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Nonabelian Gauged Linear Sigma Model*

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Abstract The gauged linear sigma model (GLSM for short) is a 2d quantum field theory introduced by Witten twenty years ago. Since then, it has been investigated extensively in physics by Hori and others. Recently, an algebro-geometric theory (for both abelian and nonabelian GLSMs) was developed by the author and his collaborators so that he can start to rigorously compute its invariants and check against physical predications. The abelian GLSM was relatively better understood and is the focus of current mathematical investigation. In this article, the author would like to look over the horizon and consider the nonabelian GLSM. The nonabelian case possesses some new features unavailable to the abelian GLSM. To aid the future mathematical development, the author surveys some of the key problems inspired by physics in the nonabelian GLSM.

Keywords GIT quotient, Stability condition, GLSM
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1 Introduction

The gauged linear sigma model (GLSM for short) in two dimensions was introduced by Witten in 1993 (see [34]), and has been a powerful tool in the study of quantum field theory in string theory for such targets as the complete intersection of GIT-quotients. In particular, it provides us with a global, precise, and yet simple picture of the stringy Kähler moduli space $\mathfrak{M}$ of theories. The space $\mathfrak{M}$ is decomposed into chambers called "phases". Each phase typically corresponds to a traditional theory such as the non-linear sigma model/Gromov-Witten theory or the Landau-Ginzburg model/FJRW-theory (see [13–15]), and wall crossing across a phase boundary is a smooth analytic continuation. The Landau-Ginzburg/Calabi-Yau (LG/CY) correspondence is an example of such a picture. The GLSM is also used to physically understand mirror symmetry for toric varieties and toric complete intersections.

It is very natural to try to develop the mathematical theories arising from the GLSM. The author and his collaborators have started to develop a mathematical theory of A-model invariants for the GLSM, which is an analogue of Gromov-Witten theory for non-linear sigma models and FJRW theory for Landau-Ginzburg orbifolds. Recently, an algebro-geometric theory for the so-called "compact type" sectors was put forth (see [16]). This algebro-geometric theory is enough for most of the examples of interest, such as Fermat Calabi-Yau 3-folds. Although

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the theory in the most general case is incomplete without an appropriate treatment of broad sectors (by analysis that is currently under development (see [17, 33])), for the purposes of computing invariants and matching with physical predictions, the algebro-geometric theory is what we need. Therefore, we can go ahead and explore the vast landscape of the GLSM. For example, there are various attempts (see [10, 31]) right now to apply it to finding a precise relationship between the FJRW and Gromov-Witten invariants for pairs of data related by the LG/CY correspondence, which was one of the motivations for Witten's original development of the GLSM.

Our knowledge of the GLSM can be roughly divided into two cases: Those with abelian gauge group and those with nonabelian gauge group. The abelian GLSM corresponds to the Gromov-Witten theory of complete intersections in toric varieties, as well as the various associated LG phases. The general picture here is relatively well understood, at least at the conjectural level. In particular, the mirror symmetry construction due to Givental-Hori-Vafa plays a critical role in our understanding in both mathematics and physics. There are plenty of problems in the abelian case, and its mathematical implications have been the focus of much investigation recently (see [3–9, 11, 27, 31]). However, in this article, we would like to look over the horizon to identify some of the key problems for possible future development; after all, the algebro-geometric theory (see [16]) by Fan-Jarvis-Ruan applies to the nonabelian case as well.

Past progress in physics has mostly been concerned with GLSMs with abelian gauge groups, and extending to models with nonabelian gauge groups has been a challenging problem. Related problems, such as mirror symmetry, have been poorly understood. More importantly, there are new features in models with nonabelian gauge groups, such as 2d-Seiberg duality, Artin phases, quiver varieties, and geometric representation theory.

We will first review the algebro-geometric theory of the GLSM by Fan-Jarvis-Ruan, in Section 2. The focus is on Sections 3–5, where we describe some important problems inspired by physics in the nonabelian GLSM. We close the paper with some general comments on mirror symmetry for the nonabelian GLSM and its applications.

2 Brief Review of the Gauged Linear Sigma Model

In this section, we briefly review the data of algebraic geometric theory of the GLSM developed by Fan-Jarvis-Ruan. For more details see the paper [18] in this same volume, or the original paper [16].

The construction of the GLSM depends on a choice of a vector space V of dimension n-over $\mathbb{C}$, a reductive algebraic group $G \subset \mathrm{GL}(V)$ over $\mathbb{C}$ with identity component G_0 such that G/G_0 is finite. We call G the gauge group. If the gauge group action on V factors through $\mathrm{SL}(V)$ then we say that it satisfies the Calabi-Yau condition.

Assume that V also admits a $\mathbb{C}^*$-action $(z_1, \cdots, z_n) \to (\lambda^{c_1} z_1, \cdots, \lambda^{c_n} z_n)$, which we denote by $\mathbb{C}^*_R$ (this is twice what the physicists call R-charge). We think of $\mathbb{C}^*_R$ as a subgroup of $\mathrm{GL}(V, \mathbb{C})$. This means we require $\gcd(c_1, \cdots, c_n) = 1$. The theory also requires a polynomial

$W\colon V \to \mathbb{C}$ of degree $d \neq 0$ with respect to the $\mathbb{C}_R^*$ action (in other words, a quasihomogeneous polynomial), invariant under the action of G. The polynomial W will be called the superpotential for our theory.

The actions of G and $\mathbb{C}_R^*$ are required to be compatible, meaning that they commute and $G \cap \mathbb{C}_R^*$ has finite order.

Definition 2.1 *We define Γ to be the subgroup of $\mathrm{GL}(V,\mathbb{C})$ generated by G and $\mathbb{C}_R^*$.*

Let $\theta : G \to \mathbb{C}^*$ and consider the GIT semistable locus of V with the G linearization determined by θ. We will call θ strongly regular if $V_G^{ss}(\theta) = V_G^{s}(\theta)$. Different choices of θ correspond to different GIT quotients (phases), and these give different, but related, theories. Inside each phase, there is another parameter $\epsilon \geq 0$ used to specify the stability condition.

In the case of $\epsilon > 0$, we need a choice of a good lift $\vartheta : \Gamma \to \mathbb{C}^*$ of θ, in the sense that $V_\Gamma^{ss}(\vartheta) = V_G^{ss}(\theta)$. The existence of a good lift is a nontrivial condition. We want to emphasis that in the important case of $\epsilon = 0$, the existence of a good lift is NOT needed. Finally, we always assume that W defines a nondegenerate holomorphic map $W\colon [V /\!\!/_\theta G] \to \mathbb{C}$.

The most common examples of the previous discussion are when G is abelian. Several examples of these are given in the paper [18] in this volume. In this paper we focus more on nonabelian cases.

2.1 State space

Given these ingredients, the GLSM gives a state space (similar to that of FJRW-theory)

$$\mathscr{H}_{W,G} = \bigoplus_{\alpha \in \mathbb{Q}} \mathscr{H}_{W,G}^{\alpha} = \bigoplus_{\Psi} \mathscr{H}_{\Psi},$$

where the sum runs over those conjugacy classes Ψ of G for which $\mathscr{X}_{\theta,\Psi}$ is nonempty, and where

$$\mathscr{H}_{W,G}^{\alpha} = H_{CR}^{\alpha+2q}(\mathscr{X}_\theta, W^\infty, \mathbb{Q}) = \bigoplus_{\Psi} H^{\alpha - 2\,\mathrm{age}\,(\gamma)+2q}(\mathscr{X}_\Psi W_\Psi^\infty, \mathbb{Q}),$$

and

$$\mathscr{H}_{\Psi} = H_{CR}^{\bullet+2q}(\mathscr{X}_{\theta,\Psi}, W^\infty, \mathbb{Q}) = \bigoplus_{\alpha \in \mathbb{Q}} H^{\alpha - 2\,\mathrm{age}\,(\gamma)+2q}(\mathscr{X}_{\theta,\Psi}, W_\Psi^\infty, \mathbb{Q}).$$

That is, the state space is the relative Chen-Ruan cohomology with an additional shift by $2q$.

For each element $g \in G$ we write $[\![g]\!] \subset G$ for the conjugacy class of g in G. We often call the factor $\mathscr{H}_{[\![g]\!]}$ the $[\![g]\!]$-sector, and we call the factor $\mathscr{H}_{[\![1]\!]}$ the untwisted sector.

Definition 2.2 *An element $\gamma \in G$ is called narrow if the corresponding component $\mathscr{X}_{[\![\gamma]\!]} \subset \mathbb{I}\mathscr{X}_\theta$ is compact. In this case we also say that the corresponding sector $\mathscr{H}_{[\![\gamma]\!]}$ is narrow. If γ is not narrow, we call it* (*and the corresponding sector*) *broad.*

The theory for narrow sectors is generally much easier to understand than for the broad sectors, but some elements of the broad sectors also behave well, namely those which are supported on the critical locus of W. We call these good sectors of compact type.

2.2 Moduli space

Associated to the ingredients described above and to a rational choice of $0 \leq \varepsilon \leq \infty$ the theory produces a moduli space that is a sort of unification of the quasimaps of [8–9, 26] with an extension of the Polishchuk-Vaintrob [29] description of the FJRW moduli space to reductive algebraic groups

$$\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta, \beta),$$

consisting of ε, ϑ-stable LG-quasimaps to the critical locus $\mathscr{CR}_\theta = [\mathrm{Crit}(W)/\!\!/_\theta G]$.

Remark 2.1 The choice of ε and ϑ determine stability conditions for the moduli space. But the choice of stability condition is by no means unique. One can find other interesting choices of stability conditions in [5, 7].

2.3 Virtual cycle and correlators

Finally, for these moduli spaces, we can construct a virtual cycle for the case where all insertions are of compact type using the cosection localization techniques of Kiem-Li [25] as applied in [3–4]. This allows us to define correlators.

Definition 2.3 *Suppose that* $\alpha_i \in \mathscr{H}_{W,G,\mathrm{comp}}$. *We define correlator*

$$\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_k}(\alpha_k)\rangle = \int_{[\mathrm{LGQ}^{\varepsilon,\vartheta}_{g,k}(\mathscr{CR}_\theta,\beta)]^{\mathrm{vir}}} \prod_i \mathrm{ev}_i^*(\alpha_i)\psi_i^{l_i}.$$

One can define the generating function in a standard fashion.

These invariants satisfy the gluing axioms whenever every insertion is of compact type. Except in the $\varepsilon = \infty$-chamber, we do not expect a forgetful morphism or string/dilaton equations.

3 Abelian GLSM and Their Mirrors

The abelian GLSM corresponds to the case of complete intersection of toric varieties which are the center of attention recently (see examples in [18] in this volume). One will not get a complete picture without considering the mirror symmetry. The abelian GLSM has a satisfactory answer for its LG mirror due to Givental-Hori-Vafa. We briefly review their construction.

Definition 3.1 *Let*

$$X_\Sigma = \frac{\mathbb{C}^n \backslash Z(\Sigma)}{(\mathbb{C}^*)^r}$$

be a compact toric variety with charge matrix $Q = (Q_{ij})$. *Then the Givental-Hori-Vafa mirror is a Landau-Ginzburg model on the toric variety*

$$\{x \in \mathbb{C}^n \mid x_1^{Q_{j1}} \cdots x_n^{Q_{jn}} = \mathrm{e}^{jt} \text{ for all } 1 \leq j \leq r\}$$

with superpotential given by the restriction of

$$W = x_1 + \cdots + x_n.$$

The above constitutes a Landau-Ginzburg model, which should be mirror to X_Σ whenever X_Σ is Fano. The failure in the non-Fano case can be observed explicitly for Hirzebruch surfaces F_n, which are Fano if and only if $n = 1$.

Let us verify in some easy examples that mirror symmetry is satisfied, at least on the cohomological level; that is,

$$H^*(X_\Sigma) \cong \mathrm{Jac}(W).$$

Example 3.1 The charge matrix of $\mathbb{P}^n$ is $(1, 1, \cdots, 1)$, so the underlying toric variety of the Givental-Hori-Vafa mirror is the subset of $\mathbb{C}^{n+1}$ defined by the equation

$$x_0 = \frac{\mathrm{e}^t}{x_1 \cdots x_n},$$

which is isomorphic to $(\mathbb{C}^*)^n$. One can check easily (see [12, Example 2.9]) that $\mathrm{Jac}(W) \cong \mathbb{C}^{n+1}$, so it does match $H^*(\mathbb{P}^n)$ as a vector space.

Remark 3.1 In recent years, some attempts have been made to adapt Hori-Vafa mirror symmetry to the non-Fano case. This involves adding higher-order terms to W to ensure that the number of its critical points coincides with the rank of the cohomology of the non-Fano variety. These correction terms have an interpretation in terms of counting holomorphic discs in X_Σ.

3.1 Equivariant Givental-Hori-Vafa mirrors of noncompact toric varieties

A simple example suffices to show that Hori-Vafa mirror symmetry fails in the noncompact case.

Example 3.2 If $X_\Sigma = \mathbb{C}$, then the Landau-Ginzburg mirror is $W = x : \mathbb{C} \to \mathbb{C}$. This has no critical points, so the duality between $H^*(X_\Sigma)$ and the Jacobian ring fails.

On the other hand, one might expect that there is an equivariant version of the Hori-Vafa mirror for which the mirror symmetry statement still holds.

Definition 3.2 *Let $X_\Sigma = \frac{\mathbb{C}^n \setminus Z(\Sigma)}{(\mathbb{C}^*)^r}$ be a toric variety with charge matrix Q. Its equivariant Givental-Hori-Vafa mirror is*

$$W = x_1 + \cdots + x_n - \sum_{i=1}^{n} \lambda_i \log(x_i),$$

subject to the constraints

$$\prod_{i=1}^{n} x_i^{Q_{bi}} = q_b$$

for nonzero parameters q_b. Here λ_i is a constant, viewed as the equivariant parameter for the ith $\mathbb{C}^*$ action.

Example 3.3 If $X_\Sigma = \mathbb{C}$, then the equivariant Givental-Hori-Vafa mirror is

$$W = x - \lambda \log(x),$$

so

$$\partial_x W = 1 - \frac{\lambda}{x}.$$

Unlike the nonequivariant case, this now has a single critical point, so we recover the correspondence between the dimension of $H^*(X_\Sigma)$ and the number of critical points.

Example 3.4 Let X_Σ be the total space of the bundle $\mathscr{O}_{\mathbb{P}^n}(-d)$, which has charge matrix $(1, 1, \cdots, 1, -d)$. In this case

$$W = x_0 + x_1 + \cdots + x_{n+1} - \sum_{i=0}^{n+1} \lambda_i \log(x_i)$$

with the constraint

$$x_0 \cdots x_n x_{n+1}^d = q.$$

Thus, we obtain

$$W = x_1 + \cdots + x_{n+1} + \frac{x_{n+1}^d}{x_1 \cdots x_n} - \sum_{i=0}^{n+1} \lambda_i \log(x_i).$$

It is straightforward to check that the number of critical points of this superpotential indeed matches the dimension of $H^*(X_\Sigma)$.

Example 3.5 It is not always necessary to modify W by all of the terms $\lambda_i \log(x_i)$ in order to achieve mirror symmetry. For example, consider the toric variety $\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$, which has charge matrix $Q = (1, 1, -1, -1)$. Its nonequivariant Givental-Hori-Vafa mirror is

$$W = x_2 + x_3 + x_4 + q\frac{x_3 x_4}{x_2},$$

defined over $(\mathbb{C}^*)^3$, while for the equivariant mirror one must subtract the terms $\sum\limits_{i=1}^{4} \lambda_i \log(x_i)$ from the above.

Even the partial modification

$$\widetilde{W} = x_2 + x_3 + x_4 + q\frac{x_3 x_4}{x_2} - \lambda_3 \log(x_3) - \lambda_4 \log(x_4)$$

of W upholds mirror symmetry, though. This makes sense from the perspective of the J-function; only the two $\mathbb{C}^*$ actions in the noncompact fiber directions of $\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$ are necessary in order to make the equivariant Gromov-Witten theory well-defined.

Despite the failure of nonequivariant Givental-Hori-Vafa mirror symmetry for noncompact toric varieties, the noncompact mirrors are still worth remembering, as they play a role in the hypersurface version of mirror symmetry considered in the next section.

3.2 Givental-Hori-Vafa mirrors of hypersurfaces in toric varieties

The Givental-Hori-Vafa construction can be adapted to give the mirror of a Fano hypersurface in a toric variety. Let us begin by studying how this works in the case of projective space.

Example 3.6 Let $X_d \subset \mathbb{P}^{N-1}$ be a degree-d hypersurface; explicitly, the Fano condition corresponds to the requirement that $d \leq N-1$. Suppose that $X_d = \{G_d = 0\}$ for the polynomial $G_d = x_1^d + \cdots + x_N^d$.

To form the Givental-Hori-Vafa mirror, one first constructs a GLSM on a noncompact toric variety for which X_d is the critical locus of the superpotential in a particular phase. Namely, let $(\mathbb{C}^*)^r$ act on $(\mathbb{C}^*)^{N+1}$ with charge matrix

$$(1, \cdots, 1, -d),$$

so that the first N factors give precisely the charge matrix for $\mathbb{P}^{N-1}$. Let the superpotential be

$$W = p \cdot G(x_1, \cdots, x_N) : \mathbb{C}^{N+1} \to \mathbb{C}$$

in coordinates $(x_1, \cdots, x_N, p)$ on $\mathbb{C}^{N+1}$, and let the moment map be

$$\mu = \frac{1}{2}\Big(\sum_{i=1}^{N} |z_i|^2 - d|p|^2\Big).$$

The GI-quotient has two chambers corresponding to $\mu > 0$ or $\mu < 0$. In the $\mu > 0$ phase of this GLSM, an easy computation shows that the critical locus of W is precisely the hypersurface X_d.

Next, construct the (non-equivariant) Hori-Vafa mirror of this noncompact toric variety containing X_d. This is sometimes called the pre-Givental-Hori-Vafa mirror of X_d. In this case, it is the Landau-Ginzburg model with superpotential

$$\widetilde{W} = x_1 + \cdots + x_N + x_{N+1}$$

on the subset of $\mathbb{C}^{N+1}$ satisfying the constraint

$$x_1 \cdots x_N x_{N+1}^{-d} = \mathrm{e}^t.$$

In other words, setting

$$x_i = u_i^d \quad \text{for } 1 \leq i \leq N$$

and

$$x_{N+1} = u_{N+1},$$

the pre-Givental-Hori-Vafa mirror becomes

$$\widetilde{W} = u_1^d + \cdots + u_N^d + \mathrm{e}^{-\frac{t}{d}} u_1 \cdots u_N$$

on the toric variety $(\mathbb{C}^*)^N$.

This Landau-Ginzburg model has a certain symmetry group. Namely, an automorphism of $(\mathbb{C}^*)^N$ of the form

$$u_i \mapsto \omega_d^{p_i} u_i$$

for which ω_d is a dth root of unity and

$$\omega_d^{p_1+\cdots+p_N}=1,$$

will preserve the superpotential $\widetilde{W}$. The group of such symmetries is denoted by $\mathrm{SL}(W_0)$, since if $W_0=u_1^d+\cdots+u_N^d$, then the automorphisms in question are precisely

$$\left\{\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_N\end{pmatrix}\,\middle|\, W_0(\lambda_1x_1,\cdots,\lambda_Nx_N)=W_0(x_1,\cdots,x_N)\right\}\cap \mathrm{SL}_N(\mathbb{C}).$$

Finally, to form the Givental-Hori-Vafa mirror of X_d, one "compactifies" the pre-Givental-Hori-Vafa mirror by adding the point $u_1=\cdots=u_N=0$ to the domain $\mathbb{C}^N$ of the Landau-Ginzburg model, and then takes the quotient by the symmetry group above. This yields

$$\begin{aligned}&\widetilde{W}:[\mathbb{C}^N/\mathrm{SL}(W)]\to\mathbb{C},\\ &\widetilde{W}=u_1^d+\cdots+u_N^d+\mathrm{e}^{-\frac{t}{d}}u_1\cdots u_N\end{aligned}$$

as the mirror.

Remark 3.2 The cohomology or state space is the simplest invariant preserved under mirror symmetry. More generally, the genus-zero A-model invariants are encoded in the so-called J-function, while the full Landau-Ginzburg B-model in genus zero can be viewed as encoding certain oscillatory integrals

$$\int_\Delta \mathrm{e}^{-\frac{W}{z}}\omega,$$

where ω is a "primitive form". In the toric case, we have seen that $\omega=\mathrm{d}\log(x_1)\wedge\cdots\wedge\mathrm{d}\log(x_n)$. On the other hand, the primitive form for a hypersurface is $\omega=\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$. For higher genus one can consider the so-called Saito-Givental theory on the LG B-model. Mirror symmetry is supposed to match it with the higher-genus Gromov-Witten invariants of the GLSM.

4 Artin Phase and Determinental Varieties

In the abelian case, we only encountered a finite isotropy group in each phase, and the orbifold model is sufficient. This is matched perfectly by our virtual cycle technique where we require the underlying moduli space to be a Deligne-Mumford stack. When we have a continuous isotropy group, we are in the context of Artin stacks. It is a long-standing problem to try to construct the virtual cycle for Artin stacks. For a nonabelian GLSM, we frequently encounter the situation that one phase is smooth or weakly coupled, while other phase is an Artin stack or strongly coupled. This is where some of deepest physical ideas come in. Hori and his collaborators argue that these models further localize on some determinantal varieties—another mysterious branch of algebraic geometry!

Recall that so far we have only discussed complete intersections of toric varieties. These are only a small class of algebraic varieties. There is a vast landscape of algebraic varieties which are not complete intersections. Unfortunately, they are much harder to access. The most common construction is via determinantal variety techniques.

Suppose that M is a matrix whose coefficients are homogeneous polynomial. The simplest example of a determinantal variety is

$$\Omega_k = \{x \in \mathbf{P}^n;\ \mathrm{rank} M(x) \leq k\}.$$

More generally, let $A : E \to F$ be a generic linear homomorphism between vector bundles E, F over a projective variety X. We can consider the determinantal variety

$$\Omega_k(A, E, F) = \{x \in X;\ \mathrm{rank}\ A(x) \leq k\}.$$

We can further generalize A to be a homomorphism between two flags (see [19] for a comprehensive introduction). The examples arising among determinantal varieties are some of oldest non-complete intersection examples in algebraic geometry. In a related story, physicists constructed millions of Calabi-Yau 3-folds in early 90's. However, extensive computation has only been done for the 14 so-called one-parameter models (the moduli space of Kähler structure is one-dimensional). Recently, this number was increased to more than 20 (see Hori-Knapp's examples). The new examples are determinantal. It is surprising that the nonabelian GLSM can access these examples. After all, GLSM is built to capture complete intersection. The physical argument to relate to determinantel varieties are highly nontrivial. We still do not yet know how to interpret it in mathematics. We should mention that the 14 older examples have known mirrors which have a one-dimensional moduli space of complex structures. It is an extremely interesting problem to find geometric mirrors of these new examples.

Example 4.1 Here we review the example (see [16, 34]) of a complete intersection in the Grassmannian $\mathrm{Gr}(k, n)$. The space $\mathrm{Gr}(k, n)$ can be constructed as the GIT quotient $M_{k,n} /\!\!/ \mathrm{GL}(k, \mathbb{C})$, where $M_{k,n}$ is the space of $k \times n$ matrices, and $\mathrm{GL}(k, \mathbb{C})$ acts as matrix multiplication on the left.

The Grassmannian $\mathrm{Gr}(k, n)$ can also be embedded into $\mathbb{P}^K$ for $K = \frac{n!}{k!(n-k)!} - 1$ by the Plücker embedding

$$A \mapsto (\cdots, \det(A_{i_1,\cdots,i_k}), \cdots),$$

where $A_{i_1,\cdots,i_k}$ is the $(k \times k)$-submatrix of A consisting of the columns $i_1, \cdots, i_k$.

The group $G = \mathrm{GL}(k, \mathbb{C})$ acts on the Plücker coordinates

$$B_{i_1,\cdots,i_k}(A) = \det(A_{i_1,\cdots,i_k})$$

by the determinant, that is, for any $U \in G$ and $A \in M_{k,n}$, we have

$$B_{i_1,\cdots,i_k}(UA) = \det(U) B_{i_1,\cdots,i_k}(A).$$

Let $F_1, \cdots, F_s \in \mathbb{C}[B_{1,\cdots,k}, \cdots, B_{n-k+1,\cdots,n}]$ be degree-d_j homogeneous polynomials such that the zero loci $Z_{F_j} = \{F_j = 0\}$ and the Plücker embedding of $\mathrm{Gr}(k, n)$ all intersect transversely in $\mathbb{P}^K$. We let

$$Z_{d_1,\cdots,d_s} = \mathrm{Gr}(k, n) \cap \bigcap_j Z_{F_j}$$

denote the corresponding complete intersection.

The analysis of $Z_{d_1,\cdots,d_s}$ is similar to the abelian case. Namely, let

$$W=\sum_j p_jF_j\colon M_{k,n}\times\mathbb{C}^s\to\mathbb{C}$$

be the superpotential. We assign an action of $G=\mathrm{GL}(k,\mathbb{C})$ on p_j by $p_j\to\det(U)^{-d_j}$.

The phase structure is similar to that of a complete intersection in projective space. The moment map is given by $\mu(A,p_1,\cdots,p_s)=\frac{1}{2}\big(A\overline{A}^{\mathrm{T}}-\sum\limits_{i=1}^{s}d_i|p_i|^2\big)$. Alternatively, to construct a linearization for GIT, the only characters of $\mathrm{GL}(k,\mathbb{C})$ are powers of the determinant, so $\theta(U)=\det(U)^{-e}$ for some e, and τ will be positive precisely when e is positive.

Let ℓ be a generator of $\mathbb{C}[\mathbf{L}^*]$ over $\mathbb{C}[V^*]$. Any element of $H^0(V,\mathbf{L}_\theta)$ can be written as a sum of monomials in the Plücker coordinates $B_{i_1,\cdots,i_k}$ and the p_j times ℓ. Any $U\in G$ will act on a monomial of the form $\prod B_{i_1,\cdots,i_k}^{b_{i_1,\cdots,i_k}}\prod p_j^{a_j}\ell^m$ by multiplication by $\det(U)^{\sum b_{i_1,\cdots,i_k}-\sum d_ja_j-me}$.

$e>0$: In order to be G-invariant, a monomial must have $\sum b_{i_1,\cdots,i_k}>0$, which implies that any points with every $B_{i_1,\cdots,i_k}=0$ must be unstable, but for each $m>0$ and each k-tuple $(i_1,\cdots,i_k)$ the monomial $B_{i_1,\cdots,i_k}^{me}\ell^m$ is G invariant, so every point with at least one nonzero $B_{i_1,\cdots,i_k}$ must be θ-semistable. Thus $[V/\!\!/_\theta G]$ is isomorphic to the bundle $\bigoplus\limits_j\mathscr{O}(-d_j)$ over $\mathrm{Gr}(k,n)$.

Furthermore, W is quasihomogeneous of degree one with respect to the following compatible $\mathbb{C}_R^*$ action:

$$\lambda(A,p_1,\cdots,p_s)=(A,\lambda p_1,\cdots,\lambda p_s).$$

The trivial lift ϑ_0 is a good lift because each monomial of the form $B_{i_1,\cdots,i_k}^{me}\ell^m$ is Γ invariant for the action induced by ϑ_0.

The critical locus (see [16, Example 7.4.1]) in this phase is given by $p_1=\cdots=p_s=0=F_1=\cdots=F_s$, so we recover the complete intersection $F_1=\cdots=F_s$ in $\mathrm{Gr}(k,n)$.

As in the toric case, we call this phase the geometric phase.

$e<0$: We call the case where $e<0$ the LG-phase. In this case, in order to be G-invariant a monomial $\prod B_{i_1,\cdots,i_k}^{b_{i_1,\cdots,i_k}}\prod p_j^{a_j}\ell^m$ must have $\sum a_j>0$, which implies that any points with every $p_j=0$ must be unstable, but for each $m>0$ and each j the monomial $p_j^{me}\ell^{md_j}$ is G-invariant, so every point with at least one nonzero p_j is θ-semistable. Therefore $V_G^{ss}(\theta)=M_{k,n}\times(\mathbb{C}^s\backslash\{\mathbf{0}\})$. Again, since the F_j and the image of the Plücker embedding are transverse, the equations $\partial_{B_{i_1,\cdots,i_k}}W=\sum\limits_j p_j\partial_{B_{i_1,\cdots,i_k}}F_j=0$ imply that the critical locus is $[(\{\mathbf{0}\}\times(\mathbb{C}^s\backslash\{\mathbf{0}\}))/\,\mathrm{GL}(k,\mathbb{C})]$ inside $[V/\!\!/_\theta G]=[(M_{k,n}\times(\mathbb{C}^s\backslash\{\mathbf{0}\}))/\,\mathrm{GL}(k,\mathbb{C})]$. This phase does not immediately fit into our theory because we have an infinite stabilizer $\mathrm{SL}(k,\mathbb{C})$ for any points of the form $(\mathbf{0},p_1,\cdots,p_s)$. This means that the quotient $[V/\!\!/_\theta G]$ is an Artin stack.

Example 4.2 A generalization of above examples is that of complete intersections in partial flag varieties (see [16]). The partial flag variety $\mathrm{Flag}(d_1,\cdots,d_k)$ parametrizes the space of partial flags

$$0\subset V_1\subset\cdots V_i\subset\cdots V_k=\mathbb{C}^n$$

such that $\dim V_i = d_i$. The combinatorial structure of the equivariant cohomology of $\mathrm{Flag}(d_1, \cdots, d_k)$ is a very interesting subject in its own right.

For our purposes, $\mathrm{Flag}(d_1, \cdots, d_k)$ can be constructed as a GIT or symplectic quotient of the vector space

$$\prod_{i=1}^{k-1} M_{d_i, d_{i+1}}$$

by the group

$$G = \prod_{i=1}^{k-1} \mathrm{GL}(d_i, \mathbb{C}).$$

The moment map sends the element $(A_1, \cdots, A_{k-1}) \in \prod_{i=1}^{k-1} M_{i,i+1}$ to the element $\frac{1}{2}(A_1 \overline{A}_1^{\mathrm{T}}, \cdots, A_{k-1}\overline{A}_{k-1}^{\mathrm{T}}) \in \prod_{i=1}^{k-1} \mathfrak{u}(d_i)$.

Let χ_i be the character of $\prod_j \mathrm{GL}(d_j)$ given by the determinant of i-th factor. Each character χ_i defines a line bundle on the vector space $M_{d_1,d_2} \times \cdots \times M_{d_{k-1},d_k}$, which descends to a line bundle L_i on $\mathrm{Flag}(d_1, \cdots, n_k)$. A hypersurface of multidegree $(\ell_1, \cdots, \ell_k)$ is a section of $\bigotimes_j L_j^{\ell_j}$. To consider the gauged linear sigma model for the complete intersection $F_1 = \cdots = F_s = 0$ of such sections, we again consider the vector space

$$V = \prod_{i=1}^{k-1} M_{d_i, d_i+1} \times \mathbb{C}^s$$

with coordinates $(p_1, \cdots, p_s)$ on $\mathbb{C}^s$ and superpotential

$$W = \sum_{j=1}^{s} p_j F_j.$$

We define an action of G on p_i by $(g_1, \cdots, g_{k-1}) \in G$ acts on p_i as $\prod_{j=1}^{k-1} \det(g_j)^{-\ell_{ij}}$, where ℓ_{ij} is the jth component of the multidegree degree of F_i.

We may describe the polarization as

$$\theta = \prod_{i=1}^{k-1} \det(g_i)^{-e_i},$$

or the moment map as

$$\begin{aligned} &\mu(A_1, \cdots, A_{k-1}, p_1, \cdots, p_s) \\ &= \frac{1}{2}\Big(A_1\overline{A}_1^{\mathrm{T}} - \sum_{i=1}^{s} \ell_{1j}|p_j|^2, \cdots, A_{k-1}\overline{A}_{k-1}^{\mathrm{T}} - \sum_{i=1}^{s} \ell_{k-1,j}|p_j|^2\Big). \end{aligned}$$

This gives a phase structure similar to the complete intersection in a product of projective spaces.

For example, when $e_i > 0$ for all $i \in \{1, \cdots, k-1\}$ we can choose a compatible $\mathbb{C}_R^*$ action with weight 1 on p_j and weight 0 on each A_i. Again, its LG-phase is an Artin stack.

4.1 The example of Hori-Tong

It is clear from the preceding examples that an Artin phase is unavoidable in the nonabelian GLSM. It is surprising that some of these are related to determinantal varieties.

Example 4.3 Hori-Tong [23] have analyzed the gauged linear sigma model of the Calabi-Yau complete intersection $Z_{1,\cdots,1} \subset \mathrm{Gr}(2,7)$, which is defined by 7 linear equations in the Plücker coordinates. Suppose that

$$F_j = \sum_{i_1 i_2} a_j^{i_1 i_2} B_{i_1 i_2},$$

where $B_{i_1,i_2} = \det[x_{i_1}, x_{i_2}]$ is a skew-symmetric quadratic expression for column x_i and $a_j^{i_1 i_2} = -a_j^{i_2 i_1}$.

$$\begin{aligned} W &= \sum_{j,i_1 i_2} p_j a_j^{i_1 i_2} \det[x_{i_1}, x_{i_2}] \\ &= \sum_{i_1 i_2} \sum_j a_j^{I_1 i_2} p_j \det[x_{i_1}, x_{i_2}] = \sum_{i_1 i_2} a_{I_1 i_2}(p) \det[x_{i_1}, x_{i_2}]. \end{aligned}$$

In this case, $A = (a_{ij}(p))$ is a skew-symmetric matrix whose coordinates are linear function of $p \in \mathbb{P}^7$. Using the so-called effective theory technique from physics, Hori-Tong gave a physical derivation that its LG-phase is equivalent to the Gromov-Witten theory of the Pfaffian variety

$$\mathrm{Pf}\Big(\bigwedge^2 \mathbb{C}^7\Big) = \Big\{A \in \bigwedge^2 \mathbb{C}^7; A \wedge A \wedge A = 0\Big\}.$$

The Pfaffian $\mathrm{Pf}(A)$ is an example of a determinantal variety such that $\mathrm{rank}(A) \le 4$. It is well-known that the Pfaffian $\mathrm{Pf}(A)$ is not a complete intersection. This provides a physical explanation to some of early work of Rodland [32].

4.2 Hori-Knapp construction

In [24], Hosono and Takagi studied the derived equivalence of some interesting examples of Calabi-Yau 3-folds which are double covers of some singular variety (such as a determinantal quintic). Inspired by their work, Hori-Knapp [22] has further generalized the $\mathrm{Pf}(A)$ example into two broad classes of examples. One of feature of those examples is the usage of non-connected Lie group such as $\mathrm{O}(n)$ to capture the double cover. Let us describe their models.

Let $G = \mathrm{U}(1) \times H/\mathrm{Ker}$ with $H = \mathrm{USp}(k), \mathrm{O}(k),$ or $\mathrm{SO}(k)$ and $V = V_p \times V_x$, where $\mathrm{U}(1)$ acts on $V_p \times V_x$, and H only acts on V_x as a fundamental representation. Ker is the finite subgroup which is the kernel of action of $\mathrm{U}(1) \times H$. We assume that $\mathrm{U}(1)$ acts on V_p with negative weight and V_x with positive weight, as in our previous example of a complete intersection of flag manifolds. For the $\mathrm{O}(k)$-series, one has an additional freedom to introduce a so-called discrete torsion. This leads to a theory with different Hodge number. This additional freedom of discrete torsion was well-known in the old orbifold theory (see [1]). We use $\mathrm{O}_+(k)$ to denote the usual orbifold theory and $\mathrm{O}_-(k)$ to denote the theory with a discrete torsion.

In the symplectic case, we use the superpotential

$$W = \sum_{ij=1}^{N} A^{ij}(p)[x_i x_j],$$

where $A^{ij}(p)$ is skew-symmetric and $[x_i x_j]$ is a skew-symmetric expression of $x_i \in V_x$. In the orthogonal case, we use the superpotential

$$W = \sum_{ij=1}^{N} S^{ij}(p)(x_i x_j),$$

where S^{ij} is symmetric and $(x_i x_j)$ is a symmetric expression of $x_i \in V_x$. Moreover, we assume that $A^{ij}(p), S^{ij}(p)$ are generic.

These models have two phases: One is a geometric phase and the other is an Artin phase. Hori-Knapp's physical argument shows that the symplectic series are equivalent to certain Pfaffian varieties, while the symmetric series are equivalent to the double cover of certain symmetric determinantal varieties.

Example 4.4 Let $G = (\mathrm{U}(1) \times \mathrm{O}(2))/\{(\pm 1, \pm \mathbf{1}_2)\}$ with weights $(-2,1)$ on p^1, p^2, p^3, p^4, p^5 and weights $(1,2)$ on x_1, x_2, x_3, x_4, x_5. Moreover, we have the superpotential

$$W = \sum_{i,j,k=1}^{5} S_k^{ij} p^k (x_i x_j),$$

such that $S_k^{ij} = S_k^{ji}$ is generic in a certain sense. This is a form of an $\mathrm{O}_+(2)$-model.

This model has two phases. The geometric phase $r > 0$ is the $\mathbb{Z}_2$-quotient of a smooth complete intersection of five hypersurfaces of bi-degree $(1,1)$ in $\mathbb{P}^4 \times \mathbb{P}^4$. The Artin/strongly coupled phase $r < 0$ is supposed to be equivalent to the double cover of a determinantal variety

$$Y_S = \{p \in \mathbb{P}^4;\ \mathrm{rank}\ S(p) \leq 4\}.$$

Example 4.5 The previous example has a Seiberg-dual model (see the next section). Let $G = (\mathrm{U}(1) \times \mathrm{SO}(4))/\{(\pm 1, \pm \mathbf{1}_4)\}$ with weights $(-2,1)$ on $p^1, \cdots, p^5$, weights $(-1,4)$ on $\widetilde{x}^1, \cdots, \widetilde{x}^5$ and weights $(2,1)$ on $(s_{ij})_{1 \leq i < j \leq 5}$. Moreover, we have the superpotential

$$W = \sum_{ij} s_{ij}(\widetilde{x}^i \widetilde{x}^j) + \sum_{i,j,k} S_k^{ij} p^k s_{ij}.$$

Hori [21] showed that this model is Seiberg-dual to the previous one in the sense that it reverses the phases. Namely, it maps the $r > 0$ phase to the Artin phase of the previous example, and the $r < 0$ phase to the geometric phase of the previous example.

Example 4.6 The Hori-Tong example also has a Seiberg-dual model. Let $G = (\mathrm{U}(1) \times \mathrm{USp}(4))/\{(\pm 1, \pm \mathbf{1}_4)\}$ with weights $(-2,1)$ on $p^1, \cdots, p^7$, weights $(-1,4)$ on $\widetilde{x}^1, \cdots, \widetilde{x}^7$ and weights $(2,1)$ on $(a_{ij})_{1 \leq i < j \leq 7}$. Moreover, we have the superpotential

$$W = \sum_{i,j=1}^{7} a_{ij}[\widetilde{x}^i \widetilde{x}^j] + \sum_{i,j,k=1}^{7} A_k^{ij} p^k a_{ij}.$$

This model is supposed to reverse the phases of Hori-Tong example, i.e., mapping r to $-r$.

The following is a list of interesting examples of one-parameter Kähler moduli studied by Hori-Knapp. We follow their notation.

$A^2_{(-2)^7,1^7}$ $h^{1,1}=1$, $h^{2,1}=50$; Pfaffian in $\mathbb{P}^6$ ($r<0$ phase); Int in $G(2,7)$ ($r>0$ phase),

$S^{8,0}_{(-2)^{32},1^8}$ $h^{1,1}=1$, $h^{2,1}=65$; Sym-det in $\mathbb{P}^{31}$ ($r<0$ phase); $\mathbb{Z}_2$-quad(8) on $\mathbb{P}^3$ ($r>0$ phase),

$S^{2,+}_{(-2)^5,1^5}$ $h^{1,1}=1$, $h^{2,1}=26$; Double sym-det in $\mathbb{P}^4$ ($r<0$ phase); Reye congruence ($r>0$ phase),

$A^2_{(-1)^4,(-2)^3,1^5}$ $h^{1,1}=1$, $h^{2,1}=51$; Pfaffian in $\mathbb{P}^6_{1111222}$ ($r<0$ phase); $\mathbb{Z}_2$ quad(4) on B_5 ($r>0$ phase),

$A^2_{(-1)^6,(-2),1^4,0}$ $h^{1,1}=1$, $h^{2,1}=59$; Pfaffian in $\mathbb{P}^6_{1111112}$ ($r<0$ phase); $\mathbb{Z}_2$ quad(4) on B_4 ($r>0$ phase),

$A^2_{(-2)^7,3,1^4}$ $h^{1,1}=1$, $h^{2,1}=61$; Pfaffian in $\mathbb{P}^6$ ($r<0$ phase); $\mathbb{Z}_2$ quad(4) on $\mathbb{P}^3$ ($r>0$ phase),

$A^2_{(-2)^5,(-4)^2,3^2,1^3}$ $h^{1,1}=1$, $h^{2,1}=61$; Pfaffian in $\mathbb{P}^6_{1111122}$ ($r<0$ phase); Pseudo Hybrid ($r>0$ phase),

$S^{2,+}_{(-1)^2,(-2)^3,1^4}$ $h^{1,1}=1$, $h^{2,1}=23$; Double sym-det in $\mathbb{P}^4_{11222}$ ($r<0$ phase); $\mathbb{Z}_2$ quad(2) on B/τ ($r>0$ phase).

Let us explain the Hori-Knapp's notation of the previous examples. In $A^k_q, S^{k,\bullet}_q$, the superscript k encodes the rank of the group H and the subscript q denotes the U(1) charges; $\bullet=\pm,0$ labels the type of theory in S-class. Sym-det denotes the symmetric determinantal variety. Double sym-det denotes the certain double cover of symmetric determinantal variety. $\mathbb{Z}_2$ quad(n) on M means a hybrid model fibered over M of a $\mathbb{Z}_2$ LG orbifold of n-variables with a quadratic superpotential. B_5 is the intersection of three hyperplanes in $G(2,5)$. B_4 is a hyperplane in $G(2,4)$. B is the intersection of three symmetric bilinears in $\mathbb{P}^3\times\mathbb{P}^3$; τ is the exchange of the two $\mathbb{P}^3$s.

The work of Hori and his collaborators suggests the following mathematical problem.

Artin/Strong Coupled-Phase Problem (1) Define the virtual cycle for the Artin phase; (2) Show that it is equivalent to the Gromov-Witten theory of the Pfaffian/double cover of symmetric determinantal varieties in the Hori-Knapp examples.

4.3 Jockers-Kummar-Lapan-Morrison-Romo construction

Built on early work of Hori-Tong, Jockers-Kummer-Lapan-Morrison-Romo gave a different GLSM construction of determinantal varieties. Instead of localizing the GLSM to these examples in some LG-phase, they directly realize them in the geometric phase. The key idea is that one can desingularize them as a complete intersection of a Grassmanian bundle. Recall the set-up of the determinantal variety

$$A: E\to F,$$

where we have a generic linear homomorphism A between two vector bundles E, F over X. The determinantal variety is

$$\Omega_k(A, E, F) = \{\phi \in X; \operatorname{rank} A(\phi) \leq k\}.$$

The key observation is that we can realize the rank condition by requiring

$$A(p)x = 0$$

for an $(n-k)$-dimensional subspace of E_p. This leads to a desingularization

$$\widetilde{\Omega}_k(A, E, F) = \{(\phi, x) \in \mathrm{Gr}_{n-k}(E);\ A(\phi)x = 0\},$$

where $\mathrm{Gr}_{n-k}(E)$ is the Grassmannian bundle of $(n-k)$-planes of E. By the definition, $\widetilde{\Omega}_k(A, E, F)$ is a complete intersection of $\mathrm{Gr}_{n-k}(E)$, defined by the equation $A(p)x = 0$.

Suppose that $X = \frac{\mathbb{C}^{D+s} - F}{(\mathbb{C}^*)^s}$ is a toric variety defined by charge matrix $Q = (Q_a^l)$ for $l = 1, \cdots, s$ and $a = 1, \cdots, D+s$. Jockers-Kummar-Lapan-Morrison-Romo gave two equivalent GLSM descriptions of $\widetilde{\Omega}_k(A, E, F)$, which they called the PAX-model and the PAXY-model.

Example 4.7 In the PAX-model, we introduce additional $(n-k)\times k$-matrix-valued variables $(P_{\alpha i})$ and a superpotential

$$W = \sum_{i,j=1}^{n} \sum_{\alpha=1}^{n-k} P_{\alpha i} A(\Phi)_{ij} X_{j\alpha} = \mathrm{Tr}(PAX).$$

The GIT-quotient $V/\!\!/G$ will have $V = M_{n-k,n} \times M_{k,n} \times \mathbb{C}^{D+s}$ and $G = \mathrm{GL}(n-k) \times (\mathbb{C}^*)^s$, where $\mathrm{GL}(n-k)$ acts on X and P on left, and $(\mathbb{C}^*)^s$ acts on X_i, P_i with weights $-(q_i^l(x)), -(q_i^l(p))$ for $l = 1, \cdots, s$ and $i = 1, \cdots, n$. W is clearly $\mathrm{GL}(n-k)$-invariant. For W to be $(\mathbb{C}^*)^s$-invariant, we require

$$\deg_l(A(\phi)_{ji}) = q_i^l(x) + q_i^l(p).$$

Moreover, we have an R-charge of 1 on P and zero on the others. One can check that the geometric phase of the PAX-model realizes $\widetilde{\Omega}_k(A, E, F)$.

Example 4.8 The PAXY-model is based on the observation that $A(\phi)$ can be factored as the product of two matrices $\widetilde{x}$ and $\widetilde{y}$ of dimension $k \times n$ and $n \times k$, respectively

$$A(\phi) = \widetilde{y}x.$$

Moreover, this decomposition is unique up to $\mathrm{GL}(k)$ transformations of the form

$$\widetilde{x} \to M\widetilde{x}, \quad \widetilde{y} \to \widetilde{y}M^{-1}$$

for $M \in \mathrm{GL}(k)$. Then, we introduce an $n \times n$-matrix $\widetilde{P}_{ji}$ and a superpotential

$$W = \sum_{i,j=1}^{n} \widetilde{P}_{ji}\Big(A(\Phi)_{ij} - \sum_{\alpha=1}^{k} \widetilde{Y}_{i\alpha}\widetilde{X}_{\alpha j}\Big) = \mathrm{Tr}(\widetilde{P}(A(\Phi) - \widetilde{Y}\widetilde{X})).$$

$\mathrm{GL}(k)$ will act trivially on P. Suppose that $(\mathbb{C}^*)_l$ acts on $\widetilde{X}_i, \widetilde{Y}_i$ with weight $q_i^l(x), q_i^l(y)$. We require $q_{ij}^l(p) = q_i^l(x) + q_j^l(y)$. In this situation W is invariant under $\mathrm{GL}(k) \times (\mathbb{C}^*)^s$. Moreover, we have an R-charge of 1 on P and zero on the others.

The authors argue that the geometric phase of the PAXY-model also realizes $\widetilde{\Omega}_k(A, E, F)$, and that, in fact, they are equivalent. Generally, both models have complicated phase structures.

Here are several explicit examples studied by their technique.

Example 4.9 This example is the crepant resolution of the determinantal quintic. Here $A = \sum_{a=1}^{5} A^a \phi_a$ for constant 5×5-matrix A^a. If A is linearly dependent on ϕ_a, we call it a linear determinantal variety. Let $\mathscr{E} = \mathscr{F} = \mathscr{O}(1)^5$. Its resolution is

$$\widetilde{\Omega}_4(A, \mathscr{E}, \mathscr{F}) = \Big\{(\phi, x) \in \mathbb{P}^4 \times \mathbb{P}^4;\ \sum_a A^a \phi_a x = 0\Big\}.$$

$\widetilde{\Omega}_4(A, \mathscr{E}, \mathscr{F})$ is a Calabi-Yau 3-fold with Hodge number $h^{1,1} = 2,\ h^{2,1} = 52$. Its PAX-model has gauge group $G = \mathrm{U}(1) \times \mathrm{U}(1)$. The variables X_i, P_i, ϕ_a have weights $(-1, 0), (1, -1), (0, 1)$. Its geometric phase is supposed to describe the GW-theory of $\widetilde{\Omega}(A, \mathscr{E}, \mathscr{F})$.

Example 4.10 The next example again is a resolution of a linear determinantal variety of codimension 4 in $\mathbb{P}^7$. We have $A = \sum_{a=1}^{8} A^a \phi_a$ for constant 4×4 matrices. The determinantal variety $\Omega_2(A, \mathscr{E}, \mathscr{F})$ is the locus of $\mathbb{P}^7$ such that $\mathrm{rank}(A) \leq 2$. Its crepant resolution

$$\widetilde{\Omega}_2(A, \mathscr{E}, \mathscr{F}) = \Big\{(\phi, x) \in \mathbb{P}^4 \times Gr(2, 4);\ \sum_a A^a \phi_a x = 0\Big\}$$

is a Calabi-Yau 3-fold with Hodge number $h^{1,1} = 2,\ h^{2,1} = 34$. This example was first studied by Gulliksen and Negard. Its PAX-model has gauge group $G = \mathrm{U}(2) \times \mathrm{U}(1)$. The variables X_i, P_i, ϕ_a have weights $(\overline{\mathbf{2}}, 0), (\mathbf{2}, -1), (\mathbf{1}, 1)$. Again, its geometric phase realizes $\widetilde{\Omega}_2(A, \mathscr{E}, \mathscr{F})$.

Example 4.11 Let $\mathscr{E} = \mathscr{O}^{\oplus 3},\ \mathscr{F} = \mathscr{O}(1)^{\oplus 2} \oplus \mathscr{O}(2)$ and $A : \mathscr{E} \to \mathscr{F}$. We consider the determinantal variety

$$\Omega_1(A, \mathscr{E}, \mathscr{F}) = \{p \in \mathbb{P}^7;\ \mathrm{rank}\ (A) \leq 1\}.$$

For generic A, the variety $\Omega_1(A, \mathscr{E}, \mathscr{F})$ is smooth (hence isomorphic to $\widetilde{\Omega}_1(A, \mathscr{E}, \mathscr{F})$) and is called the Bertin Calabi-Yau 3-fold. It has Hodge number $h^{1,1} = 2,\ h^{2,1} = 58$. Its PAX-model has gauge group $G = \mathrm{U}(2) \times \mathrm{U}(1)$. Its PAXY-model has gauge group $\mathrm{U}(1) \times \mathrm{U}(1)$. The variables $\widetilde{X}_i, \widetilde{P}_{i1}, \widetilde{P}_{i2}, \widetilde{P}_{i3}, \phi_a$ have weights $(0, 1), (-1, 0), (-1, 0), (-2, 0), (1, 0)$. The variables $\widetilde{Y}_{1,2}$ and $\widetilde{Y}_3$ have weights $(1, -1), (2, -1)$. This example has a more complicated phase structure due to the presence of two copies of $\mathrm{U}(1)$. Its geometric phase realizes $\widetilde{\Omega}_1(A, \mathscr{E}, \mathscr{F}) = \Omega_1(A, \mathscr{E}, \mathscr{F})$.

One can also fits Hori-Tong-Knapp's examples in the above construction.

Suppose that $A(\phi)$ is a symmetric $n \times n$-matrix of rank k. Then

$$A = \widetilde{x}^{\mathrm{T}} \widetilde{x}$$

for a rank-k, $k \times n$ matrix $\widetilde{x}$, which is unique up to a complexified $\mathrm{O}(k)$-symmetry.

Suppose that $A(\phi)$ is a skew-symmetric, $n \times n$ matrix of rank $k = 2l$ (skew-symmetric matrices can only have even rank). In this case

$$A = \widetilde{x}^{\mathrm{T}} \begin{pmatrix} \mathbf{0}_l & \mathbf{1}_l \\ -\mathbf{1}_l & \mathbf{0}_l \end{pmatrix} \widetilde{x},$$

where $\widetilde{x}$ is unique up to the action of a complexified $\mathrm{USp}(k)$-action. It is clear that one can realize both cases using a PAXY-model.

5 Quiver Varieties and Seiberg Duality

Quiver varieties consist of a rich class of nonabelian GIT quotients that can be described combinatorially. It has a deep connection to geometric representations via the work of Nakajima. We will refer readers to a beautiful book on the foundation of quiver representation and quiver varieties.

Definition 5.1 *A quiver diagram is a finite oriented graph* $(Q_{fr} \subset Q_0, Q_1)$ *such that* Q_0 *is the set of vertices/nodes* ($Q_{fr} \subset Q_0$ *is a subset called the frame vertices* (*often denoted by* $\square$)*. The non-frame vertices are called the gauge vertices* (*often denoted by* $\bigcirc$)) *and* Q_1 *is the set of edges/arrows. A potential of a quiver is a finite sum of cycles of the quiver. Given a quiver diagram with an assignment of a complex vector space* V_i *to each vertex* $i \in Q_0$ *and a reductive Lie group* $G_i \subset \mathrm{GL}(V_i)$ *for each gauge vertex, the correpsonding quiver variety is a GIT quotient* $V/\!\!/_\theta G$ *where* $V = \bigoplus\limits_{i \to j \in Q_1} \mathrm{Hom}(V_i, V_j)$, $G = \prod\limits_{i \in Q_0 - Q_{fr}} G_i$ *and* θ *is some polarization character.*

Given a cycle $i_1 \to i_2 \to \cdots \to i_k \to i_0$, *we can define a* G*-invariant function*

$$\mathrm{tr}(A_{i_1 \to i_k} A_{i_k \to i_{k-1}} \cdots A_{i_2 \to i_1})$$

for $A_{i_j \to i_{j-1}} \in \mathrm{Hom}(V_{i_j}, V_{i_{j-1}})$. *A potential naturally induces a* G*-invariant holomorphic function* W *on* V, *which we can use as the superpotential in the GLSM. An R-charge is a* $\mathbb{C}^*$*-action on each* $\mathrm{Hom}(V_i, V_j)$ *such that* W *is of a degree* 1. *In this case, we call* W *a graded potential.*

Quiver varieties parametrize the moduli space of quiver representations. They also appear as the vacuum moduli space in physics of the quiver gauged theory.

Example 5.1 Flag varieties are an example of the A_n-type quiver

$$\bigcirc \to \bigcirc \cdots \to \square,$$

where the last one is a framed vertex. Here we assign $\mathbb{C}^{d_i}$ to each vertex and $\mathrm{GL}(d_i)$ to each gauge vertex. One can define its hypersurface in a similar fashion as that of flag variety. Namely, any character of G_i defines a line bundle over the quiver variety. A hypersurface is a zero locus of a product of sections of these line bundles. One can formulate the GLSM for its complete intersection in the same way.

Other famous examples are the ADE-quivers or affine ADE-quivers.

Example 5.2 A Nakajima quiver is a quiver where we double all the arrows in the opposite direction. Namely, we start from a usual quiver and then add an opposite arrow $\overline{Q}_1$. Then, we add an additional arrow on each gauge node. The resulting quiver variety is $V \oplus V^* /\!\!/_\theta G$. A Nakajima quiver is naturally a holomorphic symplectic manifold and is extremely important in geometric representation theory. In physical language, a Nakajima quiver corresponds to the $N = 4$ gauged theory, while a usual quiver corresponds to the $N = 2$ gauged theory.

5.1 Local Seiberg duality

As we mentioned previously, quiver varieties arise in physics as the vacuum moduli space of quiver gauge theory. The famous Seiberg duality predicted that certain pairs of quiver gauge theory should be dual to each other. One can formulate it as a mathematical conjecture that the corresponding GLSMs are equivalent. The full range of Seiberg duality in the GLSM is not yet understood. Let us first describe some local cases by [2, 21].

The easiest example of Seiberg duality is based on the following construction.

Example 5.3 Consider a quiver with two nodes $k \to \boxed{N}$ with rank N on the frame vertex and k on the gauge node. We assume $N > k$. The corresponding quiver variety is the Grassmannian $G(k, N)$. We invert the arrow $k \leftarrow \boxed{N}$. The resulting quiver variety is $G(N - k, N)$, which is canonically isomorphic to $G(k, N)$. After replacing k by $N - k$, we get back to itself.

The simplest Seiberg duality is the following example.

Example 5.4 A slight generalization of the above example is the quiver $\boxed{N_a} \to N \to \boxed{N_f}$, where the middle vertex is a gauge vertex and other two are framed nodes. Suppose that $N_f \geq N_a > N$. The corresponding GIT quotient has two phases. The positive phase $r > 0$ is the total space $S^{\oplus N_a} \to G(N, N_f)$, where $S \to G(N, N_f)$ is the tautological bundle. The negative phase $r < 0$ is $Q^{\oplus N_f} \to G(N_a - N, N_a)$ for the quotient bundle Q. S and Q are related by the following exact sequence

$$0 \to S \to \mathscr{O}^N \to Q \to 0.$$

Seiberg duality consists of (1) adding an arrow from $N_a \to N_f$ and thinking of it as the composition of arrows from $N_a \to N \to N_f$; (2) inverting the direction of arrows $N_a \to N, N \to N_f$ and changing N to $N_f - N$ to obtain $N_a \leftarrow N_f - N \leftarrow N_f$; (3) introducing a superpotential from 3-cycle $N_f \to N_f - N \to N_a \to N_f$.

Example 5.5 For Hori-Knapp examples, Hori proposed the following dualities:

For $N \geq k$,

$$\mathrm{O}_+(k) \Leftrightarrow \mathrm{SO}(N - k + 1),$$

$$\mathrm{SO}(k) \Leftrightarrow \mathrm{O}_+(N - k + 1),$$

$$\mathrm{O}_-(k) \Leftrightarrow \mathrm{O}_-(N - k + 1);$$

for $N \geq k+3$,

$$\mathrm{USp}(k) \Leftrightarrow \mathrm{USp}(N-k-1).$$

Furthermore, the preceding duality interchanges the geometric/weakly coupled phase with the Artin/strongly coupled phase.

5.2 Global Seiberg duality

The global case can be cast into operations on quiver diagrams. A beautiful example by Benini-Park-Zhao is mutation for $\mathrm{U}(n)$ quivers. Cases other than $\mathrm{U}(n)$ have not yet been worked out.

We restrict ourselves to the case where the gauge group at each gauge node is $\mathrm{GL}(N)$. For these quiver varieties, the only additional data beyond the quiver diagram is the rank N for each node.

Suppose that k is a node with dimension N_k and no 1-cycles or 2-cycle involving k. Let $i_1, \cdots, i_l$ be the incoming nodes that have an arrow starting at i_j and ending at k. Let a_j be the number of arrows between i_i and k. We define $N_{\text{in}} = \sum_j a_j N_j$. In a similar fashion, we can also consider outgoing nodes and define N_{out}. Assume $\max\{N_{\text{in}}, N_{\text{out}}\} > N_k$. A mutation at a gauge node k consists of the following steps:

Step 1 For each "path" (a sequence of two arrows) $i \to k \to j$ passing through k, add an arrow $i \to j$ (denoted by $i \xrightarrow{k} j$). Moreover, the R-charge on $i \xrightarrow{k} j$ is the sum of R-charges on $i \to k$ and $k \to j$. Suppose that we have a cycle containing this path in the superpotential W. We have a new cycle of the same degree to replace $i \to k \to j$ by $i \xrightarrow{k} j$.

Step 2 We replace N_k by $N_k' = \max\{N_{\text{in}}, N_{\text{out}}\} - N_k$.

Step 3 Invert the direction of all arrows that start or end at k (denoted by $(i \to k)^*, (k \to j)^*$). Furthermore, introduce a cubic term of superpotential for the cycle $i \xrightarrow{k} j, (k \to j)^*, (i \to k)^*$ by assigning the R-charge of $\frac{1}{2}$ of the old R-charge for arrows $(k \to j)^*, (i \to k)^*$.

Remark 5.1 Usually, the mutation is performed for so-called cluster quiver which has no 1-cycles or 2-cycles. We also require the removal of a pair of opposite arrows between two vertices to kill all possible 2-cycles. If we start from a potential containing appropriate cubic terms, the new potential may have some quadratic term. The last step can be realized as the restriction to the critical locus of quadratic terms of new superpotential. The GLSM is equivalent to its restriction on the critical locus of the quadratic terms of the superpotential. Furthermore, mutation is closely related to the cluster algebra.

We apply Fan-Jarvis-Ruan's theory to quiver varieties, before and after mutations, to obtain their generating functions $\mathscr{F}_g^{bf}, \mathscr{F}_g^{af}$. Recall that we have a quantum variable q_i for each vertex. Benini-Park-Zhao's physical analysis suggests the following mathematical conjecture.

Mutation Conjecture $\mathscr{F}_g^{bf}$ and $\mathscr{F}_g^{af}$ are equivalent up to the change of variables

$$\widetilde{q}_i = \begin{cases} q_k^{-1}, & \text{if } \ i = k, \\ q_i q^{|a_{ki}|}(q_k+1)^{-a_{ki}}, & \text{if } \ \text{Otherwise}, \end{cases}$$

where a_{ki} is the number of arrows (with a sign) between k and i. Namely, we count $+1$ for an arrow $k \to i$ and -1 for an arrow $i \to k$.

Recall that we can think of the LG-phase as being opposite to the geometric phase. In the conjecture, z^{-1} means to go to the opposite phase.

6 Mirror Symmetry and Others

In the abelian case, the mirror symmetry construction of Givental-Hori-Vafa plays a very important role in our understanding of the GLSM. We naturally expect the same for the nonabelian case as well. In many ways, mirror symmetry for the nonabelian case should be even more interesting. For example, early work of Givental-Kim [20] showed that the quantum cohomology of the flag manifold is related to the Toda lattice. Recently, there are some remarkable works by Nekrasov-Shatashvili from the physical side and Maulik-Okounkov from mathematical side to connect the equivariant quantum cohomology of Nakajima quiver varieties to quantum groups in the form of a quantum integrable system/gauge theory correspondence. This is part of larger story in physics called the AGT-correspondence. In many ways, integrable systems should be treated as a B-model object. This naturally leads to the following question.

Nonabelian B-model Problem (1) Build a B-model mirror of nonabelian GIT-quotients and their complete intersections; (2) Incorporate quantum integrable systems such as representations of a quantum group as the part of the B-model theory for Nakajima quiver varieties.

Once we have done this, all these remarkable results are the consequence of mirror symmetry. I believe that this new strategy will provide us a deeper understanding of existing results. Moreover, it will open a door for a representation-theoretic interpretation of higher genus GW-invariants of these targets.

Building an appropriate B-model mirror for a general GIT-quotient such as quiver varieties is still a challenging problem for both mathematicians and physicists. A good starting point is a series of works [28, 30] of Rietsch and her collaborators where she proposed a B-model mirror for general flag varieties G/P. Unfortunately, the author's limited background in Lie theory prevents him from giving an in-depth survey of the construction. It is certainly on the list of paper he wants to study carefully.

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