthermore, let P-fields GW invariants be defined by

$$N^p_{g,d}=\int_{[\overline{M}_g(\mathbb{P}^4,d)^p]^{\mathrm{vir}}_{\mathrm{loc}}}1\in\mathbb{Q},$$

then

$$N_{g,d}=(-1)^{d+g+1}N^p_{g,d}.$$

This result opens a route to realize Witten's vision that GW invariants of quintic CY threefolds relate to Witten's spin class via a phase transition. On one hand, the generating function of GW invariants $F_g(q)=\sum\limits_d N_{g,d}q^d$, after this theorem, becomes a topological string amplitude of a field theory valued in $((\mathbb{C}^5-\vec{0})\times\mathbb{C})/\mathbb{C}^*$. On the other hand, it is known that Witten's spin class is a field theory valued in $[\mathbb{C}^5/\mathbb{Z}_5]$. As $((\mathbb{C}^5-\vec{0})\times\mathbb{C})/\mathbb{C}^*$ and $[\mathbb{C}^5/\mathbb{Z}_5]$ are two GIT quotients of $\big[(\mathbb{C}^5\times\mathbb{C})/\mathbb{C}^*\big]$, the mentioned phase transition might be realizable in the realm of algebraic geometry.

Let us recall the field theory valued in $[\mathbb{C}^5/\mathbb{Z}_5]$. This theory originated from Witten's spin class (see [29]). Its algebraic constructions (in narrow case) were given by Polishchuk-Vaintrob [28], and Chiodo [10]. The full theory was developed by Fan-Jarvis-Ruan [12–13], known as FJRW theory.

In [29], Witten considered the moduli space of 5-spin curves $(\Sigma^{\mathcal{C}},\mathcal{C},\mathcal{L})$ and constructed a class (Witten's spin class) by solving an elliptic system of C^∞ sections s of the (hermitian) line bundle $\mathcal{L}$:[2]

$$\overline{\partial}s+\overline{\partial_x W(s)}=0,\quad W(x)=x^5. \tag{2.2}$$

Fan, Jarvis and Ruan studied the corresponding system for general quasi-homogeneous polynomials W with broad markings, constructed the analogous Witten's top Chern class of W, and used such class to construct FJRW invariants of W.

Toward using variation of GIT quotients of $[\mathbb{C}^6/\mathbb{C}^*]$ to realize Witten's vision, we will use Witten's 5-spin class constructed as the cosection localized virtual cycle (see [6]), analogous to that of $[\overline{M}_g(\mathbb{P}^4,d)^p]^{\mathrm{vir}}_{\mathrm{loc}}$. Like before, a field in

$$[(\mathbb{C}^5\times(\mathbb{C}-0))/\mathbb{C}^*]=[\mathbb{C}^5/\mathbb{Z}_5]$$

[2] A 5-spin curve consists of a twisted nodal curve with markings $\Sigma^{\mathcal{C}}\subset\mathcal{C}$, and a line bundle $\mathcal{L}$ satisfying $\mathcal{L}^{\otimes5}\cong\omega^{\log}_{\mathcal{C}}$, where $\omega^{\log}_{\mathcal{C}}=\omega_{\mathcal{C}}(\Sigma^{\mathcal{C}})$.

consists of

$$\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \varphi_1, \cdots, \varphi_5, \rho),$$

where $(\Sigma^{\mathcal{C}}, \mathcal{C})$ is a pointed twisted curve with possibly stacky markings $\Sigma^{\mathcal{C}}$, $\mathcal{L}$ is an invertible sheaf on $\mathcal{C}$, $\varphi_i \in H^0(\mathcal{L})$, and $\rho \in H^0(\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}}^{\log})$. Since we deleted the origin in $\mathbb{C}$, the section ρ must be nowhere vanishing and hence $\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}} \cong \mathcal{O}_{\mathcal{C}}$, or equivalently $\mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}}^{\log}$. Therefore $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L})$ is a 5-spin curve. $(\varphi_1, \cdots, \varphi_5)$ gives five fields. Thus, we get a moduli space of 5-spin curves with five fields:

$$\overline{M}_{g,\gamma}^{\frac{1}{5},5p} = \{(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \varphi_1, \cdots, \varphi_5, \rho) \mid \rho \text{ is nowhere zero}\}.$$

Here γ is the monodromy data: If Σ_j is a stacky marking on $\mathcal{C}$, then $\boldsymbol{\mu}_5$ acts on $\mathcal{L}|_{\Sigma_j}$ with weight $\gamma_j = \exp\left(\frac{2\pi \mathrm{i} r}{5}\right)$ where $0 \leq r \leq 4$. We call γ narrow when $0 < r \leq 4$ (if Σ_j is a scheme-marking, γ_j is taken to be 1).

Similar to the GW case, the moduli stack $\overline{M}_{g,\gamma}^{\frac{1}{5},5p}$, relative to the stack $\mathcal{D} = \{(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L})\}$, has a perfect obstruction theory. There exists a cosection

$$\sigma\colon \mathcal{O}b \to \mathcal{O}_{\overline{M}_{g,\gamma}^{\frac{1}{5},5p}},$$

which has the same expression as that in (2.1) and whose degeneracy locus is

$$D(\sigma) = \{\xi \in \overline{M}_{g,\gamma}^{\frac{1}{5},5p} \,|\, \varphi_i = 0 \text{ for all } i\} = \overline{M}_{g,\gamma}^{\frac{1}{5}} = \{(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}) \,|\, \mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}}^{\log}\},$$

which is the moduli space of 5-spin curves.

The following result holds for non-degenerate quasi-homogeneous polynomials W and narrow γ. For the sake of convenience, we only state it in the case $W = W_5$.

Theorem 2.2 (see [6]) *Assume that γ is narrow and*

$$W = W_5.$$

Using the theory of cosection localized virtual cycles, one obtains a cycle

$$[\overline{M}_{g,\gamma}^{\frac{1}{5},5p}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_* \overline{M}_{g,\gamma}^{\frac{1}{5}}.$$

Furthermore, the class $[\overline{M}_{g,\gamma}^{\frac{1}{5},5p}]^{\mathrm{vir}}_{\mathrm{loc}}$ is identical to that constructed in [10, 28], *and under the tautological homomorphism $\iota_*\colon A_* \overline{M}_{g,\gamma}^{\frac{1}{5}} \to H_*(\overline{M}_{g,\gamma}^{\frac{1}{5}}, \mathbb{Q})$, it is identical to that constructed in* [12–13].

There is an important subclass of FJRW invariants: Those with the insertion $\frac{2}{5}$. Let $\mathcal{C}$ have k markings with all $\gamma_j = \zeta^2$ where $\zeta = \exp\left(\frac{2\pi \mathrm{i}}{5}\right)$. Define

$$\Theta_{g,k} \colon = \int_{[\overline{M}_{g,(\gamma_j)}^{\frac{1}{5},5p}]^{\mathrm{vir}}_{\mathrm{loc}}} 1 \in \mathbb{Q}.$$

Note that it is zero when $k + 2 - 2g \neq 0 \bmod 5$, as then $\overline{M}_{g,(\gamma_j)}^{\frac{1}{5},5p} = \emptyset$. It is shown (see [8]) that $\{\Theta_{g,k}\}_{g,k}$ determine all FJRW invariants with descendent (for the quintic singularity), where an explicit formula will be given in [9]. For this reason we call $\{\Theta_{g,k}\}_{g,k}$ the primary FJRW invariants.

3 Master Space Technique and Mixed Spin Fields

In the previous section, we discussed the LG-field theoretic description of GW theory of the quintic and FJRW theory of $(\mathbb{C}^5, W_5)$. Witten's vision is to link these two theories via a phase transition with respect to some complexified parameter. The approach by the authors is to develop a field theory valued in the master space to geometrically realize the "wall-crossings" of these two field theories.

3.1 Master space technique

Now we explain the master space technique to understand the wall-crossings between $K_{\mathbb{P}^4}$ and $[\mathbb{C}^5/\mathbb{Z}_5]$.

Consider a $\mathbb{C}^*$-action on $\mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1$: For $t \in \mathbb{C}^*$,

$$(x_1, \cdots, x_5, p, [u_1, u_2])^t \colon = (tx_1, \cdots, tx_5, t^{-5}p, [tu_1, u_2]).$$

It has a GIT quotient

$$W \colon = (\mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1 - \mathcal{S})/\mathbb{C}^*,$$

where $\mathcal{S} \colon = \{(x_i = 0 = u_1) \cup (\rho = 0 = u_2)\}$.

Consider a $\mathbb{C}^*$-action on W and, to avoid confusions, we call this action T-action. For $t \in T = \mathbb{C}^*$,

$$(x_1, \cdots, x_5, p, [u_1, u_2])^t = (x_1, \cdots, x_5, p, [tu_1, u_2]).$$

The T-fixed locus is

$$W^T = K_{\mathbb{P}^4} \sqcup [\mathbb{C}^5/\mathbb{Z}_5] \sqcup \{\mathrm{pt}\}.^3 \tag{3.1}$$

Let $D \subset W$ be the T-divisor defined by $p = 0$. Then $D \cong \mathbb{P}^5$ is compact. Let $I \subset H_T^*(W; \mathbb{Q})$ be the ideal generated by the T-equivariant Poicaré dual of D. The compactness of D implies the existence of a $\mathbb{Q}[\alpha]$-linear pushfoward map:

$$\int_W : I \to H_T^*(\{\mathrm{pt}\}; \mathbb{Q}) = \mathbb{Q}[\alpha].$$

The domain of $\int_W$ cannot be extended to $H_T^*(W; \mathbb{Q})$ since W is non-compact. Moreover, $\int_W$ decreases the degree by 12: If $\mu \in I \cap H_T^{2k}(W; \mathbb{Q})$ then

$$\int_W \mu \in H_T^{2(k-6)}(\{\mathrm{pt}\}; \mathbb{Q}) = \begin{cases} \mathbb{Q}\alpha^{k-6}, & k \geq 6, \\ \{0\}, & k < 6. \end{cases}$$

Similarly, there are $\mathbb{Q}[\alpha]$-linear pushforward maps:

$$\int_{K_{\mathbb{P}^4}} : \iota_1^* I \to \mathbb{Q}[\alpha], \quad \int_{[\mathbb{C}^5/\mathbb{Z}_5]} : \iota_2^* I \to \mathbb{Q}[\alpha],$$

where

$$\iota_1^* : H_T^*(W; \mathbb{Q}) \to H_T^*(K_{\mathbb{P}^4}; \mathbb{Q}) = \mathbb{Q}[H, \alpha]/\langle H^5 \rangle$$

and

$$\iota_2^* : H_T^*(W; \mathbb{Q}) \to H_T^*(\mathbb{C}^5/\mathbb{Z}_5; \mathbb{Q}) = \mathbb{Q}[\alpha]$$

[3] Explicitly, $K_{\mathbb{P}^4} = K_{\mathbb{P}^4} \times \{0\}$, $[\mathbb{C}^5/\mathbb{Z}_5] = [\mathbb{C}^5/\mathbb{Z}_5] \times \{\infty\}$, and $\{\mathrm{pt}\} = \vec{0} \times ((\mathbb{P}^1 - \{0, \infty\})/\mathbb{C}^*)$.

are induced by the inclusions $\iota_1 : K_{\mathbb{P}^4} \hookrightarrow W$ and $\iota_2 : \mathbb{C}^5/\mathbb{Z}_5 \hookrightarrow W$, respectively. Both $\int_{K_{\mathbb{P}^4}}$ and $\int_{[\mathbb{C}^5/\mathbb{Z}_5]}$ decrease the degree by 10.

Let $\mathbf{1}$ be the T-linearized trivial line bundle with weight 1 and let $\alpha = c_1(\mathbf{1})$. If $\mu \in I \cap H_T^{10}(W;\mathbb{Q})$ then

$$\int_W \mu \cup \alpha = \alpha \int_W \mu = 0.$$

Applying the localization formula to this vanishing, we obtain

$$0 = \int_{K_{\mathbb{P}^4}} \frac{(\iota_1^*\mu) \cup \alpha}{\alpha + H} + \int_{[\mathbb{C}^5/\mathbb{Z}_5]} \frac{(\iota_2^*\mu) \cup \alpha}{-\alpha} + \int_{\{\mathrm{pt}\}} \frac{(\mu|_{\{\mathrm{pt}\}}) \cup \alpha}{5\alpha(-\alpha)^5},$$

where $\alpha + H$, $-\alpha$ and $5\alpha(-\alpha)^5$ are the T-equivariant Euler classes of the normal bundles to $K_{\mathbb{P}^4}$, $[\mathbb{C}^5/\mathbb{Z}_5]$ and $\{\mathrm{pt}\}$ in W, respectively. We obtain the following wall-crossing formula:

$$\int_{K_{\mathbb{P}^4}} \frac{(\iota_1^*\mu) \cup \alpha}{\alpha + H} - \int_{[\mathbb{C}^5/\mathbb{Z}_5]} \iota_2^*\mu = \frac{\mu|_{\{\mathrm{pt}\}}}{5\alpha^5},$$

where $\mu\big|_{\{\mathrm{pt}\}} \in H_T^{10}(\mathrm{pt};\mathbb{Q}) = \mathbb{Q}\alpha^5$.

3.2 Mixed spin P-fields

Now we consider a field theory valued in W. Similar to the case of the field theory of GW valued in $K_{\mathbb{P}^4}$, the authors introduced the notion of mixed spin P-fields (MSP for short) (see [7]). An MSP field is

$$\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi_1, \cdots, \varphi_5, \rho, \nu = [\nu_1, \nu_2]).$$

$(\Sigma^{\mathcal{C}}, \mathcal{C})$ is a pointed twisted curve. $\mathcal{L}$ and $\mathcal{N}$ are invertible sheaves on $\mathcal{C}$. $\mathcal{L}$ is as before but $\mathcal{N}$ is new due to the extra factor of $\mathbb{P}^1$ in the master space technique. The sections $\varphi_i \in H^0(\mathcal{L})$ and $\rho \in H^0(\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}}^{\log})$ as before. The sections $\nu_1 \in H^0(\mathcal{L} \otimes \mathcal{N})$ and $\nu_2 \in H^0(\mathcal{N})$. The sections $\nu = [\nu_1, \nu_2]$ is a new field. We also have a narrow condition: $\varphi_i|_{\Sigma^{\mathcal{C}}} = 0$. There are combined GIT-like stability requirements: The section $(\varphi_1, \cdots, \varphi_5, \nu_1)$ is nowhere vanishing, coming from excluding $\{(x_i = 0 = u_1)\}$ in W; the section (ρ, ν_2) is nowhere vanishing, coming from excluding $\{(\rho = 0 = u_2)\}$ in W; and the section (ν_1, ν_2) is nowhere vanishing, coming from $[u_1, u_2] \in \mathbb{P}^1$. We say ξ is stable if $\mathrm{Aut}(\xi)$ is finite. For simplicity, we use φ to represent $(\varphi_1, \cdots, \varphi_5)$.

In order to understand why the moduli space of MSP fields geometrically contains the moduli space of stable maps with P-fields and the moduli space of spin curves with five P-fields, we examine the moduli space of MSP fields in details.

Let ξ be an MSP field. When $\nu_1 = 0$, since $(\varphi_1, \cdots, \varphi_5, \nu_1)$ is nowhere zero, we must have that $(\varphi_1, \cdots, \varphi_5)$ is nowhere zero. Since (ν_1, ν_2) is nowhere zero, ν_2 must be nowhere zero. Since ν_2 is a section of $\mathcal{N}$, $\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$. There is no restriction on ρ. Thus, we have $\xi \in \overline{M}_g(\mathbb{P}^4, d)^p$ and we get GW theory of the quintic Q.

When $\nu_2 = 0$, since (ρ, ν_2) is nowhere zero, ρ must be nowhere vanishing. Since ρ is a section of $\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}}^{\log}$, we must have $\mathcal{L}^5 \cong \omega_{\mathcal{C}}^{\log}$. Also ν_1 must be nowhere zero. Thus, we have $\mathcal{L} \otimes \mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$, i.e., $\mathcal{N} \cong \mathcal{L}^\vee$. The sections $\varphi_1, \cdots, \varphi_5$ can be arbitrary. Thus, we have $\xi \in \overline{M}_{g,(\gamma_j)}^{\frac{1}{5},5p}$ and get FJRW theory.

When $\rho = 0$ and $\varphi_i = 0$ for $1 \le i \le 5$, ν_1, ν_2 must be nowhere zero. Thus, we have $\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$ and $\mathcal{L} \cong \mathcal{O}_{\mathcal{C}}$. Hence, we get stable curves.

Theorem 3.1 (see [7]) *The moduli stack $W_{g,\gamma,\mathbf{d}}$ of stable MSP fields of genus g, monodromy $\gamma = (\gamma_1, \cdots, \gamma_\ell)$ of $\mathcal{L}$ along $\Sigma^{\mathcal{C}}$, and degree $\mathbf{d} = (d_0, d_\infty)$ of $\mathcal{L} \otimes \mathcal{N}$ and $\mathcal{N}$ respectively, is a separated DM stack of locally finite type.*

The moduli stack $W_{g,\gamma,\mathbf{d}}$ admits a natural T-action: For $t \in T$,

$$(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu_1, \nu_2)^t := (\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, t\nu_1, \nu_2).$$

$W_{g,\gamma,\mathbf{d}}$ is not proper since φ and ρ are sections of invertible sheaves. Thus, we cannot integrate on this stack. However, there exists a cosection of its obstruction sheaf. Using the arguments similar to GW case and LG case, we have the following theorem.

Theorem 3.2 (see [7]) *The moduli stack $W_{g,\gamma,\mathbf{d}}$ has a T-equivariant perfect obstruction theory, an equivariant cosection σ of its obstruction sheaf, and thus carries an equivariant cosection localized virtual cycle*

$$[W_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}} \in A^T_* W^-_{g,\gamma,\mathbf{d}},$$

where $W^-_{g,\gamma,\mathbf{d}}$ is the degeneracy locus of σ, i.e.,

$$W^-_{g,\gamma,\mathbf{d}} : = (\sigma = 0) = \{\xi \in W_{g,\gamma,\mathbf{d}} \mid \mathcal{C} = (\varphi = 0) \cup (\varphi_1^5 + \cdots + \varphi_5^5 = 0 = \rho)\}.$$

In order to integrate on $W^-_{g,\gamma,\mathbf{d}}$, one needs it to be proper. In fact, we have the following theorem.

Theorem 3.3 (see [7]) *The degeneracy locus $W^-_{g,\gamma,\mathbf{d}}$ is a proper T-DM stack of finite type.*

From the proof of the properness, we see a phenomenon which creates line bundles' spin structures in the LG-phase via a limit of a family of P-fields in CY-phase. We give this phenomenon the name "Landau-Ginzburg transition"(or CY-to-LG transition). It is under this phenomenon that FJRW theory captures the ghosts' contributions in GW theory in the realm of MSP moduli. Here, a map f from a curve C to $\mathbb{P}^4$ or Q is called a ghost if there are positive-genus components of C that are contracted to points by f. Over ghosts, P-fields can be nonvanishing, and such P-fields contribute to GW invariants of the quintic as "counting ghosts". For example, in Li-Zinger formula $N_{1,d} = N^{\mathrm{red}}_{1,d} + \frac{1}{12} N_{0,d}$, the number $\frac{1}{12}$ comes from the contribution by P-fields. When genus increases, such a contribution is difficult to locate. The MSP program provides a platform by which the ghosts' contributions can be captured in another phase (LG-phase) instead.

Example 3.1 The graph of T-fixed points of $W^-_{1,\emptyset,(1,0)}$.

The curves are elliptic curves without markings, $\deg \mathcal{L} = 1$, and $\deg \mathcal{N} = 0$. The graph type of fixed points which have contributions to the computations are of the four types Γ_1, Γ_2, Γ_3, Γ_4 depicted in Figure 1.

Some explanations of the figure are in order. The bottom horizontal line corresponds to $\nu_1 = 0$. The middle horizontal line corresponds to $\varphi = 0$ and $\rho = 0$. The top horizontal line corresponds to $\nu_2 = 0$. A vertex represents a connected curve if it is stable (called a stable vertex), or a node if it has two edges, or, if it has only one edge attached to it, a single non-nodal point on the curve (the edge). For each vertex, g represents the genus of the curve. If it is not a stable vertex, we use $g = 0$ here even though it doesn't represent a rational curve. An edge is a rational curve. The number near an edge is the degree of $\mathcal{L}$ on the curve. A stable vertex on the horizontal lines $0, 1, \infty$ means $\nu_1 = 0$, $\varphi = 0 = \rho$, or $\nu_2 = 0$ on the curve respectively.

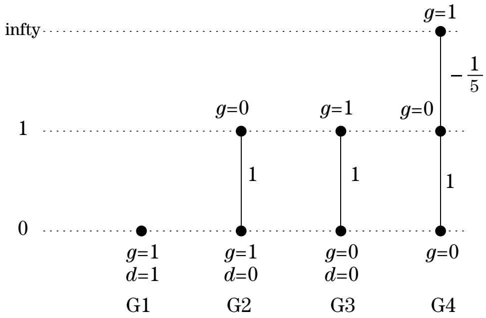

Figure 1 Graphs for $g = 1$, $\gamma = \emptyset$, $\mathbf{d} = (1, 0)$

To be more precise, the graph Γ_1 represents an elliptic curve with degree 1 line bundle $\mathcal{L}$ on the curve and $\nu_1 = 0$ on the whole curve. So it corresponds to stable maps from elliptic curves to the quintic Q with degree 1. The graph Γ_2 represents a union of an elliptic curve E with a rational curve E_0 intersecting at one node. E is a stable vertex on the bottom horizontal line. E_0 is the edge. The degree of $\mathcal{L}$ is 0 on the elliptic curve and 1 on the rational curve. The graph Γ_3 is similar to Γ_2, a union of an elliptic curve E and a rational curve E_0 intersecting at one node. E is a stable vertex on the middle horizontal line and E_0 is the edge. On E, $\varphi = 0 = \rho$ and both $\mathcal{L}$ and $\mathcal{N}$ are trivial. Thus, it represents the moduli space of elliptic curves with one marking coming from the node. The graph Γ_4 represents a union of two rational curves E_0 and E_∞ and an elliptic curve E. On the lower edge E_0, $\deg(\mathcal{L}|_{E_0}) = 1$ and $\rho|_{E_0} = 0$. Note that on each irreducible component, either $\rho = 0$ or $\varphi = 0$. On E_∞, $\deg(\mathcal{L}|_{E_\infty}) = -\frac{1}{5}$ and $\varphi|_{E_\infty} = 0$. E_∞ and E_0 intersect at one node. E_∞ is a twisted curve intersecting the elliptic curve E at a stacky point. Thus, $\mathcal{L}|_{E_\infty}$ is an invertible sheaf on the twisted curve E_∞. E is a spin elliptic curve with one marking from the node. Thus, we have $\deg(\mathcal{L}|_E) = \frac{1}{5}$.

Example 3.2 The graph of T-fixed points of $W^-_{1,\emptyset,(2,0)}$.

In this case the curve is an elliptic curve without markings, $\deg \mathcal{L} = 2$ and $\deg \mathcal{N} = 0$. The graphs of fixed points which have contributions to the computations have 15 types, listed from Γ_1 to Γ_{15} in Figure 2.

4 Vanishing and Polynomial Relations

How can we extract information of GW and/or FJRW invariants from the cycle $[W_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}}$? Let us consider a less general case to illustrate key ideas. Take $\gamma = \emptyset$, i.e., no markings. Then by virtual dimension counting, we have

$$[W_{g,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}} \in H^T_{2(d_0+d_\infty+1-g)}(W^-_{g,\mathbf{d}}, \mathbb{Q}).$$

When $d_0 + d_\infty + 1 - g > 0$, letting $u = c_1(\mathbf{1}|_{wt=1})$, i.e., u is the parameter for $H^*_T(pt)$, we have

$$[u^{d_0+d_\infty+1-g} \cdot [W_{g,\mathbf{d}}]^{\mathrm{vir}}_{\mathrm{loc}}]_0 = 0.$$

Here $[\cdot]_0$ is the degree zero term in the variable u.

Let Γ be a graph associated to fixed points of the T-action of $W_{g,\mathbf{d}}$ and F_Γ be a connected component of $W^T_{g,\mathbf{d}}$ of the graph type Γ. Applying the cosection localized version (proved in

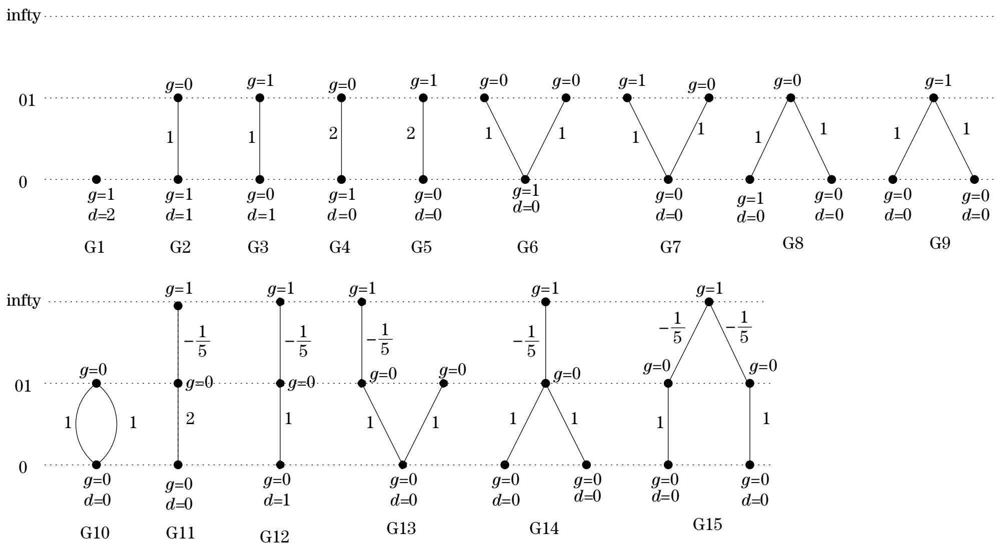

Figure 2 Graphs for $g = 1$, $\gamma = \emptyset$, $\mathbf{d} = (2, 0)$

[4]) of the virtual localization formula in [17], we have

$$\sum_{\Gamma} \left[u^{d_0+d_\infty+1-g} \frac{[F_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N_{F_\Gamma})} \right]_0 = 0. \tag{4.1}$$

To deal with $[F_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}$, we need a decomposition result, which we explain below.

Let, again, $\xi = (\mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu_1, \nu_2) \in (W_{g,d})^{\mathrm{T}}$ be an MSP field fixed by the T-action. We set

(1) $\mathcal{C}_0$ to be the part of $\mathcal{C}$ where $\nu_1 = 0$;

(2) $\mathcal{C}_1$ to be the part of $\mathcal{C}$ where $\varphi = 0 = \rho$ and hence $\nu_1 = 1 = \nu_2$, i.e., ν_1 and ν_2 are nowhere zero;

(3) $\mathcal{C}_\infty$ to be the part of $\mathcal{C}$ where $\nu_2 = 0$.

Thus, the restriction $\xi|_{(\text{connected component of } \mathcal{C}_0)}$ is in

$$\overline{M}_{g',n'}(\mathbb{P}^4, d')^p,$$

which gives Gromov-Witten invariants. Here, marked points appear from nodes on $\mathcal{C}_0$. The restriction $\xi|_{(\text{connected component of } \mathcal{C}_1)}$ is in $\overline{M}_{g',n'}$, which gives Hodge integrals. The restriction $\xi|_{(\text{connected component of } \mathcal{C}_\infty)}$ is in $\overline{M}^{\frac{1}{5},5p}_{g',\gamma'}$, which gives FJRW invariants where γ' appears because of stacky nodes on $\mathcal{C}_\infty$.

We have the following decomposition result: For a constant c,

$$[F_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}} = c \prod [\text{moduli of } \xi|_{\mathcal{C}_0}]^{\mathrm{vir}}_{\mathrm{loc}} \cdot [\text{moduli of } \xi|_{\mathcal{C}_1}]^{\mathrm{vir}}_{\mathrm{loc}} \cdot [\text{moduli of } \xi|_{\mathcal{C}_\infty}]^{\mathrm{vir}}_{\mathrm{loc}}.$$

Here the first factor gives GW invariants of stable maps to $\mathbb{P}^4$ with P-fields; the second factor gives Hodge integrals on $\overline{M}_{g',n'}$, and the third factor gives FJRW invariants of insertions $\frac{2}{5}$ (after proving a vanishing). After calculating $e(N_{F_\Gamma})$, using the polynomial relations (4.1) we obtain the following results about GW invariants of the quintic.

Theorem 4.1 (see [8]) *Letting $d_\infty = 0$, the relations* (4.1) *provide an effective algorithm to evaluate GW invariants $N_{g,d}$ provided the following are known*
(1) *$N_{g',d'}$ for (g', d') such that $g' < g$, and $d' \le d$;*
(2) *$N_{g,d'}$ for $d' < g$;*
(3) *$\Theta_{g',k}$ for $g' \le g - 1$ and $k \le 2g - 4$;*
(4) *$\Theta_{g,k}$ for $k \le 2g - 2$.*

Recall that $\Theta_{g,k}$ is the genus g FJRW invariants of insertions $\frac{2}{5}$ and $\Theta_{g,k}$ may be non-zero only when $k + 2 - 2g \equiv 0(5)$. Following the theorem, we see that to calculate all $\Theta_{2,k}$ we only need to know $\Theta_{2,2}$ in the first place; to find all $\Theta_{3,k}$, we only need to know $\Theta_{3,4}$, in addition to $\Theta_{2,2}$.

The algorithm derived from this theorem takes the following form. For any $g \ge 1$, we choose $d_\infty = 0$ and $d_0 = d \ge g$. Since $d_0 + 1 - g > 0$ in this case, the formula (4.1) gives a vanishing. After analyzing the graphs Γ appearing in the summation (4.1), we see that there is one term equal to $N_{g,d}$, and all other terms are polynomial expressions of $N_{g,d'}$ and $\Theta_{g',k}$ appearing in the list (1)–(4) of the theorem. These polynomial expressions have been derived explicitly in [8].

We can also use the vanishing (4.1) to get relations among FJRW invariants.

Theorem 4.2 (see [8]) *Letting $\mathbf{d} = (0, d_\infty)$, the vanishing* (4.1) *provide relations among FJRW invariants $\Theta_{g,k}$.*

For example, for the case of genus 2, we can use these relations to show that $\{\Theta_{2,k}\}_k$ can be inductively derived from only two unknowns $\Theta_{2,2}$ and $\Theta_{2,7}$. However, at the moment, we do not know yet whether these relations can calculate all FJRW invariants.

Example 4.1 Computations of $N_{1,1}$ and $N_{1,2}$.

In Examples 3.1–3.2, we listed all the graph types of fixed loci. Using the formulae for $e(N_{F_\Gamma})$ and $[F_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}$ in [8], we can calculate every term in the summation in (4.1).

For the genus 1 degree 1 case in Example 3.1, the contributions from the four graph types are (here G_i is the contribution from the graph Γ_i in Figure 1):

$$G_1 = -N_{1,1}, \quad G_2 = \frac{9625}{6}, \quad G_3 = \frac{-4087}{12}, \quad G_4 = -1024.$$

From (4.1), the sum of these four numbers should be zero. Thus, we obtain $N_{1,1} = \frac{2875}{12}$ which agrees with the known result.

For the genus 1 degree 2 case in Example 3.2, the contributions from 15 graph types are (here G_i is the contribution from the graph Γ_i in Figure 2):

$$\begin{aligned}
&G_1 = N_{1,2}, \quad G_2 = \frac{1106875}{6}, \quad G_3 = \frac{1331125}{12}, \quad G_4 = -\frac{5334375}{2},\\
&G_5 = \frac{17206775}{12}, \quad G_6 = 355000, \quad G_7 = -\frac{1018850}{3}, \quad G_8 = \frac{6806875}{4},\\
&G_9 = -\frac{12896875}{8}, \quad G_{10} = 782000, \quad G_{11} = \frac{28966400}{3}, \quad G_{12} = \frac{4048000}{3},\\
&G_{13} = -\frac{9934400}{3}, \quad G_{14} = -\frac{23116864}{3}, \quad G_{15} = 12288.
\end{aligned}$$

From (4.1), the sum of the fifteen numbers above being zero leads to $N_{1,2} = \frac{407125}{8}$. This is the number mathematically verified by Zinger in [30].

4.1 Speculations

Let us look at Theorem 4.1 from a different aspect. Inductively we may suppose that all GW/FJRW invariants for genus less than g are known. Then for genus g, Theorem 4.1 reduces the problem of determining the infinitely many GW invariants $\{N_{g,d}\}_{d=1}^{\infty}$ to two finite sets of initial datum

$$\{N_{g,1}, \cdots, N_{g,g-1}\}, \quad \{\Theta_{g,k}\}_{k\le 2g-2}.$$

We formulate the following speculation.

By suitable choice of positive d_0 and d_∞, the relation (4.1) provides an effective algorithm to determine the first set of initial data $\{N_{g,1}, \cdots, N_{g,g-1}\}$.

If this is true, then one is left to determine the second set of initial data $\{\Theta_{g,k}\}_{k\le 2g-2}$. We propose another conjecture about fully determining all FJRW invariants for the quintic.

Conjecture 4.1 The equation (4.1) using $d_0 = 0$ and nonempty γ's (i.e., with markings) gives relations that, together with Theorem 4.2, effectively evaluate all $\Theta_{g,k}$.

We have verified this conjecture for the case $\Theta_{2,2}$. Recall that for the case of genus 2, this is the only undetermined invariant in Theorem 4.1.

4.2 Other approaches

Another approach to Witten's proposal is the recent work of Fan-Jarvis-Ruan [14]. They worked in the more general context of the gauged linear sigma model, where more general groups G were involved. In [14, Ex. 4.2.23], they took $G = \mathbb{C}^* \times \mathbb{C}^*$ and combined the quasi-map technique with the P-fields theory to set up the moduli space. Incidentally, a closed point of the moduli also consists of a pointed twisted curve $\Sigma \subset \mathcal{C}$, two line bundles $\mathcal{L}$ and $\mathcal{N}$, and a collection of sections. We point out that, despite the similarity, the approach of [14] is different from ours.

The theory in [14] uses the concept of ϵ-stability, dependent on the real parameter ϵ, similar to the case of stable quotient [26]. The moduli for $\epsilon = 0^+$ was constructed in [14]; the case for GW-theory is when $\epsilon = +\infty$, which is yet to be constructed. For the $\epsilon = 0^+$ moduli space, the stability (on a point $(\mathcal{C}, \mathcal{L}, \mathcal{N}, \cdots)$) requires that $\mathcal{L}^{-e_1} \otimes \mathcal{N}^{-e_2}$ is ample on those components of $\mathcal{C}$ for which $\omega_{\mathcal{C}}^{\log}$ has degree zero, where $0 < e_1 < e_2$. Coming back to Example 3.2, we see that in Γ_3 of Figure 2, for the edge E connecting a genus 1 curve with a genus zero curve, we have $\deg(\omega_E^{\log}) = 0$, $\mathcal{N}|_E \cong \mathcal{O}$ and $\deg(\mathcal{L}|_E) = 1$. Thus, we have $\deg((\mathcal{L}^{-e_1} \otimes \mathcal{N}^{-e_2})|_E) < 0$. So curves with the graph type Γ_3 in Figure 2 will not be in the $\epsilon = 0^+$ moduli space of [14, Ex. 4.2.23].

The θ-parameter in [14] may resemble the r-parameter in Witten's vision of CY/LQ correspondence. In our approach, we introduced the new field $\nu = [\nu_1, \nu_2]$ in order to "quantize" the Witten's parameter in his phase transition between Calabi-Yau and Landau-Ginzberg theories. We believe that MSP field theory will provide a mathematical theory to realize the vision of Witten. We hope that both approaches will be useful for the eventual understanding of CY/LG correspondence in realizing Witten's vision that "along a suitable path, there may well be a sharply defined phase transition".

Another approach to Witten's proposal is by Choi and Kiem. In [11], they introduced the moduli spaces of ϵ-stable quasi-maps with P-fields similar to [14] and introduced additional δ-stability to make each wall-crossing more manageable.

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