Chin. Ann. Math.
38B(4), 2017, 883–900
DOI: 10.1007/s11401-017-1102-9

Chinese Annals of Mathematics, Series B

Landau-Ginzburg/Calabi-Yau Correspondence via Wall-Crossing*

Jinwon CHOI[1] Young-Hoon KIEM[2]

Abstract Landau-Ginzburg/Calabi-Yau correspondence claims the equivalence between the Gromov-Witten theory and the Fan-Jarvis-Ruan-Witten theory. The authors survey recently developed wall-crossing approach to the Landau-Ginzburg/Calabi-Yau correspondence for a quintic threefold.

Keywords Gromov-Witten invariant, Fan-Jarvis-Ruan-Witten invariant, Moduli space, Quasimap, Wall-crossing
2000 MR Subject Classification 14D23

1 Introduction

In 1990s, physicists claimed that there exists a correspondence between two theories defined from a Landau-Ginzburg model (see [31, 33]). The Landau-Ginzburg model consists of an affine space $\mathbb{C}^N$ acted on by a finite subgroup G of $\mathbb{C}^*$ and a homogeneous polynomial W of degree N. Two theories which are called the Calabi-Yau (CY for short) side and the Landau-Ginzburg (LG for short) side can be associated to a Landau-Ginzburg model. On the CY side, W is regarded as an equation for a hypersurface Y in $\mathbb{P}^{N-1} = (\mathbb{C}^N \setminus \{0\})/\mathbb{C}^*$, which is a Calabi-Yau manifold. The invariant we consider on the CY side is the Gromov-Witten (GW for short) invariant which virtually counts stable maps from a curve to Y. On the LG side, one considers W as an equation for the singularity in $\mathbb{C}^N$. What physicists claimed is that the GW invariants can be computed from the corresponding LG singularity theory. The theory on the LG side was recently developed in [17]. The invariant on the LG side is the Fan-Jarvis-Ruan-Witten (FJRW for short) invariant which virtually counts twisted spin curves.

In [7], Chiodo and Ruan proved the equivalence between the GW theory for the quintic Calabi-Yau threefold and the corresponding FJRW theory in genus zero. They showed that the generating functions for the invariants on the two sides are equivalent after taking change of variables, symplectic transformations and analytic continuations. Recently in [12], Clader studied the LG/CY correspondence for Calabi-Yau complete intersections, in which case one considers a collection of several homogeneous polynomials W defining the complete intersections.

Manuscript received December 30, 2015.
[1]Department of Mathematics, Sookmyung Women's University, Seoul 04310, Korea.
E-mail: jwchoi@sookmyung.ac.kr
[2]Department of Mathematics, Seoul National University, Seoul 08826, Korea. E-mail: kiem@snu.ac.kr
*This work was supported by the Sookmyung Women's University Research Grants (No. 1-1503-0232).

In the present article, we survey recent development (see [8]) in the LG/CY correspondence by means of wall-crossing. We focus on the case where $N = 5$ and $W = \sum_{i=1}^{5} x_i^5$, the Fermat quintic polynomial. To compare the two moduli spaces by wall-crossing, we use the definitions of the GW and FJRW invariants by cosection localization (see [3–4]). This approach enables us to study the two sides symmetrically. Namely, the moduli stacks on the two sides are both substacks of the stack of quadruples (C, L, x, p) where C is a pointed (twisted) curve with at worst nodal singularities, L is a line bundle on C, x is a section in $H^0(L^{\oplus 5})$, p is a section in $H^0(L^{-5}\omega_C^{\mathrm{tw}})$ and ω_C^{tw} is the dualizing sheaf with poles allowed at orbifold marked points. The difference for the two sides is the stability condition. So, it is natural to ask whether we can interpolate the two stability conditions by wall-crossing.

There are well-known wall-crossing theories studied in [9–10, 28] for the CY side and in [29] for the LG side, which we call the ϵ-wall-crossing. Roughly, the ϵ-wall-crossing is obtained by allowing maps to be undefined at finitely many points. In [8], a further wall-crossing, called the δ-wall-crossing is suggested to relate spaces after the ϵ-wall-crossing. The key point is that by the δ-wall-crossing the conditions on the sections x and p can be replaced by the conditions on the line bundle L. After the δ-wall-crossing, the two sides can be related by forgetting the sections and applying the virtual localization formula over the stack of curves with stable line bundles. Due to a degree reason, there are no direct wall-crossing formulas between the two sides. But the virtual localization formulas are of the same form, which looks like an analytic continuation. Their rigorous relation can be the subject of a future project.

This article is organized as follows. In Sections 2–3, we survey the wall-crossings on the CY side and the LG side respectively. In Section 4, we describe the torus localization formula after wall-crossings. We finish by a few examples in Section 5.

2 Wall-Crossing on the CY Side

2.1 GW theory

The GW invariant is defined by the virtual cycle of the moduli space of stable maps. Let Y be a smooth projective algebraic variety. Let $\beta \in H_2(Y, \mathbb{Z})$ be a curve class in Y.

Definition 2.1 (1) *An m-marked genus g prestable curve is a (connected) curve of genus g with m smooth marked points having at worst nodal singularities. A node or a marked point is called a special point.*

(2) *A map of genus g and class β is a holomorphic map $f : C \to Y$ where C is a genus g prestable curve and $f_*[C] = \beta$.*

(3) *A map $f : C \to Y$ is called stable if all genus zero components contracted by f have at least three special points and all genus one components contracted by f have at least one special point.*

Note that the condition for genus one components in item 3 is always satisfied unless $(g, m) = (1, 0)$. The stability condition is equivalent to the condition that the map f has only finitely many automorphisms.

The moduli space of stable maps is a proper Deligne-Mumford stack denoted by $\overline{M}_{g,m}(Y, \beta)$. In [1, 27], a perfect obstruction theory on $\overline{M}_{g,m}(Y, \beta)$ and hence a virtual fundamental cycle

$[\overline{M}_{g,m}(Y,\beta)]^{\mathrm{vir}} \in A_{\mathrm{vdim}}(\overline{M}_{g,m}(Y,\beta))$ have been constructed, where the virtual dimension is

$$\mathrm{vdim} = (1-g)(\dim Y - 3) - \int_\beta \omega_Y + m.$$

In the case where Y is a Calabi-Yau threefold and $m = 0$, the virtual dimension is zero and the GW invariant is defined to be the degree of $[\overline{M}_{g,m}(Y,\beta)]^{\mathrm{vir}}$.

When the virtual dimension is positive, to get a numerical invariant certain cohomology classes need to be integrated against the virtual cycle. These cohomology classes can be taken from the target Y or from the source curves of the stable maps. Let $ev_i : \overline{M}_{g,m}(Y,\beta) \to Y$ be the evaluation map sending the stable map f to its value at the i-th marked point. Then we may consider $ev_i^*(\gamma)$ for $\gamma \in H^*(Y)$. If γ can be represented by a subvariety $\Gamma \subset Y$, integrating $ev_i^*(\gamma)$ against the virtual cycle is intuitively imposing the condition that the i-th marked point is sent to Γ by the map f.

The other cohomology class we consider is the ψ-class, whose definition involves only the source curve. Let $\pi : \overline{M}_{g,m+1}(Y,\beta) \to \overline{M}_{g,m}(Y,\beta)$ be the forgetful morphism and let ω_π be the relative dualizing sheaf on $\overline{M}_{g,m+1}(Y,\beta)$. We then have the sections $\sigma_i : \overline{M}_{g,m}(Y,\beta) \to \overline{M}_{g,m+1}(Y,\beta)$ recording the i-th marked point. Consider the line bundle $\sigma_i^*\omega_\pi$. Its fiber at each point in $\overline{M}_{g,m}(Y,\beta)$ is the cotangent line of the source curve of the corresponding stable map. We define $\psi_i := c_1(\sigma_i^*\omega_\pi)$. The (descendent) GW invariants are now defined by

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_m}(\gamma_m)\rangle := \int_{[\overline{M}_{g,m}(Y,\beta)]^{\mathrm{vir}}} \psi_1^{k_1} ev_1^*(\gamma_1)\cdots\psi_m^{k_m} ev_m^*(\gamma_m),$$

where $\gamma_1, \cdots, \gamma_m$ are cohomology classes in $H^*(Y)$.

2.2 GW theory by cosection localization

The GW invariants are known to be very hard to compute. In [25], Li J. and the second named author developed a new technique to compute virtual invariants. Given a moduli space M with a perfect obstruction theory, a cosection is a morphism

$$\sigma : Ob_M \to \mathcal{O}_M$$

from the obstruction sheaf to the structure sheaf. The degeneracy locus of σ is the locus in M where σ is not surjective. Then the virtual cycle $[M]^{\mathrm{vir}}$ is shown to be supported on the degeneracy locus. This is called the cosection localization.

The cosection localization technique not only simplifies the computations but also gives us a new way to define a virtual invariant. Namely, even when the moduli space itself is not proper, we may define a virtual invariant whenever we have a cosection whose degeneracy locus is proper. In [3], Chang and Li used this property to give a new way to define the GW invariant for the quintic Calabi-Yau threefold Y, which we will review here.

Assume that Y is the Fermat quintic Calabi-Yau threefold, that is, the zero set in $\mathbb{P}^4$ of the equation $\sum_{i=1}^5 x_i^5 = 0$. Let β be given by degree d hypersurface section. Recall that a map f from a curve C to Y is equivalent to the triple (C, L, x) where $L = f^*\mathcal{O}_{\mathbb{P}^4}(1)$ and $x = (x_1, \cdots, x_5) \in H^0(C, L)^{\oplus 5}$ such that x is nonzero and satisfies the equation $\sum_{i=1}^5 x_i^5 = 0$.

It is elementary to show that f is a stable map if and only if $\omega_C^{\log} \otimes L^3$ is ample. Here, the logarithmic dualizing sheaf $\omega_C^{\log}$ is by definition $\omega_C(\sum p_i)$ where p_i's are the smooth marked points on C. One can construct the stack $\mathfrak{Q}_+$ parameterizing triples (C, L, x) as an algebraic stack by using the direct image cone construction in [3] as follows: The stack $\mathfrak{P}_+$ parameterizing pairs (C, L), where C is a prestable curve and L is a line bundle of degree d, is a smooth algebraic stack. Let $\pi : \mathcal{C} \to \mathfrak{P}_+$ be the universal curve and consider the universal line bundle $\mathcal{L}$ on $\mathcal{C}$. Then the stack of triples (C, L, x) is constructed by

$$C(\pi_*\mathcal{L}^{\oplus 5}) := \mathrm{Spec}_{\mathfrak{P}_+}(\mathrm{Sym}R^1\pi_*[(\mathcal{L}^\vee)^{\oplus 5} \otimes \omega_{\mathcal{C}/\mathfrak{P}_+}]),$$

which is called the direct image cone.

Thus, it remains to impose the condition $\sum_{i=1}^{5} x_i^5 = 0$ to the moduli space. Imposing such a condition can be handled by the cosection localization. For this, Chang and Li [3] introduced the notion of a p-field. A p-field is a section $p \in H^0(L^{-5}\omega_C)$. Since all marked points are smooth, ω_C^{tw} defined in Section 1 is the same as ω_C. In [3], the moduli stack $\mathfrak{X}_+^{\epsilon=\infty}$ parameterizing quadruples (C, L, x, p) was constructed by using the direct image cone construction again and it was shown that $\mathfrak{X}_+^{\epsilon=\infty}$ admits a perfect obstruction theory.[1] The relative obstruction sheaf $Ob_{\mathfrak{X}_+/\mathfrak{P}_+}$ at a point (C, L, x, p) has the fiber $H^1(L^{\oplus 5}) \oplus H^1(L^{-5}\omega_C)$. Then a cosection is defined by

$$\sigma|_{(C,L,x,p)}(\dot{x}, \dot{p}) = \dot{p}\sum_{i=1}^{5} x_i^5 + p\sum_{i=1}^{5} 5x_i^4\dot{x}_i, \tag{2.1}$$

where $\dot{x} = (\dot{x}_i) \in H^1(L^{\oplus 5})$ and $\dot{p} \in H^1(L^{-5}\omega_C)$. By [3, Proposition 3.5], it is lifted to a global cosection $\sigma : Ob_{\mathfrak{X}_+^{\epsilon=\infty}} \to \mathcal{O}_{\mathfrak{X}_+^{\epsilon=\infty}}$. The moduli stack $\mathfrak{X}_+^{\epsilon=\infty}$ itself is not proper. However, one can show that the degeneracy locus is precisely the locus where $p = 0$ and $\sum_{i=1}^{5} x_i^5 = 0$, that is, the locus of stable maps to the quintic threefold Y, which is proper. Hence we have the cosection localized virtual cycle $[\mathfrak{X}_+^{\epsilon=\infty}]^{\mathrm{vir}}_{\mathrm{loc}}$ supported on the moduli space of stable maps $\overline{M}_{g,m}(Y, \beta)$.

Theorem 2.1 (see [3]) *The localized virtual cycle $[\mathfrak{X}_+^{\epsilon=\infty}]^{\mathrm{vir}}_{\mathrm{loc}}$ agrees with $[\overline{M}_{g,m}(Y, \beta)]^{\mathrm{vir}}$ defined in* [1, 27] *up to sign, for $m = 0$.*

2.3 ϵ-wall crossing on the CY side

We describe the ϵ-wall-crossing for the CY side following (see [9–10]). Let $V \subset \mathbb{C}^{n+1}$ be a complex affine algebraic variety acted upon by a reductive group $G = \mathbb{C}^*$. By choosing a character θ of G, we have a linearized line bundle on V and hence a GIT quotient $V/\!\!/G$. In the case of the quintic Calabi-Yau threefold, we may take V to be an affine variety in $\mathbb{C}^5$ defined by the quintic equation and $G = \mathbb{C}^*$ where the chosen character is the identity map.

The GIT quotient $V/\!\!/G$ can be regarded as a substack in the stack quotient $[V/G]$. Consider the moduli stack of maps from a marked prestable curve C of genus g to the stack quotient $[V/G]$. Such maps are equivalent to the triple (C, P, x), where P is a principal G-bundle on C and x is a section of a fiber bundle $P \times_G V$. Note that in case of $V/\!\!/G = \mathbb{P}^4$, (C, P, x) is

[1] The superscript $\epsilon = \infty$ is used because the GW theory corresponds to the $\epsilon = \infty$-stability in the ϵ-wall-crossing which will be explained in Subsection 2.3.

equivalent to (C, L, x) in Subsection 2.2 where $L = P \times_G \mathbb{C}_\theta$ is the line bundle whose associated principal $\mathbb{C}^*$-bundle is P. The triple (C, P, x) is called a quasimap.

The section x can be regarded as a map $[x] : C \to [V/G]$. The class of the quasimap (C, L, x) is defined to be the group homomorphism

$$\beta : \mathrm{Pic}^G(V) \to \mathbb{Z}, \quad \beta(M) = \deg_C(x^*(P \times_G M)).$$

In case of $V /\!\!/ G = \mathbb{P}^4$, β is nothing but the degree of the image curve in $\mathbb{P}^4$. The base points of (C, P, x) are the points on C where the image of the section x is in the unstable locus V^{us} of V, in other words, where the image of the corresponding map f is not in $V /\!\!/ G$. Let $\mathcal{J}$ be the ideal sheaf of the unstable locus $P \times_G V^{\mathrm{us}}$ in $P \times_G V$. For $z \in C$, define $\ell(z) = \mathrm{length}_z(\mathrm{coker}(x^*\mathcal{J} \to \mathcal{O}_C))$.

Definition 2.2 *Let $\epsilon > 0$. A quasimap (C, P, x) is ϵ-stable if*
(1) *all base points are disjoint from nodes and marked points of C;*
(2) *$\omega_C^{\log} \otimes L^\epsilon$ is ample, where $L = P \times_G \mathbb{C}_\theta$;*
(3) *$\epsilon\ell(z) < 1$ for all $z \in C$.*

The moduli space $Q^\epsilon_{g,m}(V /\!\!/ G, \beta)$ of ϵ-stable quasimaps was constructed in [11]. It is a Deligne-Mumford stack equipped with a canonical obstruction theory. Moreover, the obstruction theory is perfect when V has at most loci singularities. Since the base points are away from the marked points, the evaluation maps are well-defined and hence one can define a numerical invariant similarly as before by integrating the cohomology classes against the virtual cycle.

By the numerical constraint in the definition, it is straightforward that as ϵ varies the moduli space changes only at finitely many values of ϵ, called walls. When ϵ is sufficiently large (denoted by $\epsilon = \infty$) there should be no base points and hence ϵ-stable quasimaps are precisely stable maps. Thus we recover the GW theory. When ϵ approaches to zero (denoted by $\epsilon = 0^+$), we allow arbitrarily many base points [11].

In the special case that $V /\!\!/ G \simeq \mathbb{P}^n$, $\epsilon = 0^+$-stable quasimaps are the MOP stable quotients (see [28]). The ϵ-stable quasimaps are the ϵ-stable quotients studied in [28, 30]. Therefore the ϵ-wall-crossing is the wall-crossing between the GW and MOP theories. In this case, there is a contraction morphism $c : \overline{M}_{g,m}(\mathbb{P}^n, \beta) \to Q^{\epsilon=0^+}_{g,m}(\mathbb{P}^n, \beta)$, which geometrically contracts rational tails with no marked points (see [28]). Moreover, their virtual cycles are related by $[Q^{\epsilon=0^+}_{g,m}(\mathbb{P}^n, \beta)]^{\mathrm{vir}} = c_*[\overline{M}_{g,m}(\mathbb{P}^n, \beta)]^{\mathrm{vir}}$.

When the target space $V /\!\!/ G$ is the quintic threefold Y, there is an alternative way to define a virtual cycle for the moduli space of ϵ-stable quasimaps by the cosection localization as in Section 2.2. We first consider the moduli space $Q^\epsilon_{g,m}(\mathbb{P}^4, \beta)$ of ϵ-stable triples (C, L, x). By the direct image cone construction of [3], we can enlarge the moduli space to include the p-field (see [8]). Let $\mathfrak{X}^\epsilon_+$ be the moduli space of quadruples (C, L, x, p) where (C, L, x) is an ϵ-stable quasimap. The construction in [3] shows that $\mathfrak{X}^\epsilon_+$ is equipped with a natural perfect obstruction theory and (2.1) can also be used to define a cosection on $\mathfrak{X}^\epsilon_+$ such that its degeneracy locus is precisely the locus of quasimaps to the quintic threefold Y. Therefore, as in the GW theory, we get a localized virtual cycle $[\mathfrak{X}^\epsilon_+]^{\mathrm{vir}}_{\mathrm{loc}}$ supported on $Q^\epsilon_{g,m}(Y, \beta)$.

Conjecture 2.1 The localized virtual cycle $[\mathfrak{X}^\epsilon_+]^{\mathrm{vir}}_{\mathrm{loc}}$ is equal to the virtual cycle $[Q^\epsilon_{g,m}(Y, \beta)]^{\mathrm{vir}}$ defined in [11] up to sign.

This conjecture holds when $g = 0$. In this case, there are no nonzero p-fields and hence by

(2.1) the cosection is zero. Therefore the cosection localized virtual cycle is the Euler class of the vector bundle on $Q^{\epsilon}_{0,m}(\mathbb{P}^4,\beta)$ whose fiber at (C,L,x) is $H^1(L^{-5}\omega_C)$. So by [11, Proposition 6.2.2], the two cycles agree up to sign, where the sign change is due to taking the dual of tangent-obstruction complex. We expect that Conjecture 2.1 can be proved using a similar technique as in [3].

2.4 δ-wall-crossing on the CY side

There is a similar theory of ϵ-stability invariants on the LG side as we will see in Subsection 3.2. To connect the $\epsilon = 0^+$-stability invariants on the CY side with the $\epsilon = 0^-$-stability invariants on the LG side, a further wall-crossing called the δ-wall-crossing was introduced in [8]. Consider the stack $\mathfrak{X}_+$ of quadruples (C,L,x,p) defined above. For $\delta > 0$ we define the δ-stability on $\mathfrak{X}_+$ as follows.

Definition 2.3 *A quadruple $(C,L,x,p) \in \mathfrak{X}_+$ with C a prestable curve, $L \in \mathrm{Pic}^d(C)$, $x \neq 0 \in H^0(L^{\oplus 5})$ and $p \in \mathrm{Hom}_C(L^5,\omega_C)$ is said to be δ-stable if*

(1) $\omega_C^{\log} \otimes L^a$ is ample for all $a > 0$;

(2) $(\overline{L},\overline{x})$ is a δ-stable pair on $\overline{C}$ with respect to the ample line bundle $\omega_{\overline{C}}^{\log}$, where $\rho : C \to \overline{C}$ is the stabilization morphism, $\overline{L} = \rho_* L$ and $\overline{x} = \rho_* x$.

The theory of δ-stable pairs is due to Le Potier [26]. Let C be a prestable curve with a fixed ample line bundle $\mathcal{O}_C(1)$. Given a sheaf F on C, the Hilbert polynomial P_F is defined as $P_F(k) := \chi(F(k))$. Denote by $r(F)$ the leading coefficient of P_F.

Definition 2.4 *A pair of a one-dimensional sheaf F on C and a nonzero homomorphism $x : \mathcal{O}_C^{\oplus n} \to F$ with P_F fixed is δ-semistable if*

(1) F is pure, that is, F does not have a 0-dimensional subsheaf;

(2) for any nontrivial subsheaf $F' \neq F$,

$$\frac{\chi(F') + \eta(F',x)\delta}{r(F')} \leq \frac{\chi(F) + \delta}{r(F)},$$

where $\eta(F',x)$ is 1 if x factors through F' and 0 otherwise.

We get δ-stability if $\leq$ is replaced by $<$.

Note that Definition 2.3 does not impose any condition on p. We will also call the triple (C,L,x) δ-stable if it satisfies Definition 2.3. An immediate consequence of the δ-stability condition is that (see [8, Section 5])

(1) the curve C cannot have rational tails with at most one marked point and

(2) on rational bridges with no marked point the line bundle L must have degree 1.

Here a rational bridge means a rational component $\mathbb{P}^1$ which has exactly two nodes while a tail means a connected subcurve C' which meets its complement $\overline{C - C'}$ at only one point. Note that rational tails with at most one marked point and rational bridges with no marked point are contracted by the stabilization morphism ρ.

The motivation for the δ-wall-crossing is to replace conditions on the section x by conditions on the line bundle L. When δ is large enough (denoted by $\delta = \infty$), the quadruple (C,L,x,p) is δ-stable if the cokernel of x has support in a union of rational bridges and finitely many points.

Since the cokernel of x consists of base points in the ϵ-stability, the $\delta = \infty$-stability is similar to the $\epsilon = 0^+$-stability. The only difference is that

(1) the base points are allowed to coincide with nodes and marked points;

(2) the base locus can contain rational bridges with no marked points;

(3) C does not have rational tails with at most one marked point;

(4) L has degree 1 on a rational bridge with no marked point.

In [8], the moduli stacks for $\delta = \infty$ and $\epsilon = 0^+$ are related by birational morphisms.

On the other hand, when δ is sufficiently small (denoted by $\delta = 0^+$), under the assumption that there are no strictly 0^+-semistable quadruples, (C, L, x, p) is δ-stable if and only if $\overline{L}$ itself is a stable rank 1 sheaf on $\overline{C}$. To have no strictly 0^+-semistable quadruples, we need an assumption that $r(\overline{L}) = 2g - 2 + m$ is coprime to $\chi(\overline{L}) = d - g + 1$.

For the purpose of computing the GW invariants of Y, we may assume that $2g - 2 + m$ is coprime to $d - g + 1$ because we can add a marked point and cancel its effect by the dilaton equation

$$(2g - 2 + m) \cdot [\overline{M}_{g,m}(Y, d)]^{\mathrm{vir}} = [\overline{M}_{g,m+1}(Y, d)]^{\mathrm{vir}} \cap \psi_{m+1}. \tag{2.2}$$

Hence we may assume that $2g - 2 + m$ is coprime to $d - g + 1$ on the CY side.

The following are proved in [8].

(1) As in the ϵ-wall-crossing, the moduli space of δ-stable quadruples (C, L, x, p) changes only at finitely many values of δ, called walls.

(2) When $d + \delta \geq g - 1$ and δ is not a wall, the moduli space $Q^{\delta}_{g,m}(\mathbb{P}^4, d)$ of δ-stable triples (C, L, x) is constructed as a proper Deligne-Mumford stack.

(3) By the direct image cone construction, the moduli space $\mathfrak{X}^{\delta}_{+}$ of δ-stable quadruples (C, L, x, p) is constructed as a separated Deligne-Mumford stack and is equipped with a perfect obstruction theory and a cosection defined by (2.1).

(4) If furthermore

$$g = 0 \quad \text{or} \quad d + \delta \geq 3(g - 1) + m, \tag{2.3}$$

the degeneracy locus of the cosection is proper and is precisely the locus of quadruple (C, L, x, p) where $p = 0$ and $\sum_{i=1}^{5} x_i^5 = 0$. Hence we get a virtual invariant defined by the cosection localized virtual cycle.

The wall-crossing formula for the cosection localized virtual invariant was studied in [2].

3 Wall-Crossing for the LG Side

3.1 FJRW theory

The LG model we consider is $([\mathbb{C}^5/\mathbb{Z}_5], W)$, where $[\mathbb{C}^5/\mathbb{Z}_5]$ is the quotient stack of the affine space $\mathbb{C}^5$ by the action of the finite cyclic group $\mathbb{Z}_5 \subset \mathbb{C}^*$ and W is defined by $W(x_1, \cdots, x_5) = \sum_{i=1}^{5} x_i^5$. In [17], Fan, Jarvis and Ruan developed a mathematical theory of the curve-counting invariants for the LG model. The moduli space on the LG side is the moduli space of spin curves, which we will review in this section. Let $\zeta := \mathrm{e}^{\frac{2\pi \mathrm{i}}{5}}$.

Definition 3.1 *An m-pointed twisted curve is a proper one-dimensional Deligne-Mumford stack with at worst nodal singularities together with m distinct marked points away from nodes such that*

(1) *points with nontrivial stabilizers are marked points and nodes;*

(2) *the stabilizer at each marked point is $\mathbb{Z}_5$;*

(3) *all nodes are balanced, i.e., locally near a node $\{xy = 0\}$, the isotropy group $\mathbb{Z}_5$ acts by $(x, y) \mapsto (\zeta x, \zeta^{-1} y)$.*

For a twisted curve C, we associate its coarse moduli space $|C|$. If $|C|$ is a stable curve, then we call C a twisted stable curve. It is well-known that m-pointed twisted stable curves of genus g form a proper separated Deligne-Mumford stack (see [6]).

A spin curve is a twisted stable curve C together with an orbifold line bundle L on C which is a fifth root of the logarithmic dualizing sheaf, $\omega_C^{\mathrm{tw}} := \omega_C\big(\sum_{i=1}^{m} p_i\big)$ where p_i's are orbifold marked points on C. We use the superscript tw to emphasize that the marked points are orbifold points.

Definition 3.2 *A* (5-) *spin curve consists of an orbifold line bundle L on a twisted stable curve C and an isomorphism $p : L^5 \to \omega_C^{\mathrm{tw}}$.*

The restriction of L to an orbifold point is a multiplication by ζ^k for some $k \in \{0, 1, \cdots, 4\}$. We call k the multiplicity of L at this marked point.

Definition 3.3 *Let $\vec{k} = (k_1, \cdots, k_m)$ be an m-tuple of integers with $0 \le k_i < 5$. We define the stack of m-pointed $\mathbb{Z}_5$-spin curves as*

$$\overline{M}_{g,\vec{k}} = \{(C, L, p) \mid C \textit{ twisted stable curve, } p : L^5 \overset{\cong}{\to} \omega_C^{\mathrm{tw}},\ \mathrm{mult}_{p_i} L = k_i\}.$$

Then the stack $\overline{M}_{g,\vec{k}}$ is a smooth proper Deligne-Mumford stack and is nonempty if and only if $\big(2g - 2 + m - \sum k_i\big)$ is a multiple of 5 (see [17]). Fan, Jarvis and Ruan [17] constructed analytically Witten's top Chern class on $\overline{M}_{g,\vec{k}}$, which gives the FJRW invariant.

In [4], Chang, Li and Li gave an alternative algebro-geometric definition of the FJRW invariant by using the cosection localization technique. Let $\mathfrak{X}_-^{\epsilon=-\infty}$ be the stack parametrizing quadruples (C, L, x, p) with $(C, L, p) \in \overline{M}_{g,\vec{k}}$ and $x = (x_j) \in H^0(C, L^{\oplus 5})$.[2] By using the direct image cone construction, $\mathfrak{X}_-^{\epsilon=-\infty}$ is constructed as a separated Deligne-Mumford stack of finite type.

The multiplicity vector $\vec{k}$ is called narrow when all k_i are nonzero. Otherwise it is called broad. Chang, Li and Li [4] constructed a cosection for $\mathfrak{X}_-^{\epsilon=-\infty}$ when $\vec{k}$ is narrow. The relative obstruction sheaf $Ob_{\mathfrak{X}_-^{\epsilon=-\infty}/\overline{M}_{g,\vec{k}}}$ at (C, L, x, p) is given by $H^1(L^{\oplus 5})$. By [4, Lemma 3.2], $H^1(L^{\oplus 5}) \cong H^1\big(\big(L\big(-\sum p_i\big)\big)^{\oplus 5}\big)$ in narrow cases. The cosection σ is now defined by

$$(\dot{x_i}) \mapsto p\Sigma 5 x_i^4 \dot{x_i}. \tag{3.1}$$

Since $(\dot{x_i})$ can be regarded as an element in $H^1\big(\big(L\big(-\sum p_i\big)\big)^{\oplus 5}\big)$ and $x_i \in H^0(L)$, the right hand side of (3.1) is an element in $H^1(C, \omega_C) \cong \mathbb{C}$. By [4, Proposition 3.4], it can be lifted to a cosection $\sigma : Ob_{\mathfrak{X}_-^{\epsilon=-\infty}} \to \mathcal{O}_{\mathfrak{X}_-^{\epsilon=-\infty}}$. As before, we get a localized virtual cycle $[\mathfrak{X}_-^{\epsilon=-\infty}]^{\mathrm{vir}}_{\mathrm{loc}}$. Note that in this construction of the cosection, one considers only the obstruction for x while

[2]The superscript $\epsilon = -\infty$ is used because the FJRW theory corresponds to the $\epsilon = -\infty$-stability in the ϵ-wall-crossing which will be explained in Section 3.2.

in (2.1) one considers the obstructions for x and p simultaneously. The obstruction for p lies in $H^1(L^{-5}\omega_C^{\mathrm{tw}})$ which is isomorphic to $H^1(\mathcal{O}_C)$ when p is an isomorphism. This cancels with the deformation of the line bundle L. This explains the difference in constructions.

The degeneracy locus of σ is precisely $\overline{M}_{g,\vec{k}}$, which is proper (see [4, Lemma 3.6]). So, the invariant is defined by the localized virtual cycle $[\mathfrak{X}_-^{\epsilon=-\infty}]^{\mathrm{vir}}_{\mathrm{loc}}$.

Theorem 3.1 (see [4]) *The virtual invariants obtained from* $[\mathfrak{X}_-^{\epsilon=-\infty}]^{\mathrm{vir}}_{\mathrm{loc}}$ *agree up to sign with the FJRW invariants defined by Fan, Jarvis and Ruan in* [17].

3.2 ϵ-wall crossing on the LG side

On the LG side, the ϵ-wall-crossing parallel to that on the CY side is known (see [18, 29]). In [29], Ross and Ruan studied the ϵ-wall-crossing on the LG side for $g = 0$.

Definition 3.4 *Let C be a twisted curve; let L be a line bundle of degree d on C; let x be a section in $H^0(L^{\oplus 5})$; let p be a nonzero section in $H^0(C, L^{-5}\omega_C^{\mathrm{tw}}) = \mathrm{Hom}_C(L^5, \omega_C^{\mathrm{tw}})$.*

For $\epsilon < 0$, a quadruple (C, L, x, p) is called ϵ-stable if

(1) $\omega_C^{\mathrm{tw}} \otimes \widetilde{L}^{|\epsilon|}$ *is ample where* $\widetilde{L} = L^{-5}\omega_C^{\mathrm{tw}}$*;*

(2) *the cokernel of* $p : L^5 \to \omega_C^{\mathrm{tw}}$ *has* 0*-dimensional support disjoint from special points;*

(3) $|\epsilon| \cdot \mathrm{length}_z(\mathrm{coker}(p)) < 1$ *for all* $z \in C$.

Let $\mathfrak{X}_-^\epsilon$ denote the stack of ϵ-stable quadruples after fixing the topological data (g, m, d) and $\vec{k}$ the multiplicities of L at marked points. We always assume narrow $\vec{k}$.

When ϵ is sufficiently negative ($\epsilon = -\infty$), no base points for p are allowed and an ϵ-stable quadruple is nothing but a spin curve with x-field so that we get back to the moduli stack $\mathfrak{X}_-^{\epsilon=-\infty}$ for the FJRW invariant. Furthermore, the moduli stacks $\mathfrak{X}_-^\epsilon$ vary only at finitely many values of $\epsilon < 0$, called walls, upon fixing the topological type (g, m, d). For a non-wall $\epsilon < 0$, a cosection on $\mathfrak{X}_-^\epsilon$ can be defined by (3.1) and hence we have cosection localized virtual cycles $[\mathfrak{X}_-^\epsilon]^{\mathrm{vir}}_{\mathrm{loc}}$ whose supports are proper.

By [29, Lemma 1.5], when $g = 0$, there are no nonzero x-fields. Hence the cosection localized virtual cycle is the top Chern class of the obstruction bundle, which appears as a definition of the Witten class in [29]. Hence the cosection localized virtual cycle can be regarded as its generalization.

When $\epsilon < 0$ is close to zero (denoted by $\epsilon = 0^-$), p can have 0-dimensional cokernel of arbitrary length. As in the CY side, if we allow the cokernel to have support on special points, we get the $\delta = -\infty$ stability below.

3.3 δ-wall crossing on the LG side

We can also consider a parallel theory of the δ-wall-crossing on the LG side. What we consider on the LG side is the p-field which is a section of the line bundle $L^{-5}\omega_C^{\mathrm{tw}}$. But for any orbifold line bundle L with stabilizer $\mathbb{Z}_5$, $L^{-5}\omega_C^{\mathrm{tw}}$ has trivial orbifold structure and hence is the pullback of a line bundle on the coarse moduli space $|C|$. Thus, the theory of the δ-wall-crossing in Subsection 2.4 can be easily applied.

Definition 3.5 *Let $\delta < 0$. A quadruple (C, L, x, p) is δ-stable if*

(1) $\omega_C^{\mathrm{tw}} \otimes \widetilde{L}^a$ *is ample for any* $a > 0$ *where* $\widetilde{L} = L^{-5}\omega_C^{\mathrm{tw}}$*;*

(2) $(\overline{L},\overline{p})$ *is a* $|\delta|$*-stable pair on* $\overline{C}$ *with respect to the ample line bundle* $\omega_{\overline{C}}^{\log}$*, where* $\rho : C \to \overline{C}$ *is the composition of the stabilization morphism* $|C| \to \overline{C}$ *and the coarsening map* $\tau : C \to |C|$*,* $\overline{L} = \rho_* \widetilde{L}$ *and* $\overline{p} = \rho_* p$*.*

Note that the definition does not impose any condition on x. We will also call the triple (C, L, p) to be δ-stable if above conditions are satisfied.

As before we fix $\vec{k} = (k_1, \cdots, k_m)$ and $d := \frac{1}{5}\big(2g - 2 + m - \sum k_i\big)$. For $\delta < 0$, we let $\mathfrak{X}_-^\delta$ denote the stack of δ-stable quadruples (C, L, x, p) satisfying $\mathrm{mult}_{p_i} L = k_i$ and $\deg |L| = d$. The stack $\mathfrak{X}_-^\delta$ is constructed as follows.

Assume δ is not a wall. Let $\widetilde{d} = -5d + 2g - 2 + m = \sum k_i$. By the arguments in Subsection 2.4, we have a proper Deligne-Mumford stack $\mathfrak{Q}_-^\delta$ of δ-stable triples $(|C|, \widetilde{L}, p)$, where $|C|$ is a prestable (nonorbifold) curve, $\widetilde{L} \in \mathrm{Pic}^{\widetilde{d}}(|C|)$ and $p \in H^0(\widetilde{L})$. There is a forgetful morphism $\mathfrak{Q}_-^\delta \to \mathfrak{M}_{g,m}^{\mathrm{pre}}$ to the moduli stack of prestable curves. By taking fiber product with $\mathfrak{M}_{g,m}^{\mathrm{tw}} \to \mathfrak{M}_{g,m}^{\mathrm{pre}}$ sending a twisted curve C to its coarse moduli space $|C|$, we get a proper Deligne-Mumford stack $\widetilde{\mathfrak{Q}}_-^{\delta,\mathrm{tw}}$ parametrizing δ-stable triple $(C, \widetilde{L}, p)$.

Let $\mathfrak{P}^{\mathrm{tw}}$ be a stack parametrizing pairs (C, L) of twisted curves C and line bundles L on C. We have the finite morphism $\mathfrak{P}^{\mathrm{tw}} \to \mathfrak{P}^{\mathrm{tw}}$ sending L to $L^{-5}\omega_C^{\mathrm{tw}}$. One takes the fiber product of this morphism with the forgetful morphism $\widetilde{\mathfrak{Q}}_-^{\delta,\mathrm{tw}} \to \mathfrak{P}^{\mathrm{tw}}$, $(C, \widetilde{L}, p) \to (C, \widetilde{L})$. Upon restricting to the locus where $\mathrm{mult}_{p_i} L = k_i$, we get the stack $\mathfrak{Q}_-^{\delta,\mathrm{tw}}$ parametrizing δ-stable triple (C, L, p). We have a universal line bundle $\mathcal{L}$ on $\mathfrak{Q}_-^{\delta,\mathrm{tw}}$. By using the direct image cone construction of [3], $\mathfrak{X}_-^\delta$ is constructed as a separated Deligne-Mumford stack. Since $\mathfrak{P}^{\mathrm{tw}}$ is smooth over $\mathfrak{M}_{g,m}^{\mathrm{tw}}$ by [5], there is a perfect obstruction theory on $\mathfrak{X}_-^\delta$.

A cosection can be defined by the same formula (3.1). When

$$g = 0 \quad \text{or} \quad -5d - \delta > g - 1 + m, \tag{3.2}$$

the degeneracy locus of the cosection is proper and is precisely the locus of $x = 0$, which is $\mathfrak{Q}_-^{\delta,\mathrm{tw}}$ (see [8]). Thus we obtain a cosection localized virtual invariant.

When $|\delta|$ is sufficiently large, $\overline{p}$ is surjective away from finitely many points. So, $\mathfrak{X}_-^{\delta=-\infty}$ is related to $\mathfrak{X}_-^{\epsilon=0^-}$ by wall crossing. On the other hand, when $\delta = 0^-$, $\overline{L}$ must be stable when there are no strictly semistable sheaves on $\overline{C}$, which holds under the assumption that $2g-2+m$ and $-5d + 2g - 2 + m$ are coprime. Therefore, we have

$$\mathfrak{X}_-^{\delta=0^-} = \{(C, L, x, p) \,|\, \overline{L} \text{ is stable over } \overline{C}, p \neq 0\}.$$

Note that for $g > 0$, the numerical conditions (2.3) and (3.2) for the CY and LG sides have no intersection. Hence the invariants from both sides cannot be compared directly by a wall-crossing formula. Instead, we use a torus localization formula to compare the two sides.

4 LG/CY Correspondence

In this section, we relate the CY and LG sides after the δ-wall-crossing. The key point in relating the CY and LG sides is that when $\delta = 0^{\pm}$, the stack $\mathfrak{X}_{\pm}^{\delta=0^{\pm}}$ consists of quadruple (C, L, x, p) where the stability condition is equivalent to the stability condition on $\overline{L}$. Here, we need an assumption that there are no strictly semistable line bundles. On the CY side, we use the dilaton equation (2.2) and may assume that $\gcd(2g-2+m, d-g+1) = 1$. On the LG side, we also have the dilaton equation (see [17, Theorem 4.2.9] and [22, Section 5.2]). By using the dilaton equation we may cancel the effect on the FJRW invariant of adding a marked point by integrating a ψ-class. Hence we may assume $\gcd(2g-2+m, -5d+2g-2+m)=1$ so that there are no strictly semistable $\overline{L}$ and hence $\delta=0^{\pm}$ is equivalent to $\delta=0$. Consider the stack $\overline{P}$ of pairs $(\overline{C},\overline{L})$, where $\overline{C}$ is a stable curve and $\overline{L}$ is a rank 1 stable sheaf on $\overline{C}$ with respect to the ample line bundle $\omega_{\overline{C}}^{\log}$. By [16, Theorem 4.1], $\overline{P}$ is a smooth proper Deligne-Mumford stack.

On the CY side, when (C,L,x,p) is $\delta=0$-stable and $d\geq 3(g-1)+m$, it was shown in [8] that p is always zero and the stack $\mathfrak{X}_{+}^{\delta=0}$ is in fact the stack of (C,L,x) with $x\neq 0$. By inserting a rational bridge at a node, where $\overline{L}$ is not locally free, we see that $(\overline{C},\overline{L})$ uniquely determines (C,L). Hence $\mathfrak{X}_{+}^{\delta=0}$ is the projectivization of a cone stack over $\overline{P}$. Consider the scaling action of $\mathbb{C}^*$ on x. Applying the torus localization formula for cosection localized virtual cycles in [2], we find that

$$[\mathfrak{X}_{+}^{\delta=0}]^{\mathrm{vir}}_{\mathrm{loc}}=\mathrm{res}_{t=0}\frac{[\overline{P}]}{e(R\pi_*(\mathcal{L}^{\oplus 5}\oplus\mathcal{H}om(\mathcal{L}^5,\omega_{\mathcal{C}/\overline{P}})))},\tag{4.1}$$

where $\pi:\mathcal{C}\to\overline{P}$ denotes the universal curve, $\mathcal{L}$ is the universal sheaf on $\mathcal{C}$ and t is the equivariant parameter. Here $e(\cdot)$ stands for the equivariant Euler class of the perfect complex.

Similarly on the LG side, when (C,L,x,p) is $\delta=0$-stable and $-5d+\delta>g-1+m$, one can show that x is always zero. After fixing the multiplicity vector $\vec{k}$, we see that $(\overline{C},\overline{L})$ uniquely determines $(C,\widetilde{L})$ by a local computation. Let r be the degree of the finite morphism $\mathfrak{P}^{\mathrm{tw}}\to\mathfrak{P}^{\mathrm{tw}}$ sending L to $\widetilde{L}=L^{-5}\omega_C^{\mathrm{tw}}$. Then we obtain the same residue formula.

$$[\mathfrak{X}_{-}^{\delta=0}]^{\mathrm{vir}}_{\mathrm{loc}}=r\cdot\mathrm{res}_{t=0}\frac{[\overline{P}]}{e(R\pi_*(\mathcal{L}^{\oplus 5}\oplus\mathcal{H}om(\mathcal{L}^5,\omega_{\mathcal{C}/\overline{P}})))}.\tag{4.2}$$

The equations (4.1)–(4.2) are of the same form but have opposite range for d. So, the comparison of these two equations looks like an analytic continuation.

5 Examples

In this section, we describe examples of the calculation on the CY side for $g=0$.

Conjecture 5.1 (Clemens) Let Y be a general quintic threefold. For each degree $d\geq 1$, we have the following:

(1) There are only finitely many irreducible rational curves C in Y of degree d.

(2) These curves are all disjoint.

(3) The normalization $\nu:\mathbb{P}^1\to C$ has the normal bundle $N_\nu\simeq\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$.

In general, a rational curve on Y may not be smooth. In particular, there are examples of 17,601,000 6-nodal rational quintic plane curves on general Y due to Vainsencher [32]. Conjecture 5.1 is proved for $d\leq 11$ and the rational curves are all smooth except the above case (see [13–14, 23–24]). So, rational curves in Y of degree at most 4 are finitely many and are rigidly embedded $\mathbb{P}^1$'s.

For a rigidly embedded smooth rational curve C in Y, the moduli space of genus 0 degree d stable maps to C is $\overline{M}_{0,m}(\mathbb{P}^1,d)$, which we will study in this section. The dimension of

$\overline{M}_{0,m}(\mathbb{P}^1, d)$ is $2d-2+m$. To get a contribution to the GW invariant of degree d covers of C when $m = 0$, we need to take the Euler class of the obstruction bundle of rank $2d-2$. For detail, see [15, Section 9.2.2].

In this section, we show basic calculation, where we do not need the obstruction bundle. We show calculations with ψ-class insertions for $\overline{M}_{0,m}(\mathbb{P}^1, 1)$ by examples. We also describe the effect of the δ-wall-crossing when $g = 0$.

5.1 $(g, m, d) = (0, 1, 1)$

The moduli space $\overline{M}_{0,1}(\mathbb{P}^1, 1)$ is $\mathbb{P}^1$. The ψ_1 class is $-2H$ where H is the point class. Then the GW invariant is $\int_{\overline{M}_{0,1}(\mathbb{P}^1,1)} \psi_1 = -2$. The ϵ-wall-crossing is an isomorphism. When $m \leq 2$, there are no δ-stable quasimaps and the moduli space is empty.

5.2 $(g, m, d) = (0, 2, 1)$

The moduli space $\overline{M}_{0,2}(\mathbb{P}^1, 1)$ is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ along the diagonal Δ, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The forgetting map $\pi_2 : \overline{M}_{0,2}(\mathbb{P}^1, 1) \to \overline{M}_{0,1}(\mathbb{P}^1, 1)$ is the projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ to the first factor. Let H_i be the pullback of the point class on each factor $\mathbb{P}^1$. By [21, Lemma 26.3.1], we have $\psi_1 = \pi_2^*\psi_1 + \Delta = -H_1 + H_2$ and similarly $\psi_2 = H_1 - H_2$. So the GW invariant are

$$\int_{\overline{M}_{0,2}(\mathbb{P}^1,1)} \psi_1^2 = \int_{\mathbb{P}^1\times\mathbb{P}^1} (-H_1 + H_2)^2 = -2,$$
$$\int_{\overline{M}_{0,2}(\mathbb{P}^1,1)} \psi_1\psi_2 = \int_{\mathbb{P}^1\times\mathbb{P}^1} (H_1 - H_2)^2 = -2,$$
$$\int_{\overline{M}_{0,2}(\mathbb{P}^1,1)} \psi_1\psi_2 = \int_{\mathbb{P}^1\times\mathbb{P}^1} (-H_1 + H_2)(H_1 - H_2) = 2.$$

As in the above case, the ϵ-wall-crossing is trivial and the δ-moduli spaces are empty.

5.3 $(g, m, d) = (0, 3, 1)$

The moduli space $\overline{M}_{0,3}(\mathbb{P}^1, 1)$ is isomorphic to the Fulton-MacPherson configuration space $\mathbb{P}^1[3]$, which is isomorphic to the blowup of $(\mathbb{P}^1)^3$ along the small diagonal Δ. A stable map from $\mathbb{P}^1$ to $\mathbb{P}^1$ is unique up to projective transformations of the domain curve $\mathbb{P}^1$ and each factor $\mathbb{P}^1$ in $\mathrm{Bl}_\Delta(\mathbb{P}^1)^3$ parametrizes one of the three marked points. If any of the marked points collide, a new rational component having those marked points is added to the domain curve.

So to get the moduli space $\overline{M}_{0,3}(\mathbb{P}^1, 1)$, we first blowup $(\mathbb{P}^1)^3$ along the small diagonal Δ and then along the strict transforms of the three diagonals Δ_{12}, Δ_{23} and Δ_{13}. The latter blowups are isomorphisms because the blowup centers are divisors.

By the ϵ-wall-crossing, there is a contraction morphism $c : \overline{M}_{0,3}(\mathbb{P}^1, 1) \to Q_{0,3}^{\epsilon=0^+}(\mathbb{P}^1, 1)$ (see [28]). As we will see for more general case in Proposition 5.2, c is an isomorphism.

The moduli space $Q_{0,3}^\delta(\mathbb{P}^1, 1)$ of δ-stable quasimaps helps us to understand $\overline{M}_{0,3}(\mathbb{P}^1, 1)$ from a different point of view. By the δ-stability, the curve C cannot have rational tails with one or less marked point. So C must be irreducible $\mathbb{P}^1$ with three marked points. It is easy to check that any nonzero pair of two section $(x_1, x_2) \in H^0(\mathcal{O}(1)^{\oplus 2})$ is δ-stable. Hence the moduli space $Q_{0,3}^\delta(\mathbb{P}^1, 1)$ is isomorphic to $\mathbb{P}^3$ for all $\delta > 0$.

Now we compare the cases $\epsilon = 0^+$ and $\delta = \infty$. A quasimap $(C, L, (x_1, x_2))$ has a base point if and only if the two sections x_1 and x_2 vanish at the same point. Inside $Q^{\delta=\infty}_{0,3}(\mathbb{P}^1, 1) \simeq \mathbb{P}^3$, this locus is given by a quadric surface. The δ-stable quasimap is not $\epsilon = 0^+$-stable if and only if two sections x_1 and x_2 vanish at one of the three marked points, in which case a rational tail with that marked point is added to get an $\epsilon = 0^+$-stable quasimap. For $i = 1, 2, 3$, let Γ_i be the locus, where x_1 and x_2 vanish at the i-th marked point. Then Γ_i's are disjoint lines in the quadric surface mentioned above. Let Γ be their union. By local calculation, we have the following.

Proposition 5.1 *We have the map*

$$q : Q^{\epsilon=0^+}_{0,3}(\mathbb{P}^1, 1) \to Q^{\delta=\infty}_{0,3}(\mathbb{P}^1, 1) \simeq \mathbb{P}^3,$$

which is the blowup along Γ.

It would be a good exercise to check that $\mathrm{Bl}_\Delta(\mathbb{P}^1)^3$ is isomorphic to the blowup $\mathrm{Bl}_\Gamma\mathbb{P}^3$ of $\mathbb{P}^3$ along three lines.

We describe the cohomology ring of the moduli space $\overline{M}_{0,3}(\mathbb{P}^1, 1) \simeq Q^{\epsilon=0^+}_{0,3}(\mathbb{P}^1, 1)$ from these two different points of view.

Definition 5.1 *Let* $\{H_1, H_2, H_3, e\}$ *be the set of generators of the Picard group of* $\mathrm{Bl}_\Delta(\mathbb{P}^1)^3$, *where*

(1) H_1, H_2 *and* H_3 *are pullbacks of the blowup* $\mathrm{Bl}_\Delta(\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3$ *of the point classes in the three factors of* $(\mathbb{P}^1)^3$,

(2) e *is the exceptional divisor.*

Let $\{h, E_1, E_2, E_3\}$ *be the set of generators of the Picard group of* $\mathrm{Bl}_\Gamma\mathbb{P}^3$, *where*

(1) $E_i = q^{-1}(\Gamma_i)$ *is the exceptional divisor for* $i = 1, 2, 3$,

(2) h *is the pullback of the plane class in* $\mathbb{P}^3$.

Lemma 5.1 *We have the following relations.*

(1) (a) $H_i = h - E_i$ *for* $i = 1, 2, 3$.

(b) $e = 2h - E_1 - E_2 - E_3$.

(2) (a) $h = H_1 + H_2 + H_3 - e$.

(b) $E_i = H_j + H_k - e$, *where* $\{i, j, k\} = \{1, 2, 3\}$.

Proof By construction, E_1 is nothing but the strict transform of Δ_{23}. Since Δ_{23} contains the blowup center Δ as a divisor, local computation shows that $E_1 = q^*\Delta_{23} - e = H_2 + H_3 - e$. Thus we get 2(b).

The exceptional divisor e is the locus, where the quasimap has a base point. So, e is a strict transform of the quadric surface mentioned above. Since this quadric surface contains all the blowup centers as disjoint divisors, by the same local calculation, we get 1(b).

It is elementary to check that the other relations are derived from these two relations.

Lemma 5.2 $\psi_i = E_i$.

Proof By [21, Lemma 26.4.2], the ψ_i-class is the boundary divisor, where the domain curve has two components with p_i on the one twig and other two marked points on the other. In other words, $\psi_i = E_i$.

Lemma 5.3 *The intersection products involving E_iE_j for $i \neq j$ on $\mathrm{Bl}_\Gamma\mathbb{P}^3$ are zero and those involving H_i^2 on $\mathrm{Bl}_\Delta(\mathbb{P}^1)^3$ are zero. All other possible intersection products are listed as follows:*

$\mathrm{Bl}_\Gamma\mathbb{P}^3$	$\mathrm{Bl}_\Delta(\mathbb{P}^1)^3$
$h^3 = 1$	$H_1H_2H_3 = 1$
$h^2E_i = 0$	$H_iH_je = 0$
$hE_i^2 = -1$	$H_ie^2 = -1$
$E_i^3 = -2$	$e^3 = -4$

Proof This is a consequence of the blow-up formula [19, §6.7].

Hence nonvanishing GW invariants are $\int_{\overline{M}_{0,3}(\mathbb{P}^1,1)} \psi_i^3 = \int_{\mathrm{Bl}_\Gamma\mathbb{P}^3} E_i^3 = -2$ for $i = 1, 2, 3$. Note that the moduli space of δ-stable quasimaps is useful to understand the structure of the moduli space of stable maps.

5.4 $(g, m, d) = (0, m, 1)$ for $m \geq 4$

In the case of degree 1, the moduli space $\overline{M}_{0,m}(\mathbb{P}^1, 1)$ is isomorphic to the Fulton-MacPherson configuration space $\mathbb{P}^1[m]$, which is given by a sequence of blowups along diagonals of $(\mathbb{P}^1)^m$ (see [20]).

Proposition 5.2 *The contraction morphism $c : \overline{M}_{0,m}(\mathbb{P}^1, 1) \to Q_{0,m}^{\epsilon=0^+}(\mathbb{P}^1, 1)$ is always an isomorphism.*

Proof Given a stable map $f : C \to \mathbb{P}^1$, $c(f)$ is defined by contracting all rational tails T in C with no marked points and by adding a base point of length equal to the degree of f on T at the point $x \in C$ of incidence. In case of degree one, C can have at most one rational tail T and f is of degree one at T. Since degree one map from T to $\mathbb{P}^1$ is unique up to automorphism of T, c is bijective. Hence it is an isomorphism.

Similarly as above, there is a contraction map

$$q : Q_{0,m}^{\epsilon=0^+}(\mathbb{P}^1, 1) \to Q_{0,m}^{\delta=\infty}(\mathbb{P}^1, 1),$$

which is the blowup along the locus, where two sections x_1 and x_2 have a common zero at a marked point.

When $m \geq 4$, there is a nontrivial δ-wall-crossing. For a δ-stable quasimap (C, L, x) in $Q_{0,m}^{\delta}(\mathbb{P}^1, 1)$, L has nonnegative degree on any irreducible component of C because otherwise the restriction of (L, x) to this irreducible component is a quotient pair which violates the δ-stability. Thus L has degree one on only one component and degree zero on all the other components. When $\delta = \infty$, a quasimap (C, L, x) is $\delta = \infty$-stable if x_1 and x_2 vanish simultaneously only at finitely many points or along rational bridges with no marked point.

Lemma 5.4 *For $\delta > 0$, a $\delta = \infty$-stable quasimap (C, L, x) with $g = 0$ and $d = 1$ is δ-stable if and only if for any subcurve C' of C having k marked points such that*

(1) *$\overline{C \setminus C'} \cap C'$ consists of only one point q;*

(2) *L has degree one on the irreducible component of C' containing q, and we have $\frac{m-2k}{k-1} < \delta$.*

Proof It is enough to check the stability condition for saturated subsheaves. One can see that a saturated subsheaf having positive Euler characteristic is of the form $\mathcal{O}_{C'}$, where C' is a subcurve satisfying the above two conditions. Since (C, L, x) is $\delta = \infty$-stable, the section x does not factor through $\mathcal{O}_{C'}$. Therefore the quasimap (C, L, x) is δ-stable only if $\frac{1}{k-1} < \frac{2+\delta}{m-2}$ or $\frac{m-2k}{k-1} < \delta$.

Therefore when $2 \le k < \frac{1}{2}m$, δ-wall-crossing occurs. The effect of the δ-wall-crossing is as follows. Let $C'' = \overline{C \setminus C'}$, $x'' = x|_{C''}$ and $L'' = L|_{C''}$. The above quasimap can be written as an extension

$$0 \to (0, \mathcal{O}_{C'}) \to (x, L) \to (x'', L'') \to 0.$$

Wall-crossing exchanges the subpair and the quotient pair. Thus after wall-crossing this quasimap is replaced by a quasimap given by the extension

$$0 \to (x'', L'') \to (\widehat{x}, \widehat{L}) \to (0, \mathcal{O}_{C'}) \to 0.$$

So, $\widehat{L}$ has degree one on the component of $\overline{C \setminus C'}$ containing q and $\widehat{x}|_{C'} = 0$.

In case where L has degree one on a rational bridge R with no marked point, R is contracted after wall-crossing since $\widehat{L}$ has degree zero on R. For example, Figure 1 shows the effect of δ-wall-crossing when $m = 6$. In the picture, the numbers labeled above each component indicate the degrees of L on the component.

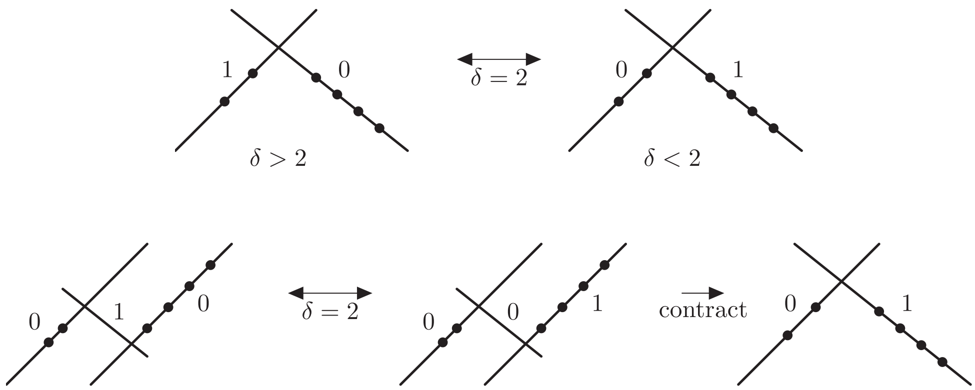

Figure 1 δ-wall-crossing when $(g, m, d) = (0, 6, 1)$

As δ approaches to zero, this process is repeated until there are no subcurves C' satisfying the above three conditions. Now we describe $Q_{0,m}^{\delta=0^+}(\mathbb{P}^1, 1)$.

Lemma 5.5 *For a $\delta = 0^+$-stable quasimap (C, L, x) with $g = 0$ and $d = 1$, the component on which L has degree one is uniquely determined by the distribution of marked points.*

Proof Consider the dual graph T whose vertices are the irreducible components of C and edges are the incidence relations between the irreducible components. Assign to a vertex of T

the number of marked points on the corresponding irreducible component of C as a weight. Then T is a connected tree. For each edge e of T, $T \setminus e$ has two connected components. Let $v \in T$ be the vertex such that L has degree one on its corresponding component of C. Then by Lemma 5.4, the quasimap (C, L, x) is $\delta = 0^+$-stable if

(∗) for each edge e of T incident to v, the sum of all weights of the vertices in the connected component of $T \setminus e$ containing v is greater than $\frac{1}{2}m$.

We want to show that the vertex of T satisfying (∗) is unique.

We use induction on the number of edges. If T contains only one edge, the assertion is trivial. For a general tree T, let w be a vertex in T having a maximal weight among the vertices incident to only one edge. If the weight of w is greater than $\frac{1}{2}m$, then w is the unique vertex satisfying (∗). If not we remove w and the unique edge incident to w from T and add the weight of w to the vertex adjacent to w. By induction the resulting tree T' has a unique vertex v satisfying (∗). It is clear that v satisfies (∗) for T as well.

Theorem 5.1 *The moduli space* $Q^{\delta=0^+}_{0,m}(\mathbb{P}^1, 1)$ *is isomorphic to a* $\mathbb{P}^3$*-bundle over* $\overline{M}_{0,m}$.

Proof Let (C, L, x) be a $\delta = 0^+$-stable quasimap. Since a rational bridge with no marked points does not satisfy (∗), C does not contain a rational bridge with no marked points and hence C is a stable curve in $\overline{M}_{0,m}$. By Lemma 5.5, the line bundle L is uniquely determined by C. Therefore, $Q^{\delta=0^+}_{0,m}(\mathbb{P}^1, 1)$ is isomorphic to a $\mathbb{P}^3$-bundle over $\overline{M}_{0,m}$, where the fiber $\mathbb{P}^3 = \mathbb{P}H^0(C, L^{\oplus 2})$ is the space of sections x.

5.5 $(g, m, d) = (0, m, d)$

We describe δ-wall-crossing for more general m and d. A quasimap (C, L, x) is $\delta = \infty$-stable if

(1) L has nonnegative degree on each irreducible component of C and

(2) x vanishes only at finitely many points and on rational bridges.

For a subcurve C' of C, let $k(C')$ be the number of marked points on C' and $\ell(C')$ be the number of points in $\overline{C \setminus C'} \cap C'$. Let d' be the degree of $L|_{C'}$. Then, the saturated subsheaf L' of L supported on C' has Euler characteristic $d' + 1 - \ell(C')$. Let

$$r = \max\left\{ \frac{d' + 1 - \ell(C')}{k(C') - 2 + \ell(C')} \middle| C' \subset C \right\}.$$

It is elementary to check that for a subcurve C' attaining the maximum r, $\ell(C') \le 1$ and L must have nonnegative degrees on all components and positive degrees on the component of C' meeting the rest of the curve C. If $r > \frac{d+1+\delta}{m-2}$, the quasimap (C, L, x) is not δ-stable. So if $r(m-2) - d - 1 > 0$, wall-crossing occurs at $\delta_0 = r(m-2) - d - 1$.

Let $C'' = \overline{C \setminus C'}$, $x'' = x|_{C''}$ and $L'' = L|_{C''}$. The above quasimap can be written as an extension

$$0 \to (0, L') \to (x, L) \to (x'', L'') \to 0.$$

Similarly as before, after crossing the wall at δ_0, this quasimap is replaced by a quasimap given by the extension

$$0 \to (x'', L'') \to (\widehat{x}, \widehat{L}) \to (0, \mathcal{O}_{C'}) \to 0.$$

The degree of $\widehat{L}$ is decreased by 1 on the components in C' meeting C'' and increased by 1 on the component in C'' meeting C'. The section $\widehat{x}$ is zero on C'. Similarly as before, if C has a rational bridge R such that $\widehat{L}$ has degree 0, R is contracted. As δ approaches to zero, this process is repeated until $r(m-2)-d-1<0$. For example, Figure 2 shows the effect of δ-wall-crossing when $m=4$ and $d=2$.

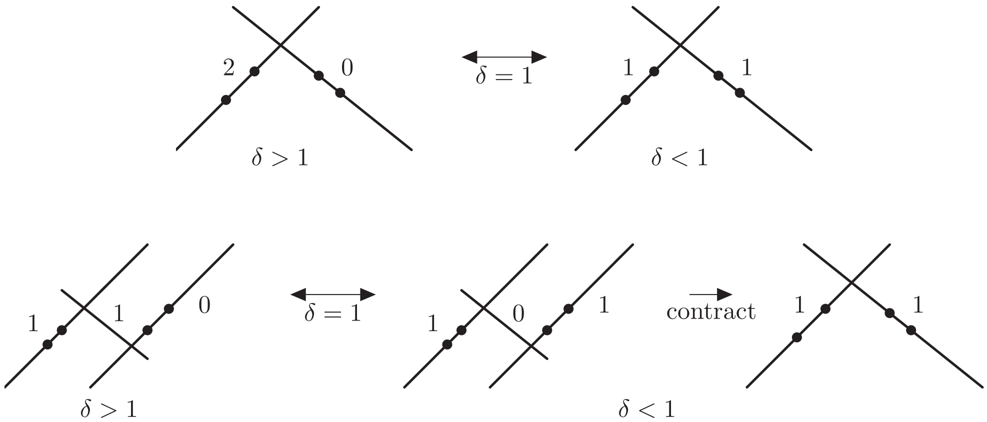

Figure 2 δ-wall-crossing when $(g,m,d)=(0,4,2)$

Acknowledgement The authors thank Huai-Liang Chang, Huijun Fan, Ionut Ciocan-Fontanine, Emily Clader, Tyler Jarvis, Bumsig Kim, Jun Li, Wei-Ping Li, Chiu-Chu Melissa Liu and Yongbin Ruan for useful discussions.

References

[1] Behrend, K. and Fantechi, B., The intrinsic normal cone, *Invent. Math.*, **129**(1), 1997, 45–88.

[2] Chang, H.-L., Kiem, Y.-H. and Li, J., Torus localization and wall crossing formulas for cosection localized virtual cycles, arXiv: 1502.00078.

[3] Chang, H.-L. and Li, J., Gromov-Witten invariants of stable maps with fields, *Int. Math. Res. Notices*, **18**, 2012, 4163–4217.

[4] Chang, H.-L., Li, J. and Li, W.-P., Wittens top Chern class via cosection localization, *Invent. Math.*, **200**(3), 2015, 1015–1063.

[5] Cheong, D., Ciocan-Fontanine, I. and Kim, B., Orbifold quasimap theory, *Math. Ann.*, **363**(3), 2015, 777–816.

[6] Chiodo, A., Stable twisted curves and their r-spin structures, *Ann. Inst. Fourier*, **58**(5), 2008, 1635–1689.

[7] Chiodo, A. and Ruan, Y., Landau-Ginzburg/Calabi-Yau correspondence for quintic three-folds via symplectic transformations, *Invent. Math.*, **182**(1), 2010, 117–165.

[8] Choi, J. and Kiem, Y.-H., Landau-Ginzburg/Calabi-Yau correspondence via quasi-maps, I, arXiv: 1103.0833.

[9] Ciocan-Fontanine, I. and Kim, B., Wall-crossing in genus zero quasimap theory and mirror maps, arXiv: 1304.7056.

[10] Ciocan-Fontanine, I. and Kim, B., Higher genus quasimap wall-crossing for semi-positive targets, arXiv: 1308.6377.

[11] Ciocan-Fontanine, I., Kim, B. and Maulik, D., Stable quasimaps to GIT quotients, *J. Geom. and Phys.*, **75**, 2014, 17–47.

[12] Clader, E., Landau-Ginzburg/Calabi-Yau correspondence for the complete intersections $X_{3,3}$ and $X_{2,2,2,2}$, arXiv: 1301.5530.

[13] Cotterill, E., Rational curves of degree 10 on a general quintic threefold, *Comm. Alg.*, **33**(6), 2005, 1833–1872.

[14] Cotterill, E., Rational curves of degree 11 on a general quintic 3-fold, *Quart. J. Math.*, **63**(3), 2012, 539–568.

[15] Cox, D. and Katz, S., Mirror symmetry and algebraic geometry, Mathematical surveys and monographs, **68**, American Mathematical Society, 1999.

[16] Faltings, G., Moduli-stacks for bundles on semistable curves, *Math. Ann.*, **304**, 1996, 489–515.

[17] Fan, H., Jarvis, T. and Ruan, Y., The Witten equation, mirror symmetry, and quantum singularity theory, *Ann. Math.*, **178**, 2013, 1–106.

[18] Fan, H., Jarvis, T. and Ruan, Y., A mathematical theory of the gauged linear sigma model, arXiv: 1506.02109.

[19] Fulton, W., Intersection Theory, second ed., Ergeb. Math. Grenzgeb, Springer-Verlag, 1998.

[20] Fulton, W. and MacPherson, R., A compactification of configuration spaces, *Ann. Math.*, **139**(1), 1994, 183–225.

[21] Hori, K., Katz, S., Klemm, A., Pandharipande, R., et al., Mirror Symmetry, Clay Mathematics Monograph, 2003.

[22] Jarvis, T., Kimura, T. and Vaintrob, A., Moduli spaces of higher spin curves and integrable hierarchies, *Comp. Math.*, **126**, 2001, 157–212.

[23] Johnsen, T. and Kleiman, S., Rational curves of degree at most 9 on a general quintic threefold, *Comm. Alg.*, **24**, 1996, 2721–2753.

[24] Katz, S., On the finiteness of rational curves on quintic threefolds, *Comp. Math.*, **60**, 1986, 151–162.

[25] Kiem, Y.-H. and Li, J., Localizing virtual cycles by cosections, *J. Amer. Math. Soc.*, **26**, 2013, 1025–1050.

[26] Le Potier, J., Systèmes cohérents et structures de niveau, Asterisque No. 214, 1993.

[27] Li, J. and Tian, G., Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties, *J. Amer. Math. Soc.*, **11**, 1998, 119–174.

[28] Marian, A., Oprea D. and Pandharipande, R., The moduli space of stable quotients. arXiv: 0904.2992.

[29] Ross, D. and Ruan, Y., Wall-crossing in genus zero Landau-Ginzburg theory, *J. Reine Angew. Math.*, 2015.

[30] Toda, Y., Moduli spaces of stable quotients and wall crossing phenomena, *Comp. Math.*, **147**, 2011, 1479–1518.

[31] Vafa, C. and Warner, N., Catastrophes and the classification of conformal field theories, *Phys. Lett. B*, **218**(1), 1989, 51–58.

[32] Vainsencher, I., Enumeration of n-fold tangent hyperplanes to a surface, *J. Algebraic Geom.*, **4**, 1995, 503–526.

[33] Witten, E., Phases of $N = 2$ theories in two dimensions, *Nucl. Phys. B*, **403**, 1993, 159–222.