Chin. Ann. Math.
38B(4), 2017, 913–936
DOI: 10.1007/s11401-017-1104-7

Chinese Annals of
Mathematics, Series B

The Moduli Space in the Gauged Linear Sigma Model*

Huijun FAN[1] Tyler JARVIS[2] Yongbin RUAN[3]

Abstract This is a survey article for the mathematical theory of Witten's Gauged Linear Sigma Model, as developed recently by the authors. Instead of developing the theory in the most general setting, in this paper the authors focus on the description of the moduli.

Keywords Landau-Ginzburg, Calabi-Yau, Gromov-Witten, GLSM, Moduli, Phase, Variation of geometric invariant theory, Symplectic reduction
2000 MR Subject Classification 53D45, 14H35, 81T13, 81T30, 14H10, 14J33, 14J81

1 Introduction

In 1991, in an effort to generalize his famous conjecture regarding the KdV-hierarchy for the intersection theory of the moduli space of Riemann surfaces (see [34–35]), Witten proposed a remarkable first-order, nonlinear, elliptic PDE associated to an arbitrary quasihomogeneous singularity. It has the simple form:

$$\overline{\partial} u_i + \overline{\frac{\partial W}{\partial u_i}} = 0, \tag{1.1}$$

where W is a quasihomogeneous polynomial with an isolated singularity at the origin, and u_i is interpreted as a section of an appropriate orbifold line bundle on an orbifold Riemann surface $\mathscr{C}$.

During the last decade we have carried out a comprehensive treatment of the Witten equation and have used it to construct a theory similar to Gromov-Witten theory (see [16–18]). This so-called FJRW-theory can be viewed as the Landau-Ginzburg phase of a Calabi-Yau hypersurface

$$X_W = \{W = 0\} \subset W\mathbb{P}^{n-1}$$

in weighted projective space. The relation between the Gromov-Witten theory of X_W and the FJRW-theory of W is the subject of the Landau-Ginzburg/Calabi-Yau correspondence, a famous duality from physics. More recently, the LG/CY correspondence has been reformulated

Manuscript received March 22, 2016.
[1]School of Mathematical Sciences, Peking University, Beijing 100871, China.
E-mail: fanhj@math.pku.edu.cn
[2]Department of Mathematics, Brigham Young University, Provo, UT 84602, USA.
E-mail: jarvis@math.byu.edu
[3]Department of Mathematics, University of Michigan Ann Arbor, MI 48105, USA.
E-mail: ruan@umich.edu
*This work was supported by the Natural Science Foundation (Nos. DMS-1564502, DMS-1405245, DMS-1564457), the National Natural Science Foundation of China (Nos. 11325101, 11271028) and the Ph.D. Programs Foundation of Ministry of Education of China (No. 20120001110060).

as a precise mathematical conjecture (see [30]), and a great deal of progress has been made on this conjecture (see [7–9, 26, 28]).

A natural question is whether the LG/CY correspondence can be generalized to complete intersections in projective space, or more generally to toric varieties. The physicists' answer is "yes". In fact, Witten considered this question in the early 90s (see [34]) in his effort to give a physical derivation of the LG/CY correspondence. In the process, he invented an important model in physics called the Gauged Linear Sigma Model (GLSM for short). From the point of view of partial differential equations, the gauged linear sigma model generalizes the Witten Equation (1.1) to the Gauged Witten Equation:

$$\overline{\partial}_A u_i + \overline{\frac{\partial W}{\partial u_i}} = 0, \tag{1.2}$$

$$*F_A = \mu, \tag{1.3}$$

where A is a connection of certain principal bundle, and μ is the moment map of the GIT-quotient, viewed as a symplectic quotient. In general, both the Gromov-Witten theory of a Calabi-Yau complete intersection X and the LG dual of X can be expressed as gauged linear sigma models. Furthermore, the LG/CY correspondence can be interpreted as a variation of the moment map μ (or a deformation of GIT) in the GLSM.

During last several years, we constructed a rigorous mathematical theory for the gauged linear sigma model (see [19–20]), and this new model seems to have many applications (see, for example [14, 28]). Our new theory is a generalization of FJRW-theory from the case of a finite gauge group to the case where the gauge group is any reductive Lie group. Surprisingly, our older theory (finite group) is one of the more difficult cases in the general theory.

We deal with the Gauged Witten Equation both analytically (see [20]) and algebraically (see [19]). But in this paper we focus on the algebraic version of the theory. In the case of a continuous group, we can treat the Gauged Witten Equation algebraically using some stability conditions. It turns out to be very convenient to incorporate the stability conditions from the quasimap theory of Ciocan-Fontanine, Kim, Maulik, and Cheong (see [6, 10–11, 24]). And indeed, it is natural to view our GLSM-theory as a union of FJRW-theory with quasimap theory. However, it is possible to impose other stability conditions (see [5, 12]).

Understanding the details of the mathematical construction of the GLSM can be daunting. This paper is an attempt to help the reader navigate past the more technical constructions and begin to understand the underlying moduli spaces. In the next section we give a brief overview of the main ingredients to the theory, and in the subsequent section we briefly review the definition of the moduli problem and the most important stability conditions. The remainder of the paper is focused on giving examples of the moduli spaces for various choices of input data.

2 The Basic Setting

The input data of our new theory consists of the following. We discuss these in more detail below.

(1) A finite dimensional vector space V over $\mathbb{C}$.

(2) A reductive algebraic group $G \subseteq \mathrm{GL}(V)$, sometimes called the gauge group.

(3) A G-character θ with the property $V_G^s(\theta) = V_G^{ss}(\theta)$. We say that it defines a strongly regular phase $\mathscr{X}_\theta = [V/\!\!/_\theta G]$.

(4) A choice of $\mathbb{C}^*$ action on V, called the R-charge and denoted by $\mathbb{C}^*_R$. This action is required to commute with the G-action, and we require $G \cap \mathbb{C}^*_R = \langle J \rangle$ to have finite order d.

(5) A G-invariant quasihomogeneous polynomial $W : V \to \mathbb{C}$, called the superpotential, having degree d with respect to the $\mathbb{C}^*_R$ action. We require that the GIT quotient $\mathrm{Crit}(W)/\!\!/_\theta G$ of the critical locus $\mathrm{Crit}(W)$ be compact.

(6) A stability parameter ε. This can be any positive rational number, but in practice, the two most useful cases are the limiting cases of $\varepsilon \to \infty$ or $\varepsilon \to 0+$. Fortunately, these cases are also easier to describe than the positive rational cases. For simplicity, in this paper we will only discuss the limiting cases of $0+$ and ∞.

(7) A Γ-character ϑ, where Γ is the subgroup of $\mathrm{GL}(V)$ generated by G and $\mathbb{C}^*_R$ by Γ. We require that ϑ define a lift of θ, meaning that $\vartheta|_G = \theta$. Except in the case of $\varepsilon = 0^+$, we also require that this lift be a good lift, meaning that it satisfy $V_\Gamma^{ss}(\vartheta) = V_G^{ss}(\theta)$. A choice of good lift affects the stability conditions for the moduli space. But in the case of $\varepsilon = 0+$ the lift need not be good, and every lift will produce the same stability conditions and the same moduli space.

With the above input data we construct a theory with the following main ingredients:

(1) A state space, which is the relative Chen-Ruan cohomology of the quotient $\mathscr{X}_\theta = [V/\!\!/_\theta G]$ with an additional shift by $2q$. For each conjugacy class $\Psi \subset G$, let

$$I(\Psi) = \{(v, g) \in V_\theta^{ss} \times G \mid g \in \Psi\}$$

and

$$\mathscr{X}_{\theta,\Psi} = [I(\Psi)/G].$$

The state space is

$$\mathscr{H}_{W,G} = \bigoplus_{\alpha\in\mathbb{Q}} \mathscr{H}^\alpha_{W,G} = \bigoplus_{\Psi} \mathscr{H}_\Psi,$$

where the sum runs over those conjugacy classes Ψ of G for which $\mathscr{X}_{\theta,\Psi}$ is nonempty, and where

$$\mathscr{H}^\alpha_{W,G} = H^{\alpha+2q}_{CR}(\mathscr{X}_\theta, W^\infty, \mathbb{Q}) = \bigoplus_{\Psi} H^{\alpha-2\,\mathrm{age}\,(\gamma)+2q}(\mathscr{X}_{\theta,\Psi}, W^\infty_\Psi, \mathbb{Q}),$$

and

$$\mathscr{H}_\Psi = H^{\bullet+2q}_{CR}(\mathscr{X}_{\theta,\Psi}, W^\infty, \mathbb{Q}) = \bigoplus_{\alpha\in\mathbb{Q}} H^{\alpha-2\,\mathrm{age}\,(\gamma)+2q}(\mathscr{X}_{\theta,\Psi}, W^\infty_\Psi, \mathbb{Q}).$$

Here $W^\infty = \mathfrak{Re}(W)^{-1}(M, \infty) \subset [V/\!\!/_\theta G]$ for some large, real M.

(2) The stack of LG-quasimaps.

We denote by $\mathscr{CR}_\theta = [\mathrm{Crit}^{ss}_G(\theta)/G] \subset [V/\!\!/_\theta G] = [V_G^{ss}(\theta)/G]$ the GIT quotient (with polarization θ) of the critical locus of W. Our main object of study is the stack

$$\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta)$$

of (ε, ϑ)-stable Landau-Ginzburg quasimaps to $\mathscr{CR}_\theta$.

(3) A virtual cycle:

$$[\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta)]^{\mathrm{vir}} \in H_*(\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta), \mathbb{Q})$$

with virtual dimension

$$\dim_{\text{vir}} = \int_\beta c_1(V/\!\!/_\theta G) + (\widehat{c}_{W,G} - 3)(1-g) + k - \sum_i (\text{age}(\gamma_i) - q),$$

where $\widehat{c}_{W,G}$ is the central charge (see Definition 2.7).

(4) Numerical invariants. Using the virtual cycle, we can define correlators

$$\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_k}(\alpha_k) \rangle = \int_{[\text{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta,\beta)]^{\text{vir}}} \prod_i ev_i^*(\alpha_i)\psi_i^{l_i}.$$

One can then define a generating function in the standard fashion. These invariants satisfy the usual gluing axioms whenever all insertions are of compact type.

In the rest of this section, we will discuss some of the input data and the state space in more detail.

2.1 GIT and symplectic quotients

The first two pieces of data consist of a reductive algebraic group G (the gauge group) acting on a finite-dimensional vector space $V \cong \mathbb{C}^n$. We do not require G to be connected, but we require that G/G_0 be finite, where G_0 is the identity component of G. If the gauge group action on V factors through $\mathrm{SL}(V)$, then we say that it satisfies the Calabi-Yau condition. But in general we do not require that G satisfy this condition.

We wish to consider the quotient stack $[Z/G]$ for a closed subvariety $Z \subseteq V$, but since the group G may not be compact, the quotient is not generally separated (Hausdorff). Geometric Invariant Theory (GIT for short) and symplectic reduction each give a way to construct separated quotients.

2.1.1 GIT quotients

The key to constructing a separated quotient using GIT is to choose a linearization of the action of G on Z, i.e., a lifting of the action of G to a line bundle $\mathbf{L}$ over Z. We always assume that the linearization on Z is induced by a linearization on V. Since $V = \mathbb{C}^n$, any line bundle $\mathbf{L}$ on V is trivial $\mathbf{L} = V \times \mathbb{C}$, and the linearization is determined by a character $\theta : G \to \mathbb{C}^*$.

Definition 2.1 *For any character $\theta : G \to \mathbb{C}^*$ we write $\mathbf{L}_\theta$ for the line bundle $V \times \mathbb{C}$ with the induced linearization. We also often write $\mathbf{L}_\theta$ to denote the corresponding line bundle on Z.*

Geometric Invariant Theory identifies an open subset $Z^{ss}(\theta)$ of $\mathbf{L}_\theta$-semistable points in Z as the set of those points $v \in Z$ for which there exists a positive integer k and a G-invariant section $f \in H^0(Z, \mathbf{L}_\theta^{\otimes k})^G$ such that $f(v) \neq 0$. We denote the set of points in Z that are semistable with respect to G and θ by $Z^{ss}_G(\theta)$. The GIT quotient stack $[Z/\!\!/_\theta G]$ is defined to be the stack

$$[Z/\!\!/_\theta G] = [Z^{ss}_G(\theta)/G].$$

Let $Z/\!\!/_{\text{aff}} G$ be the affine quotient given by $Z/\!\!/_{\text{aff}} G = \operatorname{Spec}(\mathbb{C}[Z^*]^G)$, where $\mathbb{C}[Z^*]$ is the ring of regular functions on Z. The GIT quotient stack $[Z/\!\!/_\theta G]$ is an algebraic stack with an underlying (coarse moduli) space

$$Z/\!\!/_\theta G = Z^{ss}_G(\theta)/G = \mathrm{Proj}_{Z/\!\!/_{\mathrm{aff}} G}\Big(\bigoplus_{k\geq 0} H^0(Z, \mathbf{L}^k_\theta)^G\Big).$$

The linearization $\mathbf{L}_\theta$ induces a line bundle (a.k.a. a polarization) on $[Z/\!\!/_\theta G]$, which we denote by L_θ.

Definition 2.2 *We say that a point $v \in V$ is stable with respect to the linearization θ (or θ-stable) if*

(1) *v is θ-semistable.*

(2) *The stabilizer $\mathrm{Stab}_G(v) = \{g \in G \mid gv = v\}$ is finite.*

We denote the set of θ-stable points of Z by $Z^s_G(\theta)$. We say that a point is unstable if it is not semistable.

The stable locus is important because the quotient stack $[Z^s_G(\theta)/G]$ is a Deligne-Mumford stack, whereas $[Z^{ss}_G(\theta)/G]$ is not necessarily Deligne-Mumford.

Remark 2.1 Mumford-Fogarty-Kirwan [27] use the name properly stable to describe what we call stable.

Remark 2.2 For any integers $\ell, k > 0$, each $f \in H^0(Z, \mathbf{L}^{\otimes k}_\theta)^G$ also satisfies

$$f^\ell \in H^0(Z, \mathbf{L}^{\otimes k\ell}_\theta)^G,$$

so it makes sense to extend the GIT constructions to fractional linearizations, corresponding to fractional characters in $\widehat{G}_{\mathbb{Q}} = \mathrm{Hom}(G, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$.

For a fixed Z, changing the linearization gives a different quotient. The space of (fractional) linearizations is divided into chambers, and any two linearizations lying in the same chamber have isomorphic GIT quotients. We call the isomorphism classes of these quotients phases. If the linearizations lie in distinct chambers, the quotients are birational to each other, and are related by flips (see [15, 32]).

Definition 2.3 *We say that $\theta \in \widehat{G}_{\mathbb{Q}}$ (or the corresponding linearization $\mathbf{L}_\theta$) is strongly regular if $V^{ss}_G(\theta)$ is not empty and $V^s_G(\theta) = V^{ss}_G(\theta)$.*

For purposes of this paper, all linearizations need to be strongly regular.

Remark 2.3 For any strongly regular phase θ, the complex dimension of $\mathscr{X}_\theta = [V/\!\!/_\theta G]$ is $n - \dim(G)$.

2.1.2 Symplectic quotients

One may also think of the GIT quotients as symplectic reductions. Take $Z \subseteq \mathbb{C}^n$ with the standard Kähler form $\omega = \sum\limits_i \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i$. Since G is reductive, it is the complexification of a maximal compact Lie subgroup H, acting on Z via a faithful unitary representation $H \subseteq U(n)$. Denote the Lie algebra of H by $\mathfrak{h}$.

We have a Hamiltonian action of H on Z with moment map $\mu_Z : Z \to \mathfrak{h}^*$ for the action of H on Z, given by

$$\mu_Z(v)(Y) = \frac{1}{2}\overline{v}^{\mathrm{T}} Y v = \frac{1}{2}\sum_{i,j\leq n} \overline{v}_i Y_{i,j} v_j$$

for $v \in Z$ and $Y \in \mathfrak{h}$. If $\tau \in \mathfrak{h}^*$ is a value of the moment map, then the locus $\mu^{-1}(H\tau)$ is an H-invariant set, and the symplectic orbifold quotient of Z at τ is defined as

$$[Z/\!\!/_{\tau}^{\mathrm{spl}} H] = [\mu_Z^{-1}(H\tau)/H] = [\mu_Z^{-1}(\tau)/H_\tau],$$

where H_τ is the stabilizer in H of τ. The value τ of the moment map plays the role for the symplectic quotient that the linearization θ plays for the GIT quotient. These are related by the following result.

Theorem 2.1 (see [19, Corollary 2.1.8]) *Let $\theta \in \widehat{G}$ be a character of G. Taking derivations of the character θ defines a weight $\tau_\theta \in \mathfrak{h}^*$. Whenever the coadjoint orbit of τ_θ in $\mathfrak{h}^*$ is trivial (e.g., in the case that τ is in the Lie algebra of the center of G, or if G is Abelian), then we have*

$$[Z/\!\!/_{-\tau_\theta}^{\mathrm{spl}} H] = [Z/\!\!/_\theta G].$$

As with the space of GIT linearizations, the space $\mathfrak{h}^*$ is divided into chambers; and values of τ that lie in the same chamber define isomorphic quotients. The walls between the chambers correspond to the critical points of the moment map μ. In many cases these are easier to identify in the symplectic formulation than in the GIT formulation.

2.2 Superpotential and critical locus

The next piece of data required for the GLSM is the superpotiential, which is G-invariant polynomial $W : V \to \mathbb{C}$. We are especially interested in the critical locus of the superpotential.

Definition 2.4 *Let $\theta : G \to \mathbb{C}^*$ define a strongly regular phase $\mathscr{X}_\theta = [V/\!\!/_\theta G]$. The superpotential W descends to a holomorphic function $W\colon \mathscr{X}_\theta \to \mathbb{C}$. Let $\mathrm{Crit}_G^{ss}(\theta) = \big\{v \in V_G^{ss}(\theta) \mid \frac{\partial W}{\partial x_i}$ for all $i = 1, \cdots, n\big\} \subset V^{ss}$ denote the semistable points of the critical locus. The group G acts on $\mathrm{Crit}_G^{ss}(\theta)$, and the stack quotient is*

$$\mathscr{CR}_\theta = [\mathrm{Crit}_G^{ss}(\theta)/G] = \{x \in \mathscr{X}_\theta \mid \mathrm{d}W = 0\} \subset \mathscr{X}_\theta,$$

where $\mathrm{d}W\colon T\mathscr{X}_\theta \to T\mathbb{C}^*$ is the differential of W on $\mathscr{X}_\theta$. We say that the pair (W, G) is nondegenerate for $\mathscr{X}_\theta$ if the critical locus $\mathscr{CR}_\theta \subset \mathscr{X}_\theta$ is compact.

2.3 *R*-charge and the group Γ

The gauged linear sigma model (GLSM for short) requires an additional $\mathbb{C}^*$-action on V called the R-charge. The R-charge is a $\mathbb{C}^*$-action on V of the form

$$(z_1, \cdots, z_n) \mapsto (\lambda^{c_1} z_1, \cdots, \lambda^{c_n} z_n).$$

We denote this action by $\mathbb{C}_R^*$ in order to distinguish it from other $\mathbb{C}^*$ actions (for example when $G = \mathbb{C}^*$). We think of $\mathbb{C}_R^*$ as a subgroup of $\mathrm{GL}(V, \mathbb{C})$. This means we require $\gcd(c_1, \cdots, c_n) = 1$. Unlike the case of FJRW theory, we allow the weights c_i of $\mathbb{C}_R^*$ to be zero or negative.

Remark 2.4 Our choice of $\mathbb{C}_R^*$-action differs from what the physics literature calls R-charge by a factor of 2. More precisely, the physicists' R-charge is the $\mathbb{C}^*$-action given by the weights $(2c_1/d, \cdots, 2c_n/d)$; but for our purposes, $\mathbb{C}_R^*$ is the more natural choice.

Definition 2.5 *We define the exponential grading element $J \in \mathbb{C}^*_R$ to be*

$$J = (\exp(2\pi i c_1/d), \cdots, \exp(2\pi i c_n/d)), \tag{2.1}$$

which has order d.

It is sometimes convenient to write $q_i = c_i/d$ and $q = \sum_{i=1}^{n} q_i$ so that

$$J = (\exp(2\pi i q_1), \cdots, \exp(2\pi i q_n)).$$

We require the actions of G and $\mathbb{C}^*_R$ to be compatible, by which we mean

(1) They commute: $gr = rg$ for any $g \in G$ and any $r \in \mathbb{C}^*_R$.

(2) We have $G \cap \mathbb{C}^*_R = \langle J \rangle$.

Definition 2.6 *The group Γ is the subgroup of* $\mathrm{GL}(V)$ *generated by G and $\mathbb{C}^*_R$.*

If G and $\mathbb{C}^*_R$ are compatible, then every element γ of Γ can be written as $\gamma = gr$ for $g \in G,\ r \in \mathbb{C}^*_R$; that is,

$$\Gamma = G\mathbb{C}^*_R.$$

The representation $\gamma = gr$ is unique up to an element of $\langle J \rangle$. Moreover, there is a well-defined homomorphism

$$\begin{aligned} \zeta \colon \Gamma = G\mathbb{C}^*_R &\to \mathbb{C}^*, \\ g(\lambda^{c_1}, \cdots, \lambda^{c_n}) &\mapsto \lambda^d. \end{aligned} \tag{2.2}$$

We denote the target of ζ by $H = \zeta(\mathbb{C}^*_R) = \mathbb{C}^*$, to distinguish it from $\mathbb{C}^*_R$. This gives the following exact sequence:

$$1 \to G \to \Gamma \xrightarrow{\zeta} H \to 1. \tag{2.3}$$

Moreover, there is another homomorphism

$$\begin{aligned} \xi \colon \Gamma &\to G/\langle J \rangle, \\ gr &\mapsto g\langle J \rangle. \end{aligned} \tag{2.4}$$

This is also well defined, and gives another exact sequence:

$$1 \to \mathbb{C}^*_R \to \Gamma \xrightarrow{\xi} G/\langle J \rangle \to 1.$$

Definition 2.7 *Let $N = n - \dim(G)$. We define the central charge of the theory for the choices $V, G, \mathbb{C}^*_R, W$ to be*

$$\widehat{c}_{W,G} = N - 2\sum_{j=1}^{n} c_j/d = N - 2q. \tag{2.5}$$

2.4 Lifts of the linearization to Γ

Although we are primarily interested in the GIT quotients of V by G, our constructions also depend heavily on the GIT quotients of V by Γ. For this, we need a lift of the G-linearization to a Γ-linearization. That is, we require a Γ-character ϑ that lifts θ, meaning that $\vartheta|_G = \theta$. It is not hard to prove that lifts always exist, including the trivial lift $\vartheta(gr) = \theta(g)$.

For a given lift ϑ, we always have $V^{ss}_{\Gamma}(\vartheta) \subset V^{ss}_{G}(\theta)$, but equality does not necessarily hold. If it does hold, we say that ϑ is a good lift of θ. For the stability parameter $\varepsilon = \infty$ we require the lift to be a good lift. For the choice $\varepsilon = 0+$, the lift need not be good, and every lift will produce the same theory. Unfortunately, not every $\theta \in \widehat{G}$ has a good lift for every choice of (G-compatible) $\mathbb{C}^*_R$-action, but most of the interesting examples of GLSMs have a good lift.

2.5 Choice of $\mathbb{C}^*_R$

All of our constructions ultimately depend not on $\mathbb{C}^*_R$, but rather only on the embeddings $G \subseteq \Gamma \subseteq \mathrm{GL}(V)$, on the sum $q = \sum\limits_{i=1}^{n} q_i = \sum\limits_{i=1}^{n} c_i/d$ of the $\mathbb{C}^*_R$ weights, and on a choice of a lift $\vartheta : \Gamma \to \mathbb{C}^*$ of θ.

Of course the choice of q and the embedding of Γ in $\mathrm{GL}(V)$ put many constraints on $\mathbb{C}^*_R$; but they still allow some flexibility. For example, when the gauge group G is a torus with a Calabi-Yau weight system (that is, if its weight matrix $B = (b_{ij})$ satisfies $\sum\limits_{j} b_{ij} = 0$ for each i), then we have a lot of flexibility. The following lemma is not hard to prove (see [19, Lemma 3.3.1]).

Lemma 2.1 *If the gauge group G is a torus with weight matrix $B = (b_{ij})$, and if we have a compatible $\mathbb{C}^*_R$ action with weights $(c_1, \cdots, c_n)$, such that W has $\mathbb{C}^*_R$-weight d, then for any $\mathbb{Q}$-linear combination $(b'_1, \cdots, b'_n)$ of rows of the gauge weight matrix B, we define a new choice of R-weights $(c'_1, \cdots, c'_n) = (c_1 + b'_1, \cdots, c_n + b'_n)$. Denote the corresponding $\mathbb{C}^*$ action by $\mathbb{C}^*_{R'}$.*

Since the group Γ' generated by G and $\mathbb{C}^*_{R'}$ lies inside the maximal torus of $\mathrm{GL}(n, \mathbb{C})$, it is Abelian; and so we automatically have that G and $\mathbb{C}^*_{R'}$ commute. We also have the following:

(1) *The group Γ' generated by G and $\mathbb{C}^*_R$ is the same as the group Γ generated by G and $\mathbb{C}^*_R$.*

(2) *The $\mathbb{C}^*_{R'}$-weight of W is equal to d.*

(3) *$G \cap \mathbb{C}^*_{R'} = G \cap \mathbb{C}^*_R = \langle J \rangle$, where J is the element defined by Equation (2.1) for the original $\mathbb{C}^*_R$ action.*

(4) *If B is a Calabi-Yau weight system, then for both $\mathbb{C}^*_R$ and $\mathbb{C}^*_{R'}$ the sum of the weights $q = \sum q_i = \sum c_i/d$ is the same and the central charge $\widehat{c}_W$ is the same.*

2.6 Hybrid models

A very important subclass of the toric examples—when $G = (\mathbb{C}^*)^m$—consists of the so-called hybrid models. Several examples of the hybrid model have been worked out in detail by E. Clader in [13].

Definition 2.8 *For a torus $G = (\mathbb{C}^*)^m$, a phase θ of (W, G) is called a hybrid model if*

(1) *The quotient $\mathscr{X}_\theta \to \mathscr{X}_{\mathrm{base}}$ has the structure of a toric bundle over a compact base $\mathscr{X}_{\mathrm{base}}$, and*

(2) *The $\mathbb{C}^*_R$-weights of the base variables are all zero.*

3 The GLSM Moduli Space

Given a choice of V, G, W, $\mathbb{C}^*_R$, θ, and an additional choice of stability parameters ε, ϑ, the "moduli space" for the GLSM is the stack of (ε, ϑ)-stable Landau-Ginzburg quasimaps to the critical locus $\mathscr{CR}_\theta$ of W, which we describe below. This space is naturally a substack of the stack of (ε,ϑ)-stable Landau-Ginzburg quasimaps to $\mathscr{X}_\theta$, and that larger space plays an important role in the construction of the virtual class—similar to the role of p-fields for Gromov-Witten theory.

3.1 Landau-Ginzburg quasimaps

Definition 3.1 *For any k-pointed orbicurve $\mathscr{C}, y_1, \cdots, y_k$, denote by $\omega_{\log,\mathscr{C}}$ the line bundle*

$$\omega_{\log,\mathscr{C}} = \omega_{\mathscr{C}} \otimes \mathscr{O}(y_1) \otimes \cdots \otimes \mathscr{O}(y_k) = \mathscr{O}\Big(K_{\mathscr{C}} + \sum_{i=1}^{k} y_i\Big),$$

where $\omega_{\mathscr{C}} = \mathscr{O}(K_{\mathscr{C}})$ is the canonical bundle on $\mathscr{C}$.

Also, let $\mathring{\omega}_{\log,\mathscr{C}}$ denote the principal $\mathbb{C}^*$-bundle on $\mathscr{C}$ corresponding to the line bundle $\omega_{\log,\mathscr{C}}$.

Definition 3.2 *A prestable, k-pointed, genus-g, LG-quasimap to $\mathscr{X}_\theta$ is a tuple $(\mathscr{C}, y_1, \cdots, y_k, \mathscr{P}, u, \varkappa)$ consisting of*

(A) *a prestable, k-pointed orbicurve $(\mathscr{C}, y_1, \cdots, y_k)$ of genus g,*

(B) *a principal (orbifold) Γ-bundle $\mathscr{P}\colon \mathscr{C} \to \mathrm{B}\Gamma$ over $\mathscr{C}$,*

(C) *a global section $\sigma\colon \mathscr{C} \to \mathscr{E} = \mathscr{P} \times_\Gamma V$,*

(D) *an isomorphism $\varkappa\colon \zeta_*\mathscr{P} \to \mathring{\omega}_{\log,\mathscr{C}}$ of principal $\mathbb{C}^*$-bundles,*

such that

(1) *the morphism of stacks $\mathscr{P}\colon \mathscr{C} \to \mathrm{B}\Gamma$ is representable (i.e., for each point y of $\mathscr{C}$, the induced map from the local group G_y to Γ is injective);*

(2) *the set B of points $b \in \mathscr{C}$ such that any point p of the fiber $\mathscr{P}_b$ over b is mapped by σ into an $\mathbf{L}_\theta$-unstable G-orbit of V is finite, and this set is disjoint from the nodes and marked points of $\mathscr{C}$.*

A prestable, k-pointed, genus-g, LG-quasimap to $\mathscr{CR}_\theta$ is a prestable, k-pointed, genus-g, LG-quasimap to $\mathscr{X}_\theta$ such that the image of the induced map $[\sigma]\colon \mathscr{P} \to V$ lies in $\mathrm{Crit}(W)$.

Definition 3.3 *The points b occurring in condition (2) above are called base points of the quasimap. That is, $b \in \mathscr{C}$ is a base point if at least one point (and hence every point) of the fiber $\mathscr{P}_b$ over b is mapped by σ into an $\mathbf{L}_\theta$-unstable G-orbit of V.*

Definition 3.4 *For any prestable LG-quasimap $\mathcal{Q} = (\mathscr{C}, y_1, \cdots, y_k, \mathscr{P}, \sigma, \varkappa)$, a Γ-equivaria -nt line bundle $\mathbf{L} \in \mathrm{Pic}^\Gamma(V)$ determines a line bundle $\mathscr{L} = \mathscr{P} \times_\Gamma \mathbf{L}$ over $\mathscr{E} = \mathscr{P} \times_\Gamma V$, and pulling back along σ gives a line bundle $\sigma^*(\mathscr{L})$ on $\mathscr{C}$.*

In particular, any character $\alpha \in \widehat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ determines a Γ-equivariant line bundle $\mathbf{L}_\alpha$ on V and hence a line bundle $\sigma^*(\mathscr{L}_\alpha)$ on $\mathscr{C}$.

3.2 Stability conditions for the stack of LG-quasimaps

Definition 3.5 *For any $\alpha \in \widehat{\Gamma}$ define the degree of α on $\mathcal{Q}$ to be*

$$\deg_{\mathcal{Q}}(\alpha) = \deg_{\mathscr{C}}(\sigma^*(\mathscr{L}_\alpha)) \in \mathbb{Q}.$$

This defines a homomorphism $\deg_{\mathcal{Q}} : \widehat{\Gamma} \to \mathbb{Q}$.

For any $\beta \in \mathrm{Hom}(\widehat{\Gamma}, \mathbb{Q})$ *we say that an LG-quasimap* $\mathcal{Q} = (\mathscr{C}, x_1, \cdots, x_k, \mathscr{P}, \sigma, \varkappa)$ *has degree* β *if* $\deg_{\mathcal{Q}} = \beta$.

Definition 3.6 *Given a polarization* $\theta \in \widehat{G}$, *a lift* $\vartheta \in \widehat{\Gamma}$ *of* θ *and a prestable LG-quasimap* $\mathcal{Q} = (\mathscr{C}, x_1, \cdots, x_k, \mathscr{P}, \sigma, \varkappa)$, *we say that* $\mathcal{Q}$ *is* $0+$ *stable if*

(1) *every rational component has at least two special points* (*a mark* y_i *or a node*), *and*

(2) *on every irreducible component* $\mathscr{C}'$ *with trivial* $\omega_{\log,\mathscr{C}'}$, *the line bundle* $\sigma^*(\mathscr{L}_\vartheta)$ *has positive degree.*

It turns out that condition (2) holds for one lift if and only if it holds for all lifts, because for any two lifts ϑ and ϑ' of θ, the bundles $\sigma^*(\mathscr{L}_\vartheta)$ and $\sigma^*(\mathscr{L}_{\vartheta'})$ always differ by a power of $\omega_{\log,\mathscr{C}}$ (see [19, Proposition 4.2.14]). Moreover, over $\mathbb{Q}$ the "trivial" lift, defined by setting $\vartheta(gr) = \theta(g)$, is always a valid choice of lift.

Definition 3.7 *Given a polarization* $\theta \in \widehat{G}$ *and a good lift* ϑ *of* θ (*see Subsection* 2.4) *and a prestable LG-quasimap* $\mathcal{Q} = (\mathscr{C}, x_1, \cdots, x_k, \mathscr{P}, \sigma, \varkappa)$, *we say that* $\mathcal{Q}$ *is* (∞, ϑ)*-stable if*

(1) *there are no basepoints of* σ *on* $\mathscr{C}$, *and*

(2) *for every irreducible component* $\mathscr{C}'$ *of* $\mathscr{C}$, *the line bundle* $\sigma^*(\mathscr{L}_\vartheta)$ *restricted to* $\mathscr{C}'$ *has nonnegative degree, with the degree only being allowed to vanish on components where* $\omega_{\log}$ *is ample.*

Definition 3.8 *For a given choice of compatible* G*- and* $\mathbb{C}_R^*$*-actions on a closed affine variety* $Z \subseteq V$, *a strongly regular character* $\theta \in \widehat{G}$ *and a nondegenerate* W, *we denote the corresponding stack of k-pointed, genus-g,* $0+$ *stable LG-quasimaps into* $\mathscr{CR}_\theta$ *or* $\mathscr{X}_\theta$ *of degree* β *by*

$$\mathrm{LGQ}^{0+}_{g,k}(\mathscr{CR}_\theta, \beta) \quad \textit{or} \quad \mathrm{LGQ}^{0+}_{g,k}(\mathscr{X}_\theta, \beta),$$

respectively.

If ϑ *is a good lift of* θ, *then the corresponding stack of k-pointed, genus-g,* (∞, ϑ)*-stable LG-quasimaps into* $\mathscr{CR}_\theta$ *or* $\mathscr{X}_\theta$ *of degree* β *is denoted by*

$$\mathrm{LGQ}^{\infty,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta) \quad \textit{or} \quad \mathrm{LGQ}^{\infty,\vartheta}_{g,k}(\mathscr{X}_\theta, \beta),$$

respectively.

4 A Hypersurface in Weighted Projective Space

The remainder of this paper is dedicated to giving examples of the stack of stable LG-quasimaps to $\mathscr{CR}_\theta$ and $\mathscr{X}_\theta$ for various choices of input data. We begin with an example of a hypersurface in weighted projective space.

4.1 Basic setup

Suppose that $G = \mathbb{C}^*$ and $F \in \mathbb{C}[x_1, \cdots, x_K]$ is a quasihomogeneous polynomial of G-weights $(b_1, \cdots, b_K)$ and total G-degree b. Suppose further that F has an isolated singularity at the origin. Let

$$W = pF \colon \mathbb{C}^K \times \mathbb{C} \to \mathbb{C},$$

where the variables $x_1, \cdots, x_K$ are the first K coordinates and p is the last coordinate. We assign G-weight $-b$ to the variable p, so that W is G invariant.

The critical set of W is given by the equations:

$$\partial_p W = F = 0 \quad \text{and} \quad \partial_{x_i} W = p\partial_{x_i} F = 0.$$

If $p \neq 0$, then the fact that the only singularity of F is at the origin means that $(x_1, \cdots, x_K) = (0, \cdots, 0)$. If $p = 0$, then the only constraint is $F(x_1, \cdots, x_K) = 0$. So the critical locus is

$$\operatorname{Crit}(W) = \{(0, \cdots, 0, p) \mid p \in \mathbb{C}\} \cup \{(x_1, \cdots, x_K, 0) \in \mathbb{C}^K \times \mathbb{C} \mid \ F(x_1, \cdots, x_K) = 0\}.$$

Suppose that $b_i > 0$ for $i = 1, \cdots, K$ and $b > 0$. If $b = \sum_{i=1}^{K} b_i$, then we have a Calabi-Yau weight system, but we do not assume that here.

4.2 Phases for the hypersurface

Recall that different choices of linearization θ or moment map values τ give different quotients $[V/\!\!/_\theta G]$, but any two linearizations lying in the same chamber have isomorphic quotients, called phases.

In the hypersurface case, the affine moment map

$$\mu = \frac{1}{2}\Big(\sum_{i=1}^{K} b_i|x_i|^2 - b|p|^2\Big)$$

is a quadratic function whose only critical value is $\tau = 0$, and there are two phases $\tau > 0$ or $\tau < 0$.

Case of $\boldsymbol{\tau > 0}$ We have

$$\sum_{i=1}^{K} b_i|x_i|^2 = b|p|^2 + 2\tau.$$

For each choice of p, the set of $(x_1, \cdots, x_K) \in \mathbb{C}^K$, such that $(x_1, \cdots, x_K, p) \in \mu^{-1}(\tau)$, is a nontrivial ellipsoid E, isomorphic to S^{2K-1}; and we obtain a map from the symplectic quotient $\mathscr{X}_\tau^{\mathrm{sympl}}$ of V to $[E/U(1)] = W\mathbb{P}(b_1, \cdots, b_K)$.

The resulting symplectic quotient $\mathscr{X}_\tau^{\mathrm{sympl}}$ can be expressed as the total space of the line bundle $\mathscr{O}(-b)$ over $W\mathbb{P}(b_1, \cdots, b_K)$. If $\sum_i b_i = b$, this is the canonical bundle $\omega_{W\mathbb{P}(b_1,\cdots,b_K)}$.

Alternatively, we can consider the GIT quotient $[\mathbb{C}^{K+1}/\!\!/_\theta G]$, where $\theta : G \to \mathbb{C}^*$ has weight $-e$, with $e > 0$. One can easily see that the $\mathbf{L}_\theta$-semistable points are $((\mathbb{C}^K - \{\mathbf{0}\}) \times \mathbb{C}) \subset \mathbb{C}^K \times \mathbb{C} = \mathbb{C}^{K+1}$, and the first projection $\mathrm{pr}_1 : (\mathbb{C}^K - \{\mathbf{0}\}) \times \mathbb{C} \to (\mathbb{C}^K - \{\mathbf{0}\})$ induces the map $[V/\!\!/_\theta G] \to W\mathbb{P}(b_1, \cdots, b_K)$.

The critical locus $\mathscr{CR}_\theta = \{p = 0 = F(x_1, \cdots, x_K)\}$ is a degree-b hypersurface in the image of the zero section of $[V/\!\!/_\theta G] \cong \mathscr{O}(-d) \to W\mathbb{P}(b_1, \cdots, b_K)$. We call this phase the Calabi-Yau phase or geometric phase. In this setting, we write $\mathscr{CR}_{\mathrm{geom}}$ or $\mathscr{X}_{\mathrm{geom}}$ for $\mathscr{CR}_\theta$ or $\mathscr{X}_\theta$, respectively.

Case of $\boldsymbol{\tau < 0}$ We have

$$\mu^{-1}(\tau) = \Big\{(x_1, \cdots, x_K, p)\Big| \sum_{i=1}^{K} b_i|x_i|^2 - \tau = b|p|^2\Big\}.$$

For each choice of $x_1, \cdots, x_K \in \mathbb{C}^K$ the set of $p \in \mathbb{C}$ such that $(x_1, \cdots, x_K, p) \in \mu^{-1}(\tau)$ is the circle $S^1 \subset \mathbb{C}$, and we obtain a map $\mathscr{X}_\tau^{\mathrm{sympl}} \to [S^1/U(1)]$. If we choose the generator of $U(1)$ to be λ^{-1}, then p can be considered to have positive weight b. Moreover, every p has isotropy equal to the bth roots of unity (isomorphic to $\mathbb{Z}_b$). The quotient $[S^1/U(1)]$ is $W\mathbb{P}(b) = \mathrm{B}\mathbb{Z}_b = [\mathrm{pt}/\mathbb{Z}_b]$.

In the GIT formulation of this quotient, this corresponds to $\theta : G \to \mathbb{C}^*$ of weight $-e$, and $e < 0$, the $\mathbf{L}_\theta$-semistable points are equal to $(\mathbb{C}^K \times \mathbb{C}^*) \subset \mathbb{C}^{K+1}$. The second projection $\mathrm{pr}_2 : (\mathbb{C}^K \times \mathbb{C}^*) \to \mathbb{C}^*$ induces the map $[V/\!\!/_\theta G] \to \mathrm{B}\mathbb{Z}_b$.

The toric variety $\mathscr{X}_\theta = [V/\!\!/_\theta G]$ can be viewed as the total space of a rank-K orbifold vector bundle over $\mathrm{B}\mathbb{Z}_b$. This bundle is actually just a $\mathbb{Z}_b$ bundle, where $\mathbb{Z}_b$ acts by

$$(x_1, \cdots, x_K) \mapsto (\xi_b^{b_1} x_1, \cdots, \xi_b^{b_K} x_K), \quad \xi_b = \exp(2\pi \mathrm{i}/b).$$

If we choose the $\mathbb{C}_R^*$ action such that W has $\mathbb{C}_R^*$-weight b, then this is exactly the action of the element J in FJRW-theory. So the bundle $\mathscr{X}_\theta$ is isomorphic to $[\mathbb{C}^K/\langle J\rangle]$. This is a special phase which is sort of like a toric variety with a finite group instead of $\mathbb{C}^*$.

The critical locus is the single point $\{(0, \cdots, 0)\}$ in the quotient $\mathscr{X}_\tau = [\mathbb{C}^K/\mathbb{Z}_d]$. It is clearly compact, so the polynomial W is nondegenerate. We call $\mathscr{X}_\tau$ a Landau-Ginzburg phase or a pure Landau-Ginzburg phase (see [36]). This example underlies Witten's physical argument of the Landau-Ginzburg/Calabi-Yau correspondence for Calabi-Yau hypersurfaces of weighted projective spaces.

In this setting, we write $\mathscr{CR}_{\mathrm{LG}}$ or $\mathscr{X}_{\mathrm{LG}}$ for $\mathscr{CR}_\theta$ or $\mathscr{X}_\theta$, respectively.

4.3 GLSM moduli space for the hypersurface

4.3.1 Geometric phase

For the phase $\tau > 0$, we choose $\mathbb{C}_R^*$-weights $c_{x_i} = 0$ and $c_p = 1$ (thus giving a hybrid model), so W has $\mathbb{C}_R^*$-weight $d = 1$. The element J is trivial, and the group

$$\Gamma = \{(g^{b_1}, \cdots, g^{b_K}, g^{-b} r) \mid g \in G,\ r \in \mathbb{C}_R^*\}$$

is a direct product $\Gamma \cong G \times \mathbb{C}_R^*$, with ξ and ζ just the first and second projections, respectively.

There are two ingredients to the moduli space: An Artin stack of geometric data and a more subtle stability condition.

The geometric data correspond to sections of the vector bundle $\mathscr{P} \times_\Gamma V$. This bundle can be written as a direct sum of line bundles

$$\mathscr{E} = \mathscr{P} \times_\Gamma V \cong \mathscr{A}^{\otimes b_1} \oplus \mathscr{A}^{\otimes b_2} \oplus \cdots \oplus \mathscr{A}^{\otimes b_K} \oplus (\mathscr{A}^{\otimes(-b)} \otimes \mathscr{B}),$$

where $\mathscr{A}$ corresponds to the G-action (the map $\mathscr{C} \xrightarrow{\mathscr{P}} B\Gamma \xrightarrow{\xi} BG$) and $\mathscr{B}$ corresponds to the $\mathbb{C}_R^*$ action (the map $\mathscr{C} \xrightarrow{\mathscr{P}} B\Gamma \xrightarrow{\zeta} \mathbb{C}_R^*$). And we have an isomorphism $\varkappa : \zeta_*\mathscr{P} = \mathscr{B} \to \omega_{\log,\mathscr{C}}$. Therefore, the section σ corresponds to a sequence $\sigma = (s_1, \cdots, s_K, p)$, where s_i is a section of $\mathscr{A}^{b_i}$ and p is a section of $\mathscr{A}^{-b} \otimes \omega_{\log,\mathscr{C}}$.

So LG-quasimaps to the geometric phase $\mathscr{X}_{\mathrm{geom}}$ correspond to the following data:

$$\{(\mathscr{C}, \mathscr{A}, s_1, \cdots, s_K, p) \mid s_i \in H^0(\mathscr{C}, \mathscr{A}^{b_i}),\ p \in H^0(\mathscr{C}, \mathscr{A}^{-b} \otimes \omega_{\log,\mathscr{C}})\}. \tag{4.1}$$

For all stability conditions, the LG-quasimaps can fail in at most a finite number of points to map to the θ-semistable locus. This corresponds to the locus where $s_1, \cdots, s_K$ do not simultaneously vanish. Moreover, $\mathrm{Crit}(W)^{ss}_{\mathrm{geom}}$ has $p = 0$. But if $p = 0$ at all but a finite number of points, then p is always 0.

Thus, without further specifying stability conditions, LG-quasimaps to the geometric phase $\mathscr{CR}_{\mathrm{geom}}$ correspond to the following data

$$\{(\mathscr{C}, \mathscr{A}, s_1, \cdots, s_K) \mid s_i \in H^0(\mathscr{C}, \mathscr{A}^{b_i})\}, \tag{4.2}$$

where $\mathscr{C}$ is a marked orbicurve, $\mathscr{A}$ is a line bundle over $\mathscr{C}$, and the section $\sigma = (s_1, \cdots, s_K, 0)$ maps to $\mathscr{P} \times_\Gamma \mathrm{Crit}(W)$.

Now we consider the stability conditions.

Case of $\boldsymbol{\varepsilon = \infty}$ We must find a good lift of θ. Let ℓ be a generator of $\mathbf{L}^*_\theta$ over $\mathbb{C}[V^*]$ with G acting on ℓ with weight $-e$ (and $e > 0$ in the geometric phase). The trivial lift ϑ_0 of θ corresponds to $\mathbb{C}^*_R$ acting trivially on ℓ. A monomial of the form $x_i^e \ell^{b_i}$ is Γ-invariant and does not vanish on points with $x_i \neq 0$, so every point of $\mathbb{C}^N \times \mathbb{C}$ with $x_i \neq 0$ is in $V^{ss}_\Gamma(\vartheta_0)$. Letting i range from 1 to N shows that $V^{ss}_\Gamma(\vartheta_0) = V^{ss}_G(\theta)$. Thus ϑ_0 is a good lift of the character θ.

Any other lift ϑ must have nontrivial $\mathbb{C}^*_R$ action on ℓ and thus any $\mathbb{C}^*_R$-invariant element of $\mathbb{C}[x_1, \cdots, x_K, p][\ell]$ must have each monomial containing a power of p, and hence points with $p = 0$ will not be ϑ-semistable. Therefore, ϑ_0 is the only good lift of θ.

Finally, $\sigma^* \mathscr{L}_{\vartheta_0}$ is determined by the action of Γ on $\mathbf{L}_\theta$ (or the inverse of the action of ϑ_0 on ℓ), so in this case $\sigma^* \mathscr{L}_{\vartheta_0} = \mathscr{A}^e$. Thus, the (∞, ϑ_0)-stable LG-quasimaps to $\mathscr{CR}_{\mathrm{geom}}$ or to $\mathscr{X}_{\mathrm{geom}}$ consist of those data (4.2) or (4.1), respectively, satisfying the conditions that

(1) the section σ has no basepoints (the s_i never vanish simultaneously);

(2) the line bundle $\mathscr{A}$ has positive degree on every component $\mathscr{C}'$ of $\mathscr{C}$ where $\omega_{\log,\mathscr{C}}$ has nonpositive degree.

Thus (∞, ϑ_0)-stable LG-quasimaps to $\mathscr{CR}_{\mathrm{geom}}$ correspond to stable maps to the hypersurface $X_F = \{F = 0\} \subset W\mathbb{P}(b_1, \cdots, b_K)\}$. And (∞, ϑ_0)-stable LG-quasimaps to $\mathscr{X}_{\mathrm{geom}}$ are stable maps to X_F with p-fields, studied in [3–4].

Case of $\boldsymbol{\varepsilon = 0+}$ The $0+$ stable LG-quasimaps must have the section σ take its values in $\mathrm{Crit}(W)^{ss}_{\mathrm{geom}}$ for all but a finite number of basepoints $y \in \mathscr{C}$.

Letting ϑ be the trivial lift of θ, we have $\sigma^* \mathscr{L}_\vartheta = \mathscr{A}^e$ and so the $0+$ stable LG quasimaps to $\mathscr{CR}_{\mathrm{geom}}$ or to $\mathscr{X}_{\mathrm{geom}}$ are the data of (4.2) or (4.1), respectively, satisfying the stability conditions:

(1) There are at most a finite number of basepoints (where the s_i vanish simultaneously), and these only occur away from nodes and marked points.

(2) Every rational component has at least two special points (node or marked point).

(3) On rational components with exactly two special points, the line bundle $\mathscr{A}$ has positive degree.

Thus $0+$ stable LG-quasimaps into $\mathscr{CR}_{\mathrm{geom}}$ are stable quotients into $X_F \subset W\mathbb{P}(b_1, \cdots, b_K)$, and $0+$ stable LG-quasimaps into $\mathscr{X}_{\mathrm{geom}}$ are stable quotients into X_F with p-fields.

Remark 4.1 There is a parallel theory of quasimaps into X_F. Both theories have the same moduli spaces, but the virtual cycle constructions are different. For $\varepsilon = \infty$, Chang-Li [3] proved the equivalence using a sophisticated degeneration argument. A similar argument probably works for other ε-theories.

4.3.2 LG phase

For the phase $\tau < 0$, we choose $\mathbb{C}_R^*$ to have weights $c_{x_i} = b_i$ and $c_p = 0$, which again gives a hybrid model. But now W has $\mathbb{C}_R^*$-weight $d = b$, and $J = (\xi^{b_1}, \cdots, \xi^{b_K}, 1)$, where $\xi = \exp(2\pi\mathrm{i}/d)$. We have

$$\Gamma = \{((gr)^{b_1}, \cdots, (gr)^{b_K}, g^{-b}) \mid g \in G,\ r \in \mathbb{C}_R^*\} \tag{4.3}$$
$$= \{(\alpha^{b_1}, \cdots, \alpha^{b_K}, \beta) \mid \alpha, \beta \in \mathbb{C}^*\}, \tag{4.4}$$

where $\alpha = gr$ and $\beta = g^{-b}$, with $\zeta : \Gamma \to \mathbb{C}^*$ given by $(\alpha^{b_1}, \cdots, \alpha^{b_K}, \beta) \mapsto \alpha^b\beta$.

Thus, the vector bundle $\mathscr{E} = \mathscr{P} \times_\Gamma V$ associated to an LG-quasimap is a direct sum of line bundles on $\mathscr{C}$:

$$\mathscr{E} = \mathscr{A}^{b_1} \oplus \cdots \oplus \mathscr{A}^{b_K} \oplus \mathscr{B},$$

where $\mathscr{A}$ corresponds to α and $\mathscr{B}$ corresponds to β in the presentation (4.4) of Γ. Moreover, we have $\varkappa : \mathscr{A}^b \otimes \mathscr{B} \to \omega_{\log,\mathscr{C}}$ is an isomorphism.

Thus, LG-quasimaps to $\mathscr{X}_{\mathrm{LG}}$ again consist of exactly the same data as (4.1):

$$\{(\mathscr{C}, \mathscr{A}, s_1, \cdots, s_K, p) \mid s_i \in H^0(\mathscr{C}, \mathscr{A}^{b_i}),\ p \in H^0(\mathscr{A}^{-b} \otimes \omega_{\log,\mathscr{C}})\}.$$

The base points of these quasimaps occur precisely at the zeros of p, and the base locus forms an effective divisor D in $\mathscr{C}$ with $\mathscr{A}^{-b} \otimes \omega_{\log,\mathscr{C}} \cong O(D)$, so the section p gives an isomorphism $\mathscr{A}^b \cong \omega_{\log,\mathscr{C}}(-D)$ and can be viewed as a weighted b-spin condition (see [29]). So we can reformulate the moduli problems as

$$\{(\mathscr{C}, \mathscr{A}, D, s_1, \cdots, s_K) \mid s_i \in H^0(\mathscr{C}, \mathscr{A}^{b_i}),\ \mathscr{A}^b \cong \omega_{\log,\mathscr{C}}(-D)\}, \tag{4.5}$$

where each D is an effective divisor that is disjoint from the nodes and marked points of $\mathscr{C}$.

For the LG-quasimaps to lie in the critical locus requires every $s_i = 0$, so LG-quasimaps to $\mathscr{CR}_{\mathrm{LG}}$ consist of

$$\{(\mathscr{C}, \mathscr{A}, D) \mid \mathscr{A}^b \cong \omega_{\log,\mathscr{C}}(-D)\}. \tag{4.6}$$

Case of $\varepsilon = \infty$ Again, the trivial lift is the only good lift of θ. To see this, let ℓ be a generator of $\mathbf{L}_\theta^*$ over $\mathbb{C}[V^*]$ with G acting on ℓ with weight $-e$ (and $e < 0$ in the LG phase). The trivial lift ϑ_0 corresponds to $\mathbb{C}_R^*$ acting trivially on ℓ, and a monomial of the form $p^{-e}\ell^b$ is Γ-invariant and does not vanish on points with $p \neq 0$, so $V_\Gamma^{ss}(\vartheta_0) = V_G^{ss}(\theta)$. Thus ϑ_0 is a good lift of the character θ. Any other lift ϑ must have nontrivial $\mathbb{C}_R^*$-action on ℓ, and hence any Γ-invariant function must have at least one factor of x_i in every monomial, which implies that any point of V with $x_1 = x_2 = \cdots = x_k = 0$ is not ϑ-semistable. Thus ϑ_0 is the only good lift.

The line bundle $\sigma^*\mathscr{L}_{\vartheta_0}$ is determined by the action of Γ on $\mathbf{L}_\theta$, that is by g^e in the presentation (4.3), which implies that

$$\sigma^*\mathscr{L}_{\vartheta_0} \cong \mathscr{B}^{-e/b} \cong \omega_{\log,\mathscr{C}}^{-e/b} \otimes \mathscr{A}^e.$$

Since $\varepsilon = \infty$, no base points are permitted, so $D = 0$ and $\mathscr{A}^b \cong \omega_{\log,\mathscr{C}}$. For convenience, let us assume that $e = -cb$ for some $c > 0$. The stability condition is now that

$$\omega_{\log,\mathscr{C}}^c \otimes \mathscr{A}^{-cb} \cong \omega_{\log,\mathscr{C}}^c \otimes \omega_{\log,\mathscr{C}}^{-c} = \mathscr{O}$$

can only have degree 0 on components where $\omega_{\log,\mathscr{C}}$ is ample, thus $\mathscr{C}$ must be a stable orbicurve.

So in this case (∞, ϑ_0)-stable LG-quasimaps to $\mathscr{CR}_{\mathrm{LG}}$ correspond to stable b-spin curves

$$\{\mathscr{C}, \mathscr{A} \mid \mathscr{A}^b \cong \omega_{\log,\mathscr{C}}\},$$

studied in [1, 23].

Case of $\varepsilon = 0+$ In this case basepoints are permitted, so D is not necessarily 0. The bundle

$$\sigma^* \mathscr{L}_{\vartheta_0} \cong \omega^c_{\log,\mathscr{C}} \otimes \mathscr{A}^{-bc} \cong \mathscr{O}(cD)$$

must have positive degree on any component where $\omega_{\log,\mathscr{C}}$ is not ample.

Thus 0+ stable LG quasimaps to $\mathscr{CR}_{\mathrm{LG}}$ correspond to the data of (4.6) satisfying the conditions:

(1) Every rational component has at least two special points, and

(2) On every irreducible component $\mathscr{C}'$ with trivial $\omega_{\log,\mathscr{C}'}$, there is at least one basepoint.

5 Complete Intersection in Weighted Projective Space

5.1 Basic setup

Suppose that $G = \mathbb{C}^*$ and we have several quasihomogeneous polynomials $F_1, F_2, \cdots, F_M \in \mathbb{C}[x_1, \cdots, x_K]$ of G-degree $(d_1, \cdots, d_M)$, where each variable x_i has G-weight $b_i > 0$. We assume that the F_j intersect transversely in $W\mathbb{P}(b_1, \cdots, b_K)$ and define a complete intersection. Let

$$W = \sum_i p_i F_i \colon \mathbb{C}^{K+M} \to \mathbb{C},$$

where we assign G-weight $-d_i$ to p_i. In the special case that $\sum\limits_i b_i = \sum\limits_j d_j$. Then the complete intersection defined by $F_1 = \cdots = F_M = 0$ is a Calabi-Yau orbifold in $W\mathbb{P}(b_1, \cdots, b_K)$. But we do not assume the Calabi-Yau condition here.

The critical set of W is defined by the following equations:

$$\partial_{p_j} W = F_j = 0, \quad \partial_{x_i} W = \sum_j p_j \partial_{x_i} F_j = 0. \tag{5.1}$$

Since the F_j intersect transversely, an argument similar to that for the hypersurface shows that the critical locus consists of those $(\mathbf{x}, \mathbf{p})$ where either $\mathbf{p} = \mathbf{0}$ and $\mathbf{x}$ satisfies $F_i(\mathbf{x}) = 0$ for all i, or $\mathbf{p}$ is unconstrained and $\mathbf{x} = \mathbf{0}$:

$$\mathrm{Crit}(W) = \{(\mathbf{0}, \mathbf{p}) \in \mathbb{C}^K \times \mathbb{C}^M \mid \mathbf{p} \in \mathbb{C}^M\} \cup \{(\mathbf{x}, \mathbf{0}) \in \mathbb{C}^K \times \mathbb{C}^M \mid F_i(\mathbf{x}) = 0,\ \forall i\}.$$

5.2 Phases for a complete intersection

The moment map is

$$\mu = \sum_i \frac{1}{2} b_i |x_i|^2 - \frac{1}{2} \sum_j d_j |p_j|^2.$$

Again, there are two phases, $\tau > 0$ and $\tau < 0$.

Case of $\boldsymbol{\tau} > 0$ When $\tau > 0$, we again call this the geometric phase. Any choice of $\mathbf{p} = (p_1, \cdots, p_M)$ determines a nontrivial ellipsoid $E \subset \mathbb{C}^K$ of points $\mathbf{x} = (x_1, \cdots, x_K)$ such that $(\mathbf{x},\mathbf{p})$ lies in $\mu^{-1}(\tau)$. Quotienting by $U(1)$, the first projection $\mathrm{pr}_1 : E\times\mathbb{C}^M \to E$ induces a map $\mathscr{X}_{\mathrm{geom}} \to W\mathbb{P}(b_1,\cdots,b_K)$. The full quotient is $\mathscr{X}_{\mathrm{geom}} = \bigoplus_j \mathscr{O}(-d_j)$ over $W\mathbb{P}(b_1,\cdots,b_K)$.

In the GIT formulation, this again corresponds to $\theta : G \to \mathbb{C}^*$ having weight $-e$, with $e > 0$. The semistable points of this phase are those with $\mathbf{x} \neq \mathbf{0}$, and the semistable points of the critical locus correspond to points in

$$\{F_1 = \cdots = F_M = 0\}.$$

The quotient $\mathscr{CR}_{\mathrm{geom}}$ is the locus of the complete intersection defined by all the F_i vanishing in the zero section of $\mathscr{X}_{\mathrm{geom}} \to W\mathbb{P}(b_1,\cdots,b_K)$.

Case of $\boldsymbol{\tau < 0}$ When $\tau > 0$, we again call this the Landau-Ginzburg phase. The quotient is $\mathscr{X}_{\mathrm{LG}} = \bigoplus_i \mathscr{O}(-b_i)$ over $W\mathbb{P}(d_1,\cdots,d_M)$.

In the GIT formulation, this corresponds to $\theta : G \to \mathbb{C}^*$ having weight $-e$, with $e < 0$. The semistable points of this phase are those with $\mathbf{p} \neq \mathbf{0}$, and the semistable points of the critical locus correspond to the image of the zero section of $\mathscr{X}_{\mathrm{LG}} \to W\mathbb{P}(d_1,\cdots,d_M)$.

5.3 GLSM moduli space for a complete intersection

5.3.1 Geometric phase

We choose the R-charge to act on $\mathbb{C}^K \times \mathbb{C}^M$ with weights $(0,\cdots,0,1,\cdots,1)$, which gives a hybrid model. And W has $\mathbb{C}^*_R$-weight $d = 1$. The element J is trivial, and the group

$$\Gamma = \{(g^{b_1},\cdots,g^{b_K},g^{-d_1}r,\cdots g^{-d_M}r) \mid g \in G,\ r \in \mathbb{C}^*_R\}$$

is again a direct product $\Gamma \cong G \times \mathbb{C}^*_R$, with ξ and ζ just the first and second projections, respectively.

The geometric data correspond to sections of the vector bundle $\mathscr{E} = \mathscr{P} \times_\Gamma V$, which can be written as a direct sum of line bundles

$$\mathscr{E} = \mathscr{A}^{b_1} \oplus \mathscr{A}^{b_2} \oplus \cdots \oplus \mathscr{A}^{b_K} \oplus (\mathscr{A}^{-d_1} \otimes \mathscr{B}) \oplus \cdots \oplus (\mathscr{A}^{-d_M} \otimes \mathscr{B}),$$

where $\mathscr{A}$ corresponds to the G-action and $\mathscr{B}$ corresponds to the $\mathbb{C}^*_R$ action. And we have an isomorphism $\varkappa : \zeta_*\mathscr{P} = \mathscr{B} \to \omega_{\log,\mathscr{C}}$.

So LG-quasimaps to the geometric phase $\mathscr{X}_{\mathrm{geom}}$ correspond to the data:

$$\{(\mathscr{C},\mathscr{A},s_1,\cdots,s_K,p_1,\cdots,p_M) \mid s_i \in H^0(\mathscr{C},\mathscr{A}^{b_i}),\ p_i \in H^0(\mathscr{C},\mathscr{A}^{-d_i} \otimes \omega_{\log,\mathscr{C}})\}. \tag{5.2}$$

Again $\mathrm{Crit}(W)^{ss}_{\mathrm{geom}}$ has $\mathbf{p} = 0$, so without further specifying stability conditions, LG-quasimaps to the geometric phase $\mathscr{CR}_{\mathrm{geom}}$ correspond to the following data:

$$\{(\mathscr{C},\mathscr{A},s_1,\cdots,s_K) \mid s_i \in H^0(\mathscr{C},\mathscr{A}^{b_i})\}, \tag{5.3}$$

where $\mathscr{C}$ is a marked orbicurve, $\mathscr{A}$ is a line bundle over $\mathscr{C}$, and the section $\sigma = (s_1,\cdots,s_K,\mathbf{0})$ maps to $\mathscr{P} \times_\Gamma \mathrm{Crit}(W)$.

Case of $\boldsymbol{\varepsilon = \infty}$ Again ϑ_0 is easily seen to be a good lift, and $\sigma^*\mathscr{L}_{\vartheta_0} = \mathscr{A}^e$. Thus, just as in the hypersurface case, we have that (∞,ϑ_0)-stable LG-quasimaps to $\mathscr{CR}_{\mathrm{geom}}$ correspond to stable maps to the complete intersection $X = \{F_1 = \cdots = F_M = 0\} \subset W\mathbb{P}(b_1, \cdots, b_K)\}$. And (∞, ϑ_0)-stable LG-quasimaps to $\mathscr{X}_{\mathrm{geom}}$ are stable maps to X with p-fields.

Case of $\varepsilon = 0+$ The arguments given in the hypersurface case are easily adapted to show that 0+ stable LG-quasimaps into $\mathscr{CR}_{\mathrm{geom}}$ are stable quotients into $X \subset W\mathbb{P}(b_1, \cdots, b_K)$, and 0+ stable LG-quasimaps into $\mathscr{X}_{\mathrm{geom}}$ are stable quotients into X_F with p-fields.

5.3.2 LG phase

Assume that $d_1 = \cdots = d_r = d$ and choose the R-charge weights $c_{x_i} = b_i$ and $c_{p_j} = 0$. Now W has $\mathbb{C}^*_R$-weight d, and

$$\Gamma = \{(gr)^{b_1}, \cdots, (gr)^{b_K}, g^{-d}, \cdots, g^{-d} \mid g \in G,\ r \in \mathbb{C}^*_R\} \tag{5.4}$$

$$= \{\alpha^{b_1}, \cdots, \alpha^{b_K}, \beta, \cdots, \beta) \mid \alpha, \beta \in \mathbb{C}^*\}, \tag{5.5}$$

where $\alpha = gr$ and $\beta = g^{-d}$, and the map ζ sends (α, β) to $\alpha^d\beta$.

Thus, $\mathscr{E} = \mathscr{P} \times_\Gamma V$ is a direct sum of line bundles on $\mathscr{C}$:

$$\mathscr{E} = \mathscr{A}^{b_1} \oplus \cdots \oplus \mathscr{A}^{b_K} \oplus \mathscr{B} \oplus \cdots \oplus \mathscr{B},$$

where $\mathscr{A}$ corresponds to α and $\mathscr{B}$ corresponds to β in the presentation (5.5) of Γ. Moreover, $\varkappa : \mathscr{A}^d \otimes \mathscr{B} \to \omega_{\log,\mathscr{C}}$ is an isomorphism.

Thus, LG-quasimaps to $\mathscr{X}_{\mathrm{LG}}$ consist of the data:

$$\{(\mathscr{C}, \mathscr{A}, s_1, \cdots, s_K, p_1, \cdots, p_M) \mid s_i \in H^0(\mathscr{C}, \mathscr{A}^{b_i}),\ p_i \in H^0(\mathscr{A}^{-d} \otimes \omega_{\log,\mathscr{C}})\}.$$

And LQ-quasimaps to $\mathscr{CR}_{\mathrm{LG}}$ also require that all the $s_1, \cdots, s_K$ vanish, giving

$$\{(\mathscr{C}, \mathscr{A}, p_1, \cdots, p_M) \mid p_i \in H^0(\mathscr{A}^{-d} \otimes \omega_{\log,\mathscr{C}})\}.$$

Case of $\varepsilon = \infty$ Again, the trivial lift is the only good lift of θ. The line bundle $\sigma^*\mathscr{L}_{\vartheta_0}$ is again

$$\sigma^*\mathscr{L}_{\vartheta_0} \cong \mathscr{B}^{-e/d} \cong \omega_{\log,\mathscr{C}}^{-e/d} \otimes \mathscr{A}^e.$$

And so the stability condition is that

$$\omega_{\log,\mathscr{C}} \otimes \mathscr{A}^{-d}$$

can only have degree 0 on components where $\omega_{\log,\mathscr{C}}$ is ample. And since $\varepsilon = \infty$, no base points are permitted, so the p_i cannot all simultaneously vanish.

So in this case (∞, ϑ_0)-stable LG-quasimaps to $\mathscr{CR}_{\mathrm{LG}}$ correspond to stable maps to $W\mathbb{P}(d, d, \cdots, d)$. And for each $(p_1, \cdots, p_n) \in W\mathbb{P}(d, \cdots, d)$, we have a pure LG-model of superpotential $\sum_i p_i F_i$. One can view this as a family of pure LG-theories.

Remark 5.1 An LG-phase of a complete intersection of differing degrees (i.e., $d_i \neq d_j$ for some $1 \leq i, j \leq r$) does not admit a hybrid model structure and will generally have no good lift.

Case of $\varepsilon = 0+$ Now basepoints are permitted, and every rational component must have at least two special points. Again, the stability condition is that

$$\omega_{\log,\mathscr{C}} \otimes \mathscr{A}^{-d}$$

can only have degree 0 on components where $\omega_{\log,\mathscr{C}}$ is ample.

Remark 5.2 As mentioned in Remark 5.1, without the condition $d_1 = d_2 = \cdots = d_r = d$ there is usually no good lift of θ. But in the $\varepsilon = 0+$ case, we do not need a good lift, so fixing any $d > 0$ we can take $c_{p_j} = d - d_j$, which again gives W the $\mathbb{C}^*_R$-weight of d and a corresponding proper DM stack of $0+$ stable LG-quasimaps.

6 Graph Moduli Space

The graph moduli space is very important in Gromov-Witten theory. We can construct it in the GLSM setting as follows.

Suppose that we have a phase θ, a superpotential $W : [\mathbb{C}^n/G] \to \mathbb{C}$ with a certain R-charge $\mathbb{C}^*_R$, defining Γ and a lift ϑ of θ. We construct a new GLSM as follows.

Let $V' = V \times \mathbb{C}^2$, and let $\mathbb{C}^*$ act on $\mathbb{C}^2$ with weights $(1, 1)$. Let $G' = G \times \mathbb{C}^*$ act on V' with the product action, so G acts trivially on the last two coordinates and $\mathbb{C}^*$ acts trivially on the first n coordinates.

Let $\theta' : G' \to \mathbb{C}^*$ be given by sending any $(g, h) \in G \times \mathbb{C}^*$ to $\theta(g)h^{-k}$ for some $k > 0$. The GIT quotient is the product $[V' /\!\!/_{\theta'} G'] = [V /\!\!/_\theta G] \times \mathbb{P}^1$. Let W' be defined on V' by the same polynomial as W, so that the critical locus of W' in V' is $\mathbb{C}^2$ times the critical locus of W, and the GIT quotient of the critical locus is the product of $\mathbb{P}^1$ and the corresponding quotient in the original GLSM.

Keeping the same R-charge (that is, letting $\mathbb{C}^*_R$ acts trivially on the last two coordinates of V'), we have $\Gamma' = \Gamma \times \mathbb{C}^*$, and we construct a lift ϑ' of θ' by sending $(\gamma, h) \in \Gamma \times \mathbb{C}^*$ to $\vartheta(\gamma)h^{-k}$. It is easy to see that ϑ' is a good lift of θ' if ϑ is a good lift of θ.

In the $\varepsilon = \infty$ case, no basepoints can occur, and projecting to the two new coordinates (z_0, z_1) induces a stable map $\mathscr{C} \to \mathbb{P}^1$. Therefore, the new GLSM in this case can be reformulated as the usual GLSM for $[V /\!\!/_\theta G]$ with the additional data of a stable map $f : \mathscr{C} \to \mathbb{P}^1$.

7 Generalized Graph Space

We can generalize slightly the graph moduli space to obtain a new moduli space with a remarkable property. Let us take the quintic GLSM as an example. Now, we consider a new GLSM on $\mathbb{C}^{6+2} /\!\!/ (\mathbb{C}^*)^2$, given by $G = (\mathbb{C}^*)^2$ acting on $V = \mathbb{C}^8$ with weights

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & -5 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

for an integer $d > 0$. Let the coordinates on V be $x_0, \cdots, x_4, p, z_0, z_1$, corresponding the columns in the weight matrix above. The moment maps are

$$\mu_1 = \frac{1}{2}\Big(\sum_{i=0}^{4} |x_i|^2 - 5|p|^2 + d|z_0|^2\Big), \quad \mu_2 = \frac{1}{2}(|z_0|^2 + |z_1|^2).$$

There are three chambers. We are interested primarily in the chamber $0 < \mu_1 < d\mu_2$. This corresponds to a character θ of G with weights $(-e_1, -e_2)$ and $0 < e_1 < de_2$. The θ-unstable locus for θ is

$$\{x_0 = x_1 = x_2 = x_3 = x_4 = z_0 = 0\} \cup \{p = z_1 = 0\} \cup \{z_0 = z_1 = 0\}.$$

Taking the superpotential $W = \sum_{i=1}^{5} px_i^5$ and the R-charge of weight $(0,0,0,0,0,1,0,0)$, we have $\Gamma = G \times \mathbb{C}_R^* = \{(a,a,a,a,a,ra^{-5},ba^d,b) \mid a,b,r \in \mathbb{C}^*\}$, and the map ζ takes $(a,a,a,a,a,ra^{-5},ba^d,b)$ to r.

There is no good lift of θ, so we restrict to the case of $\varepsilon = 0+$. We must choose some lift for the stability condition, so we take the trivial lift $\vartheta(a,b,r) = a^{-e_1}b^{-e_2}$. Any other lift will give the same stability conditions.

The resulting moduli problem consists of

$$\{(\mathscr{C},\mathscr{A},\mathscr{B},x_1,\cdots,x_5,p,z_1,z_2) \,| x_i \in H^0(\mathscr{C},\mathscr{A}),\ p \in H^0(\mathscr{C},\mathscr{A}^{-5}\otimes\omega_{\log,\mathscr{C}}),\\ z_1 \in H^0(\mathscr{C},\mathscr{A}^d\otimes\mathscr{B}),\ z_2 \in H^0(\mathscr{C},\mathscr{B})\},$$

satisfying the stability condition that $\sigma^*\mathscr{L}_\vartheta = \mathscr{A}^{-e_1}\mathscr{B}^{-e_2}$ is ample on all components where $\omega_{\log,\mathscr{C}}$ has degree 0.

The critical locus is

$$\mathrm{Crit}(W) = \{x_0 = \cdots = x_4 = 0\} \cup \Big\{p = \sum_{i=0}^{4} x_i^5 = 0\Big\}.$$

And so, in the chamber we are interested in, the GIT quotient of the critical locus has two components $\mathscr{CR}_\theta = C_1 \cup C_2$, where C_1 corresponds to $x_0 = \cdots = x_4 = 0$, so $z_0 \neq 0$ can be scaled to 1 by the second $\mathbb{C}^*$ action and so the quotient C_1 is isomorphic to $\mathbb{P}(5,1)$, with coordinates p, z_1. The component C_2 corresponds to the locus $\big\{p = \sum_{i=0}^{4} x_i^5 = 0\big\}$.

The critical locus admits a $\mathbb{C}^*$ action by multiplication on z_1. The fixed loci of the action are

(1) the locus $\{x_0 = \cdots = x_4 = z_1 = 0\}$, which is the point $B\mu_5$ inside $\mathbb{P}(5,d)$.

(2) the locus $\big\{p = z_0 = \sum_{i=0}^{4} x_i^5 = 0\big\}$, which is a Calabi-Yau threefold in $\mathbb{P}^4$.

(3) the locus $\{x_0 = \cdots = x_4 = p = 0\}$, which is the point $B\mu_d$ inside $\mathbb{P}(5,d)$.

The GLSMs for the three fixed loci correspond, respectively to (1) a weighted FJRW theory, (2) 0+ stable quasimaps to the quintic threefold, and (3) the theory of Hassett stable curves with light points given by the vanishing of z_0 and z_1. This remarkable property gives us the hope that we can extract a relation between Gromov-Witten theory and FJRW-theory geometrically by using localization techniques on this moduli space. A program is being carried out right now for the $\varepsilon = 0^+$ theory (see [14]).

The same theory with a different $\varepsilon = \infty$ stability condition was discovered and the localization argument was carried out independently by Chang-Li-Li-Liu (see [5]).

8 Non-Abelian Examples

The subject of gauged linear sigma models for non-Abelian groups is a very active area of research in physics and is far from complete. Here, we discuss complete intersections in a Grassmannian or flag variety.

All of this should work similarly in the setting of complete intersections of quiver varieties, although the details have not been worked out. It would be very interesting to explore mirror symmetry among Calabi-Yau complete intersections in quiver varieties.

8.1 Complete intersections in a Grassmanian

The space $\mathrm{Gr}(k,n)$ can be constructed as a GIT quotient $M_{k,n}/\!\!/\mathrm{GL}(k,\mathbb{C})$, where $M_{k,n}$ is the space of $k\times n$ matrices and $\mathrm{GL}(k,\mathbb{C})$ acts as matrix multiplication on the left.

The Grassmannian $\mathrm{Gr}(k,n)$ can also be embedded into $\mathbb{P}^K$ for $K=\frac{n!}{k!(n-k)!}-1$ by the Plücker embedding

$$A\mapsto(\cdots,\det(A_{i_1,\cdots,i_k}),\cdots),$$

where $A_{i_1,\cdots,i_k}$ is the $(k\times k)$-submatrix of A consisting of the columns $i_1,\cdots,i_k$.

The group $G=\mathrm{GL}(k,\mathbb{C})$ acts on the Plücker coordinates $B_{i_1,\cdots,i_k}(A)=\det(A_{i_1,\cdots,i_k})$ by the determinant, that is, for any $U\in G$ and $A\in M_{k,n}$, we have

$$B_{i_1,\cdots,i_k}(UA)=\det(U)B_{i_1,\cdots,i_k}(A).$$

Let $F_1,\cdots,F_s\in\mathbb{C}[B_{1,\cdots,k},\cdots,B_{n-k+1,\cdots,n}]$ be degree-d_j homogeneous polynomials such that the zero loci $Z_{F_j}=\{F_j=0\}$ and the Plücker embedding of $\mathrm{Gr}(k,n)$ all intersect transversely in $\mathbb{P}^K$. We let

$$Z_{d_1,\cdots,d_s}=\mathrm{Gr}(k,n)\cap\bigcap_j Z_{F_j}$$

denote the corresponding complete intersection.

The analysis of $Z_{d_1,\cdots,d_s}$ is similar to the Abelian case. Namely, let

$$W=\sum_j p_jF_j\colon M_{k,n}\times\mathbb{C}^s\to\mathbb{C}$$

be the superpotential. We assign an action of $G=\mathrm{GL}(k,\mathbb{C})$ on p_j by $p_j\to\det(U)^{-d_j}$.

The phase structure is similar to that of a complete intersection in projective space. The moment map is given by $\mu(A,p_1,\cdots,p_s)=\frac{1}{2}\big(A\overline{A}^{\mathrm{T}}-\sum\limits_{i=1}^{s}d_i|p_i|^2\big)$. Alternatively, to construct a linearization for GIT, the only characters of $\mathrm{GL}(k,\mathbb{C})$ are powers of the determinant, so $\theta(U)=\det(U)^{-e}$ for some e, and τ will be positive precisely when e is positive.

Let ℓ be a generator of $\mathbb{C}[\mathbf{L}_\theta^*]$ over $\mathbb{C}[V^*]$. Any element of $H^0(V,\mathbf{L}_\theta)$ can be written as a sum of monomials in the Plücker coordinates $B_{i_1,\cdots,i_k}$ and the p_j times ℓ. Any $U\in G$ will act on a monomial of the form $\prod B_{i_1,\cdots,i_k}^{b_{i_1,\cdots,i_k}}\prod p_j^{a_j}\ell^m$ by multiplication by $\det(U)^{\sum b_{i_1,\cdots,i_k}-\sum d_ja_j-me}$.

8.1.1 Geometric phase

Assume that $e>0$. In order to be G-invariant, a monomial must have $\sum b_{i_1,\cdots,i_k}>0$, which implies that any points with every $B_{i_1,\cdots,i_k}=0$ must be unstable, but for each $m>0$ and each k-tuple $(i_1,\cdots,i_k)$ the monomial $B_{i_1,\cdots,i_k}^{me}\ell^m$ is G invariant, so every point with at least one nonzero $B_{i_1,\cdots,i_k}$ must be θ-semistable. Thus $[V/\!\!/_\theta G]$ is isomorphic to the bundle $\bigoplus\limits_j\mathscr{O}(-d_j)$ over $\mathrm{Gr}(k,n)$.

As in the toric case, the critical locus in this phase is given by $p_1=\cdots=p_s=0=F_1=\cdots=F_s$, so we recover the complete intersection $F_1=\cdots=F_s$ in $\mathrm{Gr}(k,n)$, and we call this phase the geometric phase.

Just as for the toric complete intersection, we choose the C_R^*-action to have weight 0 on the space $M_{n,k}$ and weight 1 on all of the p_j, so that W has $\mathbb{C}_R^*$ weight 1 and $\Gamma\cong\mathrm{GL}(k)\times\mathbb{C}^*$.

The trivial lift ϑ_0 is a good lift because each monomial of the form $B^e_{i_1,\cdots,i_k}\ell$ is Γ invariant for the action induced by ϑ_0.

The prestable moduli problem of LG-quasimaps to the critical locus $\mathscr{CR}_{\mathrm{geom}}$ consists of maps from prestable orbicurves to the complete intersection

$$\{(\mathscr{C}, f : \mathscr{C} \to Z_{d_1,\cdots,d_s})\}. \tag{8.1}$$

If $\mathscr{E}$ denotes the tautological bundle on $\mathrm{Gr}(k,n)$, then the line bundle $\sigma^*(\mathscr{L}_{\vartheta_0})$ is the eth power $\sigma^*(\mathscr{L}_{\vartheta_0}) = \det(f^*\mathscr{E})^e$ of the determinant of the pullback—corresponding to the fact that any $U \in G$ acts on ℓ by $\det(U)^{-e}$ and $\mathbb{C}^*_R$ acts on ℓ trivially.

8.1.2 LG-phase

We call the case where $e < 0$ the LG-phase. In order to be G-invariant, a monomial $\prod B^{b_{i_1,\cdots,i_k}}_{i_1,\cdots,i_k} \prod p_j^{a_j} \ell^m$ must have $\sum a_j > 0$, which implies that any points with every $p_j = 0$ must be unstable, but for each $m > 0$ and each j the monomial $p_j^{me}\ell^{md_j}$ is G-invariant, so every point with at least one nonzero p_j is θ-semistable. Therefore $V^{ss}_G(\theta) = M_{k,n} \times (\mathbb{C}^s \backslash \{\mathbf{0}\})$. Again, since the F_j and the image of the Plücker embedding are transverse, the equations $\partial_{B_{i_1,\cdots,i_k}} W = \sum_j p_j \partial_{B_{i_1,\cdots,i_k}} F_j = 0$ imply that the critical locus is $[(\{\mathbf{0}\} \times (\mathbb{C}^s \backslash \{\mathbf{0}\}))/\,\mathrm{GL}(k,\mathbb{C})]$ inside $[V/\!\!/_\theta G] = [(M_{k,n} \times (\mathbb{C}^s \backslash \{\mathbf{0}\}))/\,\mathrm{GL}(k,\mathbb{C})]$.

This phase does not immediately fit into our theory because we have an infinite stabilizer $\mathrm{SL}(k,\mathbb{C})$ for any points of the form $(\mathbf{0}, p_1, \cdots, p_s)$. This means that the quotient $[V/\!\!/_\theta G]$ is an Artin stack (not Deligne-Mumford).

Hori-Tong [21] analyzed the gauged linear sigma model of the Calabi-Yau complete intersection $Z_{1,\cdots,1} \subset \mathrm{Gr}(2,7)$ which is defined by seven linear equations in the Plücker coordinates. They gave a physical derivation that its LG-phase is equivalent to the Gromov-Witten theory of the so-called Pfaffian variety

$$\mathrm{Pf}\left(\bigwedge^2 \mathbb{C}^7\right) = \left\{A \in \bigwedge^2 \mathbb{C}^7;\ A \wedge A \wedge A = 0\right\}.$$

It is interesting to note that the Pfaffian $\mathrm{Pf}\left(\bigwedge^2 \mathbb{C}^7\right)$ is not a complete intersection. For additional work on this example, see [2, 22, 25, 31].

8.2 Complete intersections in a flag variety

Another class of interesting examples is that of complete intersections in partial flag varieties. The partial flag variety $\mathrm{Flag}(d_1,\cdots,d_k)$ parametrizes the space of partial flags

$$0 \subset V_1 \subset \cdots V_i \subset \cdots V_k = \mathbb{C}^n$$

such that $\dim V_i = d_i$. The combinatorial structure of the equivariant cohomology of $\mathrm{Flag}(d_1, \cdots, d_k)$ is a very interesting subject in its own right.

For our purposes, $\mathrm{Flag}(d_1,\cdots,d_k)$ can be constructed as a GIT or symplectic quotient of the vector space

$$\prod_{i=1}^{k-1} M_{d_i,d_{i+1}}$$

by the group

$$G = \prod_{i=1}^{k-1} \mathrm{GL}(d_i, \mathbb{C}).$$

The moment map sends the element $(A_1, \cdots, A_{k-1}) \in \prod_{i=1}^{k-1} M_{i,i+1}$ to the element

$$\frac{1}{2}(A_1 \overline{A}_1^{\mathrm{T}}, \cdots, A_{k-1}\overline{A}_{k-1}^{\mathrm{T}}) \in \prod_{i=1}^{k-1} \mathfrak{u}(d_i).$$

Let the χ_i be the character of $\prod_j \mathrm{GL}(d_j)$ given by the determinant of ith factor. Each character χ_i defines a line bundle on the vector space $M_{d_1,d_2} \times \cdots \times M_{d_{k-1},d_k}$, which descends to a line bundle L_i on $\mathrm{Flag}(d_1, \cdots, n_k)$. A hypersurface of multidegree $(\ell_1, \cdots, \ell_k)$ is a section of $\bigotimes_j L_j^{\ell_j}$.

To consider the gauged linear sigma model for the complete intersection $F_1 = \cdots = F_s = 0$ of such sections, we again consider the vector space

$$V = \prod_{i=1}^{k-1} M_{d_i,d_i+1} \times \mathbb{C}^s$$

with coordinates $(p_1, \cdots, p_s)$ on $\mathbb{C}^s$ and superpotential

$$W = \sum_{j=1}^{s} p_j F_j.$$

We define an action of G on p_i by $(g_1, \cdots, g_{k-1}) \in G$ acts on p_i as $\prod_{j=1}^{k-1} \det(g_j)^{-\ell_{ij}}$, where ℓ_{ij} is the jth component of the multidegree degree of F_i.

We may describe the polarization as

$$\theta = \prod_{i=1}^{k-1} \det(g_i)^{-e_i},$$

or the moment map as

$$\mu(A_1, \cdots, A_{k-1}, p_1, \cdots, p_s) = \frac{1}{2}\Big(A_1\overline{A}_1^{\mathrm{T}} - \sum_{i=1}^{s} \ell_{1j}|p_j|^2, \cdots, A_{k-1}\overline{A}_{k-1}^{\mathrm{T}} - \sum_{i=1}^{s} \ell_{k-1,j}|p_j|^2\Big).$$

This gives a phase structure similar to a complete intersection in a product of projective spaces.

For example, when $e_i > 0$ for all $i \in \{1, \cdots, k-1\}$, we can choose a compatible $\mathbb{C}_R^*$ action with weight 1 on p_j and weight 0 on each A_i, and again the trivial lift ϑ_0 is a good lift of θ in this phase.

This example should be easy to generalize to complete intersections in quiver varieties. It would be very interesting to calculate the details of our theory for these examples.

9 General Comments

When G is non-Abelian or the R-charge is not integral, G and $\mathbb{C}^*_R$ interact in a nontrivial way and the description of moduli space is more complicated. For more details, we encourage readers to consult [19].

An important technique that we have not touched on here is cosection localization, which is the main tool for constructing a virtual cycle for the GLSM. Starting from the noncompact stack of LG-quasimaps to $[V/\!\!/_\theta G]$, the cosection localization technique enables us to construct a virtual cycle supported on the compact substack of LG-quasimaps to the critical locus of W.

Finally, we remark that the choice of stability condition in our paper is by no means unique. There are other choices of stability conditions that result in different moduli spaces. Please see [5, 12] for examples.

References

[1] Abramovich, D. and Jarvis, T. J., Moduli of twisted spin curves (electronic), *Proc. Amer. Math. Soc.*, **131**(3), 2003, 685–699.

[2] Addington, N., Donovan, W. and Segal, E., The Pfaffian-Grassmannian equivalence revisited, 2014, arXiv: 1401.3661.

[3] Chang, H.-L. and Li, J., Gromov-Witten invariants of stable maps with fields, *International Mathematics Research Notices*, RNR, **18**, 2012, 4163–4217.

[4] Chang, H.-L., Li, J. and Li, W., Witten's top Chern class via cosection localization, *Invent. Math.*, 2013, arXiv:1303.7126.

[5] Chang, H.-L., Li, J., Li, W. and Liu, C.-C., Mixed-spin-p-fields of Fermat quintic polynomials, 2015, arXiv:1505.07532v1.

[6] Cheong, D., Ciocan-Fontanine, I. and Kim, B., Orbifold Quasimap Theory, 2014, arXiv:math/1405.7160.

[7] Chiodo, A., Iritani, H. and Ruan, Y., Landau-Ginzburg/Calabi-Yau correspondence, global mirror symmetry and Orlov equivalence, *Publ. Math. Inst. Hautes Études Sci.*, **119**, 2014, 127–216.

[8] Chiodo, A. and Ruan, Y., Landau-Ginzburg/Calabi-Yau correspondence for quintic three-folds via symplectic transformations, *Invent. Math.*, **182**(1), 2010, 117–165.

[9] Chiodo, A. and Ruan, Y., LG/CY correspondence: The state space isomorphism, *Adv. Math.*, **227**(6), 2011, 2157–2188.

[10] Ciocan-Fontanine, I. and Kim, B., Moduli stacks of stable toric quasimaps, *Adv. Math.*, **225**(6), 2010, 3022–3051.

[11] Ciocan-Fontanine, I., Kim, B. and Maulik, D., Stable quasimaps to GIT quotients, *J. Geom. Phys.*, **75**, 2014, 17–47.

[12] Choi, J. and Kiem, Y.-H., Landau-Ginzburg/Calabi-Yau correspondence via quasi-maps, 2016, arXiv: 1103.0833 [math.AG].

[13] Clader, E., Landau-Ginzburg/Calabi-Yau correspondence for the complete intersections $X_{-}\{3,3\}$ and $X_{-}\{2,2,2,2\}$, 2013, arXiv: math/1301.5530.

[14] Clader, E., Janda, F. and Ruan, Y., Higher-genus quasimap wall-crossing via localization, 2017, arXiv:1702.03427 [math.AG].

[15] Dolgachev, I. V. and Hu, Y., Variation of geometric invariant theory quotients, *Inst. Hautes Études Sci. Publ. Math.*, (87), 1998, 5–56.

[16] Fan, H., Jarvis, T. J. and Ruan, Y., The Witten equation and its virtual fundamental cycle, 2007, arXiv:math/0712.4025.

[17] Fan, H., Jarvis, T. J. and Ruan, Y., Geometry and analysis of spin equations, *Comm. Pure Appl. Math.*, **61**(6), 2008, 745–788.

[18] Fan, H., Jarvis, T. J. and Ruan, Y., The Witten equation, mirror symmetry and quantum singularity theory, *Ann. Math.*, **178**, 2013, 1–106.

[19] Fan, H., Jarvis, T. J. and Ruan, Y., A mathematical theory of the Gauged Linear Sigma Model, 2015, arXiv:math/1506.02109.

[20] Fan, H., Jarvis, T. J. and Ruan, Y., Analytic Theory of the Gauged Linear Sigma Model, 2017, preprint.

[21] Hori, K. and Tong, D., Aspects of non-abelian gauge dynamics in two-dimensional $\mathscr{N} = (2, 2)$ theories (electronic), *J. High Energy Phys.*, **5**, 2007, 079, 41 pp.

[22] Hosono, S. and Konishi, Y., Higher genus Gromov-Witten invariants of the Grassmannian, and the Pfaffian Calabi-Yau 3-folds, *Adv. Theor. Math. Phys.*, **13**(2), 2009, 463–495.

[23] Jarvis, T., Kimura, T. and Vaintrob, A., Moduli spaces of higher spin curves and integrable hierarchies, *Compositio Math.*, **126**, 2001, 157–212.

[24] Kim, B., Stable quasimaps, 2011, arXiv:math/1106.0804.

[25] Kuznetsov, A., Lefschetz decompositions and categorical resolutions of singularities, *Selecta Math.* (*N. S.*), **13**(4), 2008, 661–696.

[26] Lee, Y., Priddis, N. and Shoemaker, M., A Proof of the Landau-Ginzburg/Calabi-Yau Correspondence via the Crepant Transformation Conjecture, 2014, arXiv:1410.5503.

[27] Mumford, D., Fogarty, J. and Kirwan, F., Geometric Invariant Theory, 3rd ed., Springer-Verlag, 1994.

[28] Priddis, N. and Shoemaker, M., A Landau-Ginzburg/Calabi-Yau correspondence for the mirror quintic, 2013, arXiv:1309.6262.

[29] Ross, D. and Ruan, Y., Wall-crossing in genus zero Landau-Ginzburg theory, 2014, arXiv:1402.6688.

[30] Ruan, Y., The Witten equation and the geometry of the Landau-Ginzburg model, *Pro. Symposia in Pure Math.*, **85**, 2012, 209–240.

[31] Rødland, E. A., The Pfaffian Calabi-Yau, its mirror, and their link to the Grassmannian $G(2, 7)$, *Compositio Math.*, **122**(2), 2000, 135–149.

[32] Thaddeus, M. Geometric invariant theory and flips, *J. Amer. Math. Soc.*, **9**(3), 1996, 691–723.

[33] Vafa, C. and Warner, N., Catastrophes and the classification of conformal theories, *Physics Letters B*, **218**, 1989, 51–58.

[34] Witten, E., The N-matrix model and gauged WZW models, *Nuclear Physics B*, **371**, 1992, 191–245.

[35] Witten, E., Algebraic Geometry Associated with Matrix Models of Two-Dimensional Gravity, Topological Methods in Modern Mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993, 235–269.

[36] Witten, E., Phases of $N = 2$ Theories in Two Dimensions, in Mirror Symmetry, II, Volume 1 of AMS/IP Stud. Adv. Math., Amer. Math. Soc., Providence, RI, 1997, 143–211.