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Beijing Lectures on the Grade Restriction Rule

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Abstract The authors describe the relationships between categories of B-branes in different phases of the non-Abelian gauged linear sigma model. The relationship is described explicitly for the model proposed by Hori and Tong with non-Abelian gauge group that connects two non-birational Calabi-Yau varieties studied by Rødland. A grade restriction rule for this model is derived using the hemisphere partition function and it is used to map B-type D-branes between the two Calabi-Yau varieties.

Keywords Gauged linear sigma model, Non-birational Calabi-Yau manifolds, D-branes, Equivalences of categories
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1 A Quick Preview

The goal of this paper is to use the non-Abelian GLSM (gauged linear sigma model) to understand relationships between categories of B-branes of different theories. In certain cases, this will lead to equivalences of derived categories

$$D^b \operatorname{Coh} X \cong D^b \operatorname{Coh} Y$$

for X and Y birationally distinct Calabi-Yau varieties. For this lecture we will focus on a particular GLSM constructed in [1] which we shall call the Rødland model. It has two phases, and in either of the two, the model reduces at low energies to a sigma model with smooth Calabi-Yau target, X for one and Y for the other. Below, we describe what X and Y are.

In addition to mathematical applications, our story is motivated by studying boundary conditions in two dimensional gauge theory with $(2,2)$ supersymmetry. This sheds light on D-branes in string theory and field theory dualities. This can be viewed as a toy model for similar phenomena in higher dimensional field theories.

Rødland Example We choose a collection of complex numbers $(A_k^{ij})_{i,j,k=1}^7$ which is anti-symmetric in the upper indices, $A_k^{ij} = -A_k^{ji}$. We assume that it is generic in a certain sense.

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(1) Complete intersection in $\mathrm{Gr}(2,7)$.

X is a complete intersection of seven hyperplanes in the Grassmannian $\mathrm{Gr}(2,7)$,

$$\sum_{i,j=1}^{7} A_k^{ij}[x_i x_j] = 0, \quad k = 1, \cdots, 7,$$

where $[x_i x_j] = x_i^1 x_j^2 - x_i^2 x_j^1$ are the Plücker coordinates for $\mathrm{Gr}(2,7)$. When (A_k^{ij}) is generic, it is a smooth Calabi-Yau threefold with invariants

$$\int_X H^3 = 42, \quad \int_X c_2 \cdot H = 85, \quad \int_X c_3 = \chi = -98.$$

(2) Pfaffian Calabi-Yau.

Y is a Pfaffian Calabi-Yau. Put $A^{ij}(p) := \sum_{k=1}^{7} A_k^{ij} p^k$ and regard them as elements of a 7×7 antisymmetric matrix $A(p)$. When (A_k^{ij}) is generic, the rank of $A(p)$ for non-zero p is either six or four, and the rank four locus defines a smooth submanifold $Y \subset \mathbb{P}^6$ of codimension three. It is a Calabi-Yau threefold with invariants

$$\int_Y H^3 = 14, \quad \int_Y c_2 \cdot H = 56, \quad \int_Y c_3 = \chi = -98.$$

(3) Smoothness.

Let V be a seven dimensional vector space. Then the collection of numbers $A^{ij}(p)$ can be viewed as the coefficients of the map $A : \wedge^2 V \to V$. The Grassmannian $\mathrm{Gr}(2,7)$ is a subvariety $\mathbb{P}(\wedge^2 V)$ under the Plücker embedding $\mathrm{Gr}(2,7) \subset \mathbb{P}(\wedge^2 V)$. The variety X is the intersection $X = \ker(A) \cap \mathrm{Gr}(2,7)$ and is smooth if $\mathbb{P}(\ker A)$ is transverse to $\mathrm{Gr}(2,7)$. This is the condition for the collection of numbers $A^{ij}(p)$ to be called generic. If $A^{ij}(p)$ is generic then Y is also smooth (see [2]).

(4) A tale of two Calabi-Yau threefolds.

X and Y are birationally distinct, but they have equivalent categories of B-branes,

$$D^b \operatorname{Coh} X \cong D^b \operatorname{Coh} Y.$$

Rødland [3] argued that the families X and Y appear to have the same mirror family. Combining this observation with Kontsevich's homological mirror symmetry conjecture predicts the equivalence of categories of B-branes on X and Y. The first proofs of the derived equivalences were given by Borisov-Caldararu [2] and Kuznetsov [4]. Recently Addington-Donovan-Segal [5] gave a new proof using ideas from Herbst-Hori-Page [6]. The equivalence was shown by contructing a Fourier-Mukai functor:

$$\Phi : D^b \operatorname{Coh} Y \to D^b \operatorname{Coh} X.$$

2 Introduction and Overview

The aim of this paper is to describe the grade restriction rule obtained in [7], for D-brane transport along paths between the two phases of the non-Abelian GLSM of [1]. This provides an equivalence $D^b \operatorname{Coh} Y \to D^b \operatorname{Coh} X$ for each homotopy class of paths (window). The new technique will be to use the hemisphere partition function in the derivation.

Outline We will start by reviewing the data of the GLSM and the description of B-branes in the GLSM. Then using the data associated to a B-brane, we will explain how to compute the hemisphere partition function. The hemisphere partition function depends on a choice of an integration contour, and only certain branes with charges satisfying a grade restriction rule will have a convergent partition function along a path connecting two phases of the GLSM. For the Rødland model we will determine the grade-restriction rule for several simple choices of paths connecting the Grassmannian and Pfaffian phases. Finally we will then describe how the grade restriction rule for a specific path relates to earlier proofs of the derived equivalence between the two different phases by Borisov-Caldararu and Kuznetsov.

3 The Gauged Linear Sigma Model

The gauged linear sigma model is a class of two-dimensional $(2,2)$ supersymmetric gauge theories introduced by Witten [8]. It can be used to construct a family of $(2,2)$ superconformal field theories which may have limits corresponding to Calabi-Yau sigma models, Landau-Ginzburg orbifolds and/or hybrids thereof.

The ingredients A 2d $(2,2)$ supersymmetric gauge theory is specified by a choice of
(1) G a compact Lie group (gauge group),
(2) V a $\mathbb{C}$-representation of G (matter content),
(3) $W(\phi)$ a G-invariant polynomial function of $\phi \in V$ (superpotential),
(4) $\widetilde{W}(\sigma)$ a G-invariant polynomial function of $\sigma \in \mathfrak{g}_{\mathbb{C}}$ (twisted superpotential).

An adjoint invariant norm $X \mapsto \frac{1}{\mathrm{e}^2}X^2$ of $i\mathfrak{g}$ and a G-invariant hermitian inner product on V must also be chosen. These induce G-invariant symplectic structures on $\mathfrak{g}_{\mathbb{C}}$ and V. The associated moment map on V that vanishes at the origin is denoted by $\phi \mapsto \mu(\phi)$. Physically, the parameter $e \in \mathbb{R}$ is the coupling constant for the vector multiplet.

Two $U(1)$ R-symmetries
(1) $U(1)_V$ R-symmetry.

The vector $U(1)$ R-symmetry is preserved if there exists a diagonalizable $R \in \mathrm{End}(V)$, called the R-charge, such that $W(\lambda^R\phi) = \lambda^2 W(\phi), \lambda \in \mathbb{C}^\times$. It possesses charge integrality if $\mathrm{e}^{\pi \mathrm{i} R}$ agrees with the action of a gauge group element, say $J \in G$. If the system has $U(1)_V$ symmetry with charge integrality, one can apply the A-twist to obtain a topological field theory.
(2) $U(1)_A$ R-symmetry.

The axial $U(1)$ R-symmetry is preserved at the classical level if the twisted superpotential is linear, $\widetilde{W}(\sigma) = -t(\sigma)$, for $t = \zeta - i\theta \in (\mathfrak{g}^*_{\mathbb{C}})^G$. ζ and θ are called the Fayet-Iliopoulos (FI) parameter and theta parameter respectively. To be precise, the theta parameter is subject to a discrete identification, $\theta \in (i\mathfrak{g}^*)^G/\sim$. It remains to be a symmetry of the quantum system under the Calabi-Yau condition: The representation maps G to $SL(V)$. Under these conditions, one can apply the B-twist to obtain a topological field theory.

We assume all of these conditions. Then, the gauge theory is expected to flow to a superconformal field theory with central charge $\frac{\mathrm{c}}{3} = \mathrm{tr}_V(1-R) - \dim G$.

Phases The classical potential is

$$U = \frac{1}{8e^2}[\sigma, \overline{\sigma}]^2 + \frac{1}{2}(|\sigma\phi|^2 + |\overline{\sigma}\phi|^2) + \frac{e^2}{2}(\mu(\phi) - \zeta)^2 + |\mathrm{d}W(\phi)|^2.$$

A zero of this potential is called a classical supersymmetric vacuum. The space of the FI-paramerter ζ is decomposed into chambers called phases according to the pattern of gauge symmetry breaking by the vacuum values of ϕ. Typically, and always if the gauge group is Abelian, deep inside a phase, the gauge group is broken to a finite subgroup and hence $\sigma = 0$. In such a "weakly coupled" phase, the nature of the low energy theory can be found by a classical analysis. Sometimes the low energy theory is a non-linear sigma model with a smooth Calabi-Yau target space (geometric, or CY, phase), sometimes the target is an orbifold (orbifold phase), sometimes the theory reduces to a Landau-Ginzburg orbifold (LG phase), and in general it is a mixture of these (hybrid phase). On the boundary between different phases, there are classical vacua with an unbroken continuous subgroup. There, σ can take any value in the Cartan subalgebra of the unbroken subgroup. The emergence of a non-compact flat direction in σ (Coulomb branch) is a sign of singularity. A typical picture of phases and their classical boundaries is shown in Figure 1.

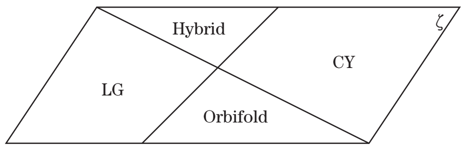


Figure 1 Classical phase boundaries

If the gauge group is non-Abelian, the unbroken gauge group can be continuous even deep inside a phase. In such a "strongly coupled" phase, a hard physical analysis is needed to understand the low energy behaviour.

Quantum Parameter Space Under the assumption of two $U(1)$ R-symmetries, we have a family of 2d $(2,2)$ superconformal field theories parametrized by the product space $\mathcal{M}_A \times \mathcal{M}_B$, where $\mathcal{M}_A$ is the space of FI-theta parameter t, and $\mathcal{M}_B$ is the space of parameters that determine the superpotential $W(\phi)$. To be precise, we need to avoid the discriminant locus, where the theory is singular. The FI-theta parameter t is in the discriminant Δ if there is a quantum Coulomb branch. When projected to the FI parameter space, Δ becomes an Amoeba (see [9–10]) with spines that asymptote to the classical phase boundaries. An illustration is shown in Figure 2. In a geometric phase, $\mathcal{M}_A$ is the space of complexified Kähler class while $\mathcal{M}_B$ is the moduli space of complex structures.

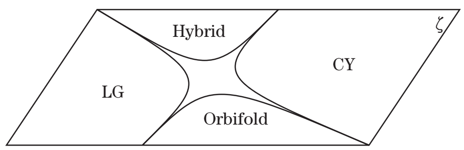


Figure 2 Quantum corrected phase boundaries

4 Examples

Quintic

(1) $G = U(1)$,

(2) $V = \mathbb{C}(-5) \oplus \mathbb{C}(1)^{\oplus 5}$ where $\mathbb{C}(j)$ is the weight j representations of $U(1)$,

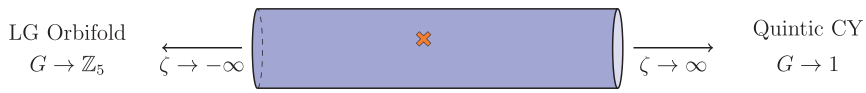
(3) $W = pf(x_1, \cdots, x_5)$ where f is a quintic polynomial,

(4) $\widetilde{W} = -t\sigma$ with $t = \zeta - i\theta \in \frac{\mathbb{C}}{2\pi i\mathbb{Z}}$.

Table 1 Charges for the quintic GLSM

	p	$x_{1,\cdots,5}$
q	-5	1
R	2	0

$\zeta \gg 0$ is a geometric phase where the gauge group is completely broken. It describes the sigma model whose target space is the quintic hypersurface $X_f \subset \mathbb{P}^4$ defined by the equation $f(x) = 0$. $\zeta \ll 0$ is the Landau-Ginzburg orbifold phase where the gauge group is broken to the $\mathbb{Z}_5$ subgroup. The discriminant locus Δ is a single point at $t \equiv 5\log(-5) \bmod 2\pi i\mathbb{Z}$.



Rødland model

(1) $G = U(2)$,

(2) $V = (\det^{-1} S)^{\oplus 7} \oplus S^{\oplus 7} \ni (p^1, \cdots, p^7, x_1, \cdots, x_7)$ where $S \cong \mathbb{C}^2$ is the fundamental representation of $U(2)$,

(3) $W = \sum_{i,j,k=1}^{7} A_k^{ij} p^k [x_i x_j]$,

(4) $\widetilde{W} = -t \cdot \mathrm{tr}_S(\sigma)$ with $t \in \frac{\mathbb{C}}{2\pi i\mathbb{Z}}$.

$\zeta \gg 0$ is a geometric phase where the gauge group is completely broken, with the Calabi-Yau manifold $X \subset G(2,7)$ as the target. $\zeta \ll 0$ is a strongly coupled phase where the unbroken subgroup is $SU(2) \subset U(2)$ at any vacuum value of ϕ. The intermediate energy theory is a

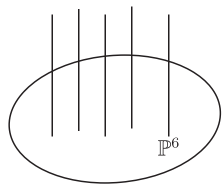


Figure 3 $SU(2)$ gauge theory fibered over $\mathbb{P}^6$

$SU(2)$ gauge theory with fundamental doublets x_i that have a mass matrix $A^{ij}(p)$, which is fibered over the space of p's. The nature of the theory depends on the number of massless doublets, which is seven minus the rank of the mass matrix. The rank can be six or four.

(1) On a generic point $p \in \mathbb{P}^6$, the rank is six and the number of massless doublet is one. Supersymmetry is broken in the $SU(2)$ gauge theory with a single massless doublet. Therefore, the low energy theory is not supported on the generic points of $\mathbb{P}^6$.

(2) On the rank four locus, we have $SU(2)$ gauge theory with three massless doublets X_1, X_2, X_3. This flows to the free theory of the three mesons

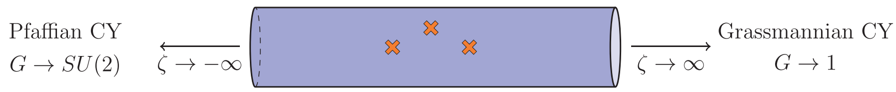
$$B_1 = [X_2X_3], \quad B_2 = [X_3X_1], \quad B_3 = [X_1X_2].$$

Thus, near this locus, the effective theory is the Landau-Ginzburg model with the superpotential

$$W_{\text{eff}} = \delta p^1 B_1 + \delta p^2 B_2 + \delta p^3 B_3.$$

(3) The theory reduces to a sigma model whose target space is the critical locus of W_{eff}, that is, the Pfaffian Calabi-Yau threefold Y.

In summary, we have a one parameter family of 2d $(2,2)$ superconformal field theories which has two limits corresponding to the sigma models with target X and Y. The discriminant locus Δ consists of the three points $\{\alpha_1, \alpha_2, \alpha_3\}$ at $t \equiv 7\log\left(2\cos\left(\frac{\pi a}{7}\right)\right) + \pi i a$ $(a = 1,2,3)$. Of course, the two limits are connected by paths that avoid these three points.



5 B-branes in the GLSM

From the general principle of 2d $(2,2)$ supersymmetry, we expect to have a category of B-branes at each point of $\mathcal{M}_A \times \mathcal{M}_B$, and it is invariant under deformations in $\mathcal{M}_A$. In a geometric phase, the category is equivalent to the derived category of coherent sheaves on the target. In a Landau-Ginzburg phase, it is the category of matrix factorizations. If a path in $\mathcal{M}_A$ between two phases is chosen, we have an equivalence of the categories at the two ends, and the equivalence depends only on the homotopy class of the paths. We would like to find such equivalences using GLSMs.

A B-brane in the GLSM is given by the following data:

(i) A $\mathbb{Z}_2$ graded vector space $M = M^{\text{ev}} \oplus M^{\text{odd}}$ with an even representation of $U(1)_V \times G$,

$$U(1)_V \times G \to GL(M), \quad (\lambda, g) \mapsto \lambda^{r_*}\rho(g).$$

We assume that the charge integrality: $e^{\pi i r_*}\rho(J)$ is the $\mathbb{Z}_2$-grading; $+1/-1$ on even/odd elements of M.

(ii) A polynomial function $Q(\phi)$ of $\phi \in V$ with values in $\text{End}^{\text{odd}}(M)$ such that

$$\begin{aligned} &Q(\phi)^2 = W(\phi)\text{id}_M \quad \text{[matrix factorization]},\\ &\rho(g)^{-1}Q(g\phi)\rho(g) = Q(\phi) \quad [G\text{-equivariance}],\\ &\lambda^{r_*}Q(\lambda^R\phi)\lambda^{-r_*} = \lambda Q(\phi) \quad [R\text{-charge } 1]. \end{aligned}$$

(iii) A Weyl invariant Lagrangian submanifold $\gamma \subset \mathfrak{t}_{\mathbb{C}}$.

Given the data $\mathfrak{B} = (M, Q, \gamma)$, we have the boundary condition on the bulk fields as well as interaction at the boundary, at the classical level. However, it is not a priori clear whether the classical data defines a D-brane in the quantum theory. If it does, we are most interested in the low energy behaviour, especially, the image of the D-brane at the infra-red fixed point. It may happen that different data flow to the same brane in the superconformal field theory. What is the full set of such relations among the classical data? Also, we would like to find the rule of D-brane transport along paths in $\mathcal{M}_A$.

In [6], such questions were studied in Abelian GLSMs, and the grade restriction rule was proposed as the key to the answer. It was found that:

(i) Deep inside a phase, there is a class of γ's such that the data (M, Q, γ) defines a D-brane in the quantum theory for an arbitrary (M, Q),

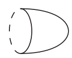
(ii) That is not the case near the phase boundary. For each homotopy class of paths that connect two phases, there is a set of allowed representations of G: One can find a family of γ along the path so that (M, Q, γ) defines a D-brane as long as M consists of representations from the allowed set. This is the grade restriction rule.

(iii) As far as the grade restricted (M, Q)'s are concerned, the two phases have a common set of relations. In one parameter models, there is no non-trivial relation at all — the category of grade restricted branes is equivalent to the category in the infra-red theory.

(i) and (ii) are obtained from the study of effective boundary potential $V_{\text{eff}}(\sigma)$ for large values of σ. (iii) is a mathematical fact whose origin goes back to Kawamata [11] and van den Bergh [12], and is refined/generalized by [13–15].

In [7], the analysis is extended to the Rødland model and its cousins. Rather than the effective boundary potential $V_{\text{eff}}(\sigma)$, the hemisphere partition function is used in the analysis. A new feature is that, in a strongly coupled phase with a continuous unbroken gauge group, there is a non-trivial grade restriction rule, unlike in an Abelian theory where no such rule is present (i).

The hemisphere partition function is, as the name suggests, the partition function on the hemisphere with a certain background that preserves two supercharges,

$Z(\mathfrak{B}) = Z\Big($  $\mathfrak{B}$ b.c. $\Big)$,

where we place the boundary condition/interaction corresponding to the data $\mathfrak{B} = (M, Q, \gamma)$ at the boundary. It was computed in [16–18] and found to agree with the central charge of the brane whenever the latter is known. Let

$$V = \bigoplus_i \mathbb{C}(R_i, Q_i)$$

be the weight decomposition of the matter representation. Then, the hemisphere partition function with the boundary condition interaction corresponding to $\mathfrak{B} = (M, Q, \gamma)$ is given by

$$Z(\mathfrak{B}) = (r\Lambda)^{\frac{c}{6}} \int_\gamma \mathrm{d}^l\sigma \prod_{\alpha>0} \alpha(\sigma)\sinh(\pi\alpha(\sigma)) \prod_i \Gamma\Big(iQ_i(\sigma) + \frac{R_i}{2}\Big) e^{it(\sigma)} f_M(\sigma)$$

with

$$f_M(\sigma) := \mathrm{tr}_M(e^{\pi i r_*}\rho(e^{2\pi\sigma})).$$

In the above expression, r is the radius of the hemisphere, Λ is the renormalization scale, $\mathrm{d}^l\sigma := \mathrm{d}\sigma_1 \wedge \cdots \wedge \mathrm{d}\sigma_l$ is the flat holomorphic volume form on $\mathfrak{t}_{\mathbb{C}}$ (l is the rank of G), $\prod_{\alpha>0}$ is the product over the positive roots of G, and $\Gamma(z)$ is Euler's gamma function.

The main concern is whether this integral is convergent. The grade restriction rule is derived from the condition of absolute convergence. The absolute convergence can be derived using Stirling's approximation for the gamma function. For an Abelian GLSM, the growth of the integrand is essentially $\mathrm{e}^{-rV_{\text{eff}}(\sigma)}$, and the rule from absolute convergence agrees with the rule derived from $V_{\text{eff}}(\sigma)$ in [6].

6 The Grade Restriction Rule in the Rødland Model

Let us summarize the result of [7]. As before, S stands for the fundamental representation of $G = U(2)$. We use the shorthand notation for other representations,

$$S^l S(i) := \mathrm{Sym}^l S \otimes (\det S)^{\otimes i}, \quad l \in \{0, 1, 2, \cdots\},\ i \in \mathbb{Z}.$$

We may also write $\mathbb{C}(i) = (\det S)^{\otimes i}$ and $S(i) = S \otimes (\det S)^{\otimes i}$.

6.1 No GRR in the weakly coupled phase

In the Grassmannian phase $\zeta \gg 0$ where the gauge group is completely broken, there is no non-trivial grade restriction rule. There is a universal contour, say, γ_+ defined by

$$\mathrm{Im}\,\sigma_1 = (\mathrm{Re}\,\sigma_1)^2, \quad \mathrm{Im}\,\sigma_2 = (\mathrm{Re}\,\sigma_2)^2$$

for which the integral is absolutely convergent for an arbitrary (M, Q).

6.2 GRR in the strongly coupled phase

In the Pfaffian phase $\zeta \ll 0$ where the $SU(2)$ subgroup of the gauge group is totally unbroken, there is a non-trivial grade restriction rule. In order for the integration in the $SU(2)$ direction to be convergent, the representation M should be a direct sum of the irreducible representations from the three series $\mathbb{C}(i)$, $S(i)$ and $S^2 S(i)$ $(i \in \mathbb{Z})$. If that is the case, the integral is convergent, say for γ_- given by

$$\mathrm{Im}\,\sigma_1 = \mathrm{Im}\,\sigma_2 = -(\mathrm{Re}\,\sigma_1 + \mathrm{Re}\,\sigma_2)^2.$$

6.3 GRR for the paths between the two phases

Let us consider the D-brane transport along paths that go between the Grassmannian phase and the Pfaffian phase, as shown in Figure 4.

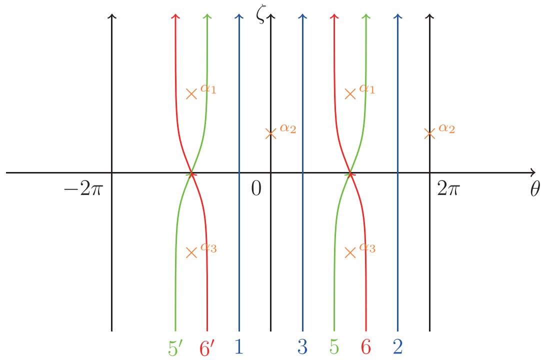


Figure 4 Paths in the complexified Kähler moduli space $\mathcal{M}_A$

When projected to the FI-parameter line, such a path goes through the phase boundary where the unbroken gauge group becomes bigger than $SU(2)$. Accordingly, there is a stronger grade restriction rule than the one found in the Pfaffian phase. Below, we describe the rule.

The allowed representations for path 3 are

$$\begin{array}{ccccc} & \mathbb{C}(1), & \cdots, & \mathbb{C}(6), & \mathbb{C}(7), \\ S, & S(1), & \cdots, & S(6), & \\ S^2S, & S^2S(1), & \cdots, & S^2S(6). & \end{array}$$

The allowed representations for path 1 are

$$\begin{array}{ccccc} & \mathbb{C}(1), & \cdots, & \mathbb{C}(6), & \mathbb{C}(7), \\ & S(1), & \cdots, & S(6), & S(7), \\ S^2S, & S^2S(1), & \cdots, & S^2S(6). & \end{array}$$

The allowed representations for path 5 are

$$\begin{array}{cccc} \mathbb{C}, & \mathbb{C}(1), & \cdots, & \mathbb{C}(6), \\ S, & S(1), & \cdots, & S(6), \\ S^2S, & S^2S(1), & \cdots, & S^2S(6) \end{array}$$

and the allowed representations for path $6'$ are

$$\begin{array}{cccccc} & & \mathbb{C}(2), & \cdots, & \mathbb{C}(6), & \mathbb{C}(7), \quad \mathbb{C}(8), \\ & S(1), & S(2), & \cdots, & S(6), & S(7), \\ S^2S, & S^2S(1), & S^2S(2), & \cdots, & S^2S(6). & \end{array}$$

7 Derived Equivalences and Borisov-Caldararu

We now show how to obtain Borisov-Caldararu's proof of the derived equivalence between the Grassmannian X and Pfaffian Calabi Yau Y threefolds as an application of the non-Abelian grade-restriction rule. The idea of the proof is to reconstruct the variety X from skyscraper sheaves. In physical terms, this is the moduli space of a D0 brane. A more precise version of this idea is the following theorem.

Theorem 7.1 (see [19]) *Let X be a smooth irreducible projective varitiety with ample canonical or anti-canonical sheaf. If $D^b\operatorname{Coh}X$ is derived equivalent to $D^b\operatorname{Coh}Y$ for some other smooth algebraic variety Y, then X is isomorphic to Y.*

A key aspect of the Borisov-Caldararu proof of the Grassmannian-Pfaffian equivalence is to determine where the skyscraper sheaf on the Pfaffian goes to under the equivalence. This will be determined by a classical incidence correspondence which we now describe.

Incidence Correspondence The heart of the Borisov-Caldararu construction is constructing a sheaf $\mathcal{I}_C$ on $X\times Y$ which we now describe in terms of an incidence correspondence following [20]. Recall that $\mathcal{Y} = Pf(4,7) = Pf(4,V^*)$ is the Pfaffian variety of 7 by 7 anti-symmetric matrices A of rank at most 4. $\mathcal{Y}$ is singular along the locus $Z = Pf(2,V^*) = Gr(2,V^*)$ where the rank of A drops to 2. $\mathcal{Y}$ has a resolution $\widetilde{\mathcal{Y}} = \{([w],[A])|\, w \in \ker A\}$.

Let $\mathcal{K} \subset V\otimes\mathcal{G}r$ be the universal subbundle of $\mathcal{G}r = \mathrm{Gr}(3,7) = \mathrm{Gr}(3,V)$ and let $\mathcal{K}^\perp \subset V^*\otimes\mathcal{G}r$ be the annihilator of $\mathcal{K}$. Then $\widetilde{\mathcal{Y}} = \mathbb{P}_{\mathcal{G}r}(\wedge^2\mathcal{K}^\perp)$. In particular, $\widetilde{\mathcal{Y}}$ is a $\mathbb{P}^5$ bundle over $\mathcal{G}r$ and is therefore smooth. Define $\Delta_0 \subset Gr(2,7)\otimes Gr(3,7)$ by

$$\Delta_0 = \{([\xi],[w])|\, \dim(\xi\cap w) \geq 1\}.$$

Let $\mathcal{I}_{\Delta_0}$ be the ideal sheaf of Δ_0 and

$$\mathcal{I} = (id\times\rho)^*\mathcal{I}_{\Delta_0}$$

be its pullback to $\mathcal{X}\times\widetilde{\mathcal{Y}}$. Finally, let $\mathcal{I}_C = \mathcal{I}|_{X\times Y}$ be the restriction of $\mathcal{I}$ from $\mathcal{X}\times\widetilde{\mathcal{Y}}$ to $X\times Y$.

$$
\begin{array}{ccc}
 & \widetilde{\mathcal{Y}} \subset Gr(3,7) \times Pf(4,7) & \\
 \rho \swarrow & & \searrow \pi \\
 Gr(3,7) = \mathcal{G}r & & Pf(4,7) = \mathcal{Y}
\end{array}
$$

Figure 5 Resolution of $\widetilde{\mathcal{Y}}$

Theorem 7.2 (see [2]) *The sheaf $\mathcal{I}_C$ in $X \times Y$ gives rise to a Fourier-Mukai functor*

$$
\begin{aligned}
&\Phi : D^b \operatorname{Coh} Y \to D^b \operatorname{Coh} X, \\
&\Phi(\mathcal{E}) = R\pi_{X*}(\pi_Y^*(\mathcal{E}) \otimes^L \mathcal{I}_C).
\end{aligned}
$$

Φ is an equivalence of categories,

$$\Phi \mathcal{O}_y = \mathcal{I}_{C_y}.$$

The functor takes the sky-skraper sheaf of a point $y \in Y$ to the ideal sheaf $\mathcal{I}_{C_y}$ of a curve $C_y \in X$, that is $\Phi(\mathcal{O}_y) = \mathcal{I}_{C_y}$. The family of curves C_y has genus 6 and degree 14. The resolution of $\mathcal{O}_{C_y}$ nearly fits into a window. Given the inclusion $i : C_y \hookrightarrow X$, the ideal sheaf of C_y is defined by the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to i_* \mathcal{O}_C \to 0. \tag{7.1}$$

The structure sheaf $\mathcal{O}_X$ of X can be obtained by taking the tensor product of the Koszul resolutions for its seven defining hyperplane sections. The weights of the Koszul resolution for $\mathcal{O}_X$ would use the representations

$$\mathbb{C}, \quad \mathbb{C}(1), \quad \cdots, \quad \mathbb{C}(6), \quad \mathbb{C}(7),$$

which is just one size too large to fit into a window. However we are interested in the weights for the ideal sheaf $\mathcal{I}_C$. To proceed we follow Borisov-Caldararu and resolve the structure sheaf of the curve $i_* \mathcal{O}_C$.

Let $G = \operatorname{Gr}(2, V)$ be the Grassmannian of two-planes in V. Let $K \subset V$ be a linear subspace. We define $\mathcal{S} \subset G$ to be the locus of two-planes $T \in V$ (points $T \in G$) which intersect K non-trivially. As defined, $\mathcal{S}$ is a Schubert cycle corresponding to the increasing sequence $(0, \dim K, 7)$. Let T be the rank two tautological subbundle on G. Assume $\dim K = 3$ and let $A = \operatorname{Ann} K$. Then there exists a resolution of the structure sheaf $\mathcal{O}_{\mathcal{S}}$ (of the Schubert cell) on G of the form:

$$0 \to \wedge^4 A \otimes \operatorname{Sym}^2 T(-1) \to \wedge^3 A \otimes T(-1) \to \wedge^2 A \otimes \mathcal{O}_G(-1) \to \mathcal{O}_G \to \mathcal{O}_{\mathcal{S}} \to 0.$$

The vector spaces $\wedge^k A$ encode the multiplicity of the representations of the gauge group $U(2)$. The representations appearing in the resolution of $\mathcal{O}_{\mathcal{S}}$ are

$$
\begin{array}{cc}
\mathbb{C}, & \mathbb{C}(1), \\
 & S(1), \\
 & S^2 S(1).
\end{array}
$$

Note that S and T have opposite $U(1)$ charges. The exact sequence for $\mathcal{O}_{C_y}$ can be obtained by tensoring the exact sequence for $\mathcal{O}_{\mathcal{S}}$ with the exact sequence for the intersection of six hyperplanes

$$
\begin{array}{cccc}
\mathbb{C}, & \mathbb{C}(1), & \cdots, & \mathbb{C}(7), \\
 & S(1), & \cdots, & S(7), \\
 & S^2 S(1), & \cdots, & S^2 S(7).
\end{array}
$$

From the exact sequence (7.1), we see that the graded character $f_M(\sigma)$ for the ideal sheaf $\mathcal{I}_{C_y}$ can be obtained from the difference in weights of $i_*\mathcal{O}_C$ and $\mathcal{O}_X$. Since the representation $\mathbb{C}$ occurs with multiplicity one in both $i_*\mathcal{O}_C$ and $\mathcal{O}_X$, its contribution to the character $f_M(\sigma)$ for $\mathcal{I}_{C_y}$ cancels, and the weights occuring in the character for $\mathcal{I}_{C_y}$ precisely fit into the rectangle of allowed representations

$$\begin{array}{cccc} \mathbb{C}(1), & \mathbb{C}(2), & \cdots, & \mathbb{C}(7), \\ S(1), & S(2), & \cdots, & S(7), \\ S^2S(1), & S^2S(2), & \cdots, & S^2S(7). \end{array}$$

This rectangular structure is a Lefschetz collection in Kuznetsov's terminology. It corresponds to path $5'$. Conjecturally, the complex can be lifted to a B-brane $\mathfrak{B} = (M, Q, \gamma)$ in the GLSM that reduces to $\mathcal{I}_{C_y}$ on X and $\mathcal{O}_y$ on Y.

8 Parting Thoughts

The convergence of the hemisphere partition function can be used to determine the grade restriction rule for branes in the GLSM. This gives a powerful new method to understand derived equivalences. The derived equivalence of Hosono-Takagi [20–21] can be analyzed in almost an identical fashion. The new pair of dual threefolds constructed by Miura [22] have a GLSM description due to Galkin [23] and Gerhardus-Jockers [24]. We hope to find the grade restriction rule in this GLSM that would yields the expected derived equivalence for Miura's example.

The new ingredient of our analysis is using analytical information to guide a purely algebraic analysis. Hopefully this interplay between algebra, analysis, and physics will be fruitful for all three subjects.

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