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Geometry of the Second-Order Tangent Bundles of Riemannian Manifolds

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Abstract Let (M,g) be an n-dimensional Riemannian manifold and T^2M be its second-order tangent bundle equipped with a lift metric $\widetilde{g}$. In this paper, first, the authors construct some Riemannian almost product structures on $(T^2M,\widetilde{g})$ and present some results concerning these structures. Then, they investigate the curvature properties of $(T^2M,\widetilde{g})$. Finally, they study the properties of two metric connections with nonvanishing torsion on $(T^2M,\widetilde{g})$: The H-lift of the Levi-Civita connection of g to T^2M, and the product conjugate connection defined by the Levi-Civita connection of $\widetilde{g}$ and an almost product structure.

Keywords Almost product structure, Killing vector field, Metric connection, Riemannian metric, Second-order tangent bundle
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1 Introduction

Given an n-dimensional manifold M, a second-order tangent bundle T^2M over M can be constructed from the equivalent classes of curves on M which agree up to their acceleration (for details, see [7, 16]). Moreover, in [7], it is proved that a second-order tangent bundle T^2M becomes a vector bundle over M if and only if M has a linear connection. The prolongations of tensor fields and connections given on M to its second-order tangent bundle T^2M were studied in [16]. Let (M,g) be an n-dimensional Riemannian manifold and T^2M be its second-order tangent bundle. A lift metric $\widetilde{g}$ of the metric g to T^2M was defined and studied by local coordinates in [10]. In this paper, we study some geometric properties of T^2M equipped with the metric $\widetilde{g}$ and almost product structures. In tensor calculations, we usually use the method of adapted frame which allows the tensor calculus to be done efficiently.

We point out here and once that all geometric objects considered in this paper are supposed to be of class C^∞.

2 Preliminaries

Let (M,g) be an n-dimensional Riemannian manifold and ∇ be the Levi-Civita connection of g. The second-order tangent bundle T^2M of M is the $3n$-dimensional manifold of 2-jets j^2f at $0\in\mathbb{R}$ of differentiable curves: $f:\mathbb{R}\to M$, where $\mathbb{R}$ denotes the set of real numbers. The canonical projection $\pi_2:T^2M\to M$ defines the natural bundle structure of T^2M over M. If we introduce the canonical projection $\pi_{12}:T^2M\to TM$, then T^2M has a bundle structure

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over the tangent bundle TM with projection π_{12}. Let (U, x^i) be a system of coordinates in M and f be a curve in U which locally expressed as $x^i = f^i(t)$. If we take a 2-jet $j^2 f$ belonging to $\pi_2^{-1}(U)$ and define

$$x^i = f^i(0), \quad y^i = \frac{\mathrm{d} f^i}{\mathrm{d} t}(0), \quad z^i = \frac{\mathrm{d}^2 f^i}{\mathrm{d} t^2}(0),$$

then the 2-jet $j^2 f$ is expressed uniquely by the set (x^i, y^i, z^i). Thus, (x^i, y^i, z^i) is the system of coordinates induced in $\pi_2^{-1}(U)$ from (U, x^i). The coordinates (x^i, y^i, z^i) in $\pi_2^{-1}(U)$ are called the induced coordinates. By defining

$$\xi^i = x^i, \quad \xi^{\bar{i}} = y^i, \quad \xi^{\bar{\bar{i}}} = z^i,$$

we write the induced coordinates (x^i, y^i, z^i) as $\{\xi^A\}$. The indices $A, B, C, \cdots$ run over the range $\{1, 2, \cdots, n;\ n+1, n+2, \cdots, 2n;\ 2n+1, 2n+2, \cdots, 3n\}$.

Let us define some lifts of vector fields. Using the connection ∇, a vector bundle isomorphism can be defined by

$$\begin{aligned} S : T^2M &\to TM \oplus TM, \\ : j^2 f &\to (\dot{f}(0), (\nabla_{\dot{f}(0)} \dot{f})(0)), \end{aligned}$$

where $TM \oplus TM$ is the Whitney sum of the tangent bundle TM with itself, the tangent vector $\dot{f}(0)$ to f at $f(0)$ is the velocity vector field of f at $f(0)$, and the covariant derivative $(\nabla_{\dot{f}(0)} \dot{f})(0)$ is the covariant acceleration of f at $f(0)$. Using the induced coordinates $\{\xi^A\}$ in T^2M and $TM \oplus TM$, we have

$$S : (x^i, y^i, z^i) \to (x^i, y^i, \omega^i),$$

where $\omega^i = z^i + y^s y^r \Gamma^i_{sr}$. Each vector field X on M defines the vector fields ${}^0X$, ${}^IX$ and ${}^{II}X$ on T^2M corresponding, respectively, in the isomorphism S to ${}^HX + {}^HX$, ${}^VX + 0$ and $0 + {}^VX$, where ${}^VX$ and ${}^HX$ are the vertical and the horizontal lifts of X to TM with respect to ∇. The vector fields ${}^0X$, ${}^IX$ and ${}^{II}X$ on T^2M are called λ-lifts ($\lambda = 0, I, II$) of a vector field X on M to T^2M. Also note that the λ-lifts ${}^\lambda X$ of a vector field X in this paper are different from those in [16]. Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the vector fields ${}^0X$, ${}^IX$ and ${}^{II}X$ on T^2M are given, with respect to the induced coordinates $\{\xi^A\}$, by (see [4])

$$ {}^0X = X^j \partial_j - y^s \Gamma^j_{sh} X^h \partial_{\bar{j}} - C^j_h X^h \partial_{\bar{\bar{j}}}, \tag{2.1}$$

$$ {}^IX = X^j \partial_{\bar{j}} - 2y^s \Gamma^j_{sh} X^h \partial_{\bar{\bar{j}}}, \tag{2.2}$$

$$ {}^{II}X = X^j \partial_{\bar{\bar{j}}} \tag{2.3}$$

with respect to the natural frame $\{\partial_A\} = \{\partial_i, \partial_{\bar{i}}, \partial_{\bar{\bar{i}}}\}$ in T^2M, where $C^j_h = z^m \Gamma^j_{hm} + y^s y^r (\partial_h \Gamma^j_{sr} + \Gamma^j_{hm} \Gamma^m_{sr} - 2\Gamma^j_{sm} \Gamma^m_{hr})$, Γ^j_{sr} are the coefficients of the connection ∇ on M and $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$, $\partial_{\bar{\bar{i}}} = \frac{\partial}{\partial z^i}$. For the Lie bracket on T^2M in terms of the λ-lifts of vector fields X, Y on M, we have the following formulas:

$$\begin{cases} [{}^0X, {}^0Y] = {}^0[X, Y] - {}^I(R(X, Y)y) - {}^{II}(R(X, Y)\omega), \\ [{}^0X, {}^IY] = {}^I(\nabla_X Y), \quad [{}^0X, {}^{II}Y] = {}^{II}(\nabla_X Y), \\ [{}^\mu X, {}^\lambda Y] = 0, \quad \mu, \lambda = I, II, \end{cases} \tag{2.4}$$

where R is the curvature tensor field of the connection ∇ on M defined by $R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$ (for details, see [6]).

With the connection ∇ of g on M, we can introduce a frame field in each induced coordinate neighborhood $\pi_2^{-1}(U)$ of T^2M. In each coordinate neighborhood (U,x^i), using $X_i=\frac{\partial}{\partial x^i}$, from (2.1)–(2.3) we have

$$\begin{aligned} E_i &= {}^0(X_i) = {}^0\Big(\frac{\partial}{\partial x^i}\Big) = \partial_i - y^s\Gamma^k_{is}\partial_{\overline{k}} - C^k_i\partial_{\overline{\overline{k}}}, \\ E_{\overline{i}} &= {}^I(X_i) = {}^I\Big(\frac{\partial}{\partial x^i}\Big) = \partial_{\overline{i}} - 2y^s\Gamma^k_{is}\partial_{\overline{\overline{k}}}, \\ E_{\overline{\overline{i}}} &= {}^{II}(X_i) = {}^{II}\Big(\frac{\partial}{\partial x^i}\Big) = \partial_{\overline{\overline{i}}} \end{aligned}$$

with respect to the natural frame $\{\partial_A\}$ in T^2M (see [4]). The $3n$ linearly independent vector fields $E_i, E_{\overline{i}}$ and $E_{\overline{\overline{i}}}$ constitute the adapted frame in $\pi_2^{-1}(U)$. We write the adapted frame $\{E_\beta\}=\{E_i,E_{\overline{i}},E_{\overline{\overline{i}}}\}$. The indices $\alpha,\beta,\gamma,\cdots$ run over the range $\{1,2,\cdots,n;\ n+1,n+2,\cdots,2n;\ 2n+1,2n+2,\cdots,3n\}$ and indicate the indices with respect to the adapted frame $\{E_\beta\}$. The matrix of frames changes $E_\beta = A_\beta{}^B\partial_B$ is in the following form:

$$A_\beta{}^B = \begin{pmatrix} \delta^k_i & 0 & 0 \\ -y^s\Gamma^k_{is} & \delta^k_i & 0 \\ -C^k_i & -2y^s\Gamma^k_{is} & \delta^k_i \end{pmatrix}.$$

The inverse of the matrix above is as follows:

$$\widetilde{A}^\alpha{}_B = \begin{pmatrix} \delta^k_i & 0 & 0 \\ y^s\Gamma^k_{is} & \delta^k_i & 0 \\ C^k_i + 2y^sy^r\Gamma^k_{sm}\Gamma^m_{ri} & 2y^s\Gamma^k_{is} & \delta^k_i \end{pmatrix}.$$

Using ${}^\lambda X^\alpha = \widetilde{A}^\alpha{}_B\,{}^\lambda X^B$, for $\lambda = 0, I, II$, from (2.1)−(2.3), we get

$${}^0X = X^iE_i, \quad {}^IX = X^iE_{\overline{i}}, \quad {}^{II}X = X^iE_{\overline{\overline{i}}} \tag{2.5}$$

with respect to the adapted frame $\{E_\beta\}$.

3 The Riemannian Metric and Its Levi-Civita Connection

A Riemannian metric on the second-order tangent bunde T^2M over a Riemannian manifold (M,g) is defined by the identities:

$$\begin{cases} \widetilde{g}({}^\lambda X, {}^\mu Y) = g(X,Y), & \lambda=\mu, \\ \widetilde{g}({}^\lambda X, {}^\mu Y) = 0, & \lambda\neq\mu \end{cases}$$

for all vector fields X, Y on M, where $\lambda,\mu = 0, I, II$. The Riemannian metric $\widetilde{g}$ and its inverse have components

$$\widetilde{g}_{\beta\gamma} = \begin{pmatrix} g_{ij} & 0 & 0 \\ 0 & g_{ij} & 0 \\ 0 & 0 & g_{ij} \end{pmatrix} \quad \text{and} \quad \widetilde{g}^{\gamma\alpha} = \begin{pmatrix} g^{jk} & 0 & 0 \\ 0 & g^{jk} & 0 \\ 0 & 0 & g^{jk} \end{pmatrix} \tag{3.1}$$

with respect to the adapted frame $\{E_\beta\}$ (see [10]).

In order to use in some calculation, we compute the structure constants of the Lie Algebra referred to the adapted frame $\{E_\beta\}$ are denoted $\Omega_{\beta\gamma}{}^{\varepsilon}$, so

$$[E_\beta, E_\gamma] = \Omega_{\beta\gamma}{}^{\varepsilon} E_\varepsilon.$$

It follows that the possible nonzero structure constants are given by

$$\begin{cases} \Omega_{ij}{}^{\overline{k}} = y^p R_{jip}{}^k, & \Omega_{i\overline{j}}{}^{\overline{k}} = \Gamma_{ij}^k, \\ \Omega_{ij}{}^{\overline{\overline{k}}} = \omega^s R_{jis}{}^k, & \Omega_{i\overline{\overline{j}}}{}^{\overline{\overline{k}}} = \Gamma_{ij}^k, \end{cases} \tag{3.2}$$

where $R_{jis}{}^k$ are the local components of the curvature tensor R of the connection ∇ on M (see [4]).

We shall examine the Lie derivatives of $\widetilde{g}$ with respect to vector fields ${}^0X$, ${}^IX$ and ${}^{II}X$. The components of $L_{\widetilde{X}}\widetilde{g}$ with respect to the adapted frame $\{E_\beta\}$ can be written as

$$(L_{\widetilde{X}}\widetilde{g})_{\gamma\beta} = \widetilde{X}^\varepsilon E_\varepsilon \widetilde{g}_{\gamma\beta} + \widetilde{g}_{\varepsilon\beta} E_\gamma \widetilde{X}^\varepsilon + \widetilde{g}_{\gamma\varepsilon} E_\beta \widetilde{X}^\varepsilon - \widetilde{X}^\varepsilon(\Omega_{\varepsilon\gamma}{}^{\delta}\widetilde{g}_{\delta\beta} + \Omega_{\varepsilon\beta}{}^{\delta}\widetilde{g}_{\gamma\delta}).$$

By virtue of (2.5) and (3.2), we get the lemma below.

Lemma 3.1 *If $X = X^h\partial_h$ is a vector field on M, then*

(i) $L_{{}^0X}\widetilde{g} = \begin{pmatrix} L_X g_{ij} & y^p X^h R_{hipj} & \omega^s X^h R_{hisj} \\ y^p X^h R_{hjpi} & 0 & 0 \\ \omega^s X^h R_{hjsi} & 0 & 0 \end{pmatrix}$,

(ii) $L_{{}^IX}\widetilde{g} = \begin{pmatrix} 0 & \nabla_i X_j & 0 \\ \nabla_j X_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

(iii) $L_{{}^{II}X}\widetilde{g} = \begin{pmatrix} 0 & 0 & \nabla_i X_j \\ 0 & 0 & 0 \\ \nabla_j X_i & 0 & 0 \end{pmatrix}$

with respect to the adapted frame $\{E_\beta\}$ in T^2M.

Recall that a vector field X is a Killing vector field or an infinitesimal isometry with respect to g if and only if $L_X g = 0$. It is also known that if X is a Killing vector field in (M, g), then $\nabla_i\nabla_p X_j + X^h R_{hipj} = 0$. As a consequence of the vanishing second covariant derivative of X, we get $X^h R_{hipj} = 0$. Conversely, from $L_X g = 0$ and $X^h R_{hipj} = 0$, we can say that the second covariant derivative of X vanishes. Thus, by Lemma 3.1, we obtain the following result.

Proposition 3.1 *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$.*

(i) *The 0-lift ${}^0X$ of a vector field X on M is a Killing vector field on $(T^2M, \widetilde{g})$ if and only if X is a Killing vector field with vanishing second covariant derivative on (M, g).*

(ii) *The I-lift ${}^IX$ and II-lift ${}^{II}X$ of a vector field X on M are both Killing vector fields on $(T^2M, \widetilde{g})$ if and only if X is a parallel vector field on (M, g).*

Remark 3.1 Relations between certain Killing vector fields of (M, g) and $(T^2M, \widetilde{g})$ were established in [9, Theorem 6]. But the result is wrong. We revise the result in [9].

In [9], the authors attempted to calculate the Chrystoffel symbols of the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ with respect to the adapted frame by $\widetilde{\Gamma}^\varepsilon_{\beta\gamma}$. But unfortunately their calculations are wrong. We shall correct the errors and obtain valid expressions for $\widetilde{\Gamma}^{\varepsilon}_{\beta\gamma}$. Since the connection $\widetilde{\nabla}$ is torsion-free, we have

$$\widetilde{\Gamma}^{\varepsilon}_{\beta\gamma} - \widetilde{\Gamma}^{\varepsilon}_{\gamma\beta} = \Omega_{\beta\gamma}{}^{\varepsilon}$$

with respect to the adapted frame $\{E_\beta\}$. Thus by virtue of $\widetilde{\nabla}\widetilde{g} = 0$, we get

$$\widetilde{\Gamma}^{\varepsilon}_{\beta\gamma} = \frac{1}{2}\widetilde{g}^{\varepsilon\alpha}(E_\beta \widetilde{g}_{\alpha\gamma} + E_\gamma \widetilde{g}_{\alpha\beta} - E_\alpha \widetilde{g}_{\beta\gamma}) + \frac{1}{2}(\Omega_{\beta\gamma}{}^{\varepsilon} + \Omega^{\varepsilon}_{\cdot\ \beta\gamma} + \Omega^{\varepsilon}_{\cdot\ \gamma\beta}),$$

where $\Omega^{\varepsilon}_{\cdot\ \beta\gamma} = \widetilde{g}^{\alpha\varepsilon}\widetilde{g}_{\delta\gamma}\Omega_{\alpha\beta}{}^{\delta}$. On taking account of (3.1)–(3.2) and writing $\widetilde{\nabla}_{E_\beta} E_\gamma = \widetilde{\Gamma}^{\varepsilon}_{\beta\gamma} E_\varepsilon$, we have the following proposition.

Proposition 3.2 *The Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ on T^2M is given by*

$$\begin{cases} \widetilde{\nabla}_{E_i} E_j = \Gamma^k_{ij} E_k + \frac{1}{2} y^p R_{jip}{}^k E_{\overline{k}} + \frac{1}{2}\omega^s R_{jis}{}^k E_{\overline{\overline{k}}}, \\ \widetilde{\nabla}_{E_{\overline{i}}} E_j = \frac{1}{2} y^p R_{pij}{}^k E_k, \quad \widetilde{\nabla}_{E_i} E_{\overline{j}} = \frac{1}{2} y^p R_{pji}{}^k E_k + \Gamma^k_{ij} E_{\overline{k}}, \\ \widetilde{\nabla}_{E_{\overline{\overline{i}}}} E_j = \frac{1}{2}\omega^s R_{sij}{}^k E_k, \quad \widetilde{\nabla}_{E_i} E_{\overline{\overline{j}}} = \frac{1}{2}\omega^s R_{sji}{}^k E_k + \Gamma^k_{ij} E_{\overline{\overline{k}}}, \\ \widetilde{\nabla}_{E_{\overline{i}}} E_{\overline{j}} = 0, \quad \widetilde{\nabla}_{E_{\overline{\overline{i}}}} E_{\overline{j}} = 0, \quad \widetilde{\nabla}_{E_{\overline{\overline{i}}}} E_{\overline{\overline{j}}} = 0, \quad \widetilde{\nabla}_{E_{\overline{i}}} E_{\overline{\overline{j}}} = 0 \end{cases}$$

with respect to the adapted frame $\{E_\beta\}$.

Recall that a vector field $X = X^j\partial_j$ on a Riemannian manifold is called incompressible if it satisfies the following condition:

$$\nabla_j X^j = 0.$$

By virtue of Proposition 3.2, we then get

$$\begin{aligned} \widetilde{\nabla}_J\ {}^0\widetilde{X}^J &= \widetilde{\nabla}_j\ (X^j E_j) = (E_j X^j)E_j + X^j \widetilde{\nabla}_{E_j} E_j \\ &= (\partial_j X^j)E_j + X^k(\Gamma^j_{jk})E_j = (\nabla_j X^j)E_j, \\ \widetilde{\nabla}_J\ {}^I\widetilde{X}^J &= \widetilde{\nabla}_{\overline{j}}\ (X^j E_{\overline{j}}) = (E_{\overline{j}} X^j)E_{\overline{j}} + X^j \widetilde{\nabla}_{E_{\overline{j}}} E_{\overline{j}} = 0, \\ \widetilde{\nabla}_J\ {}^{II}\widetilde{X}^J &= \widetilde{\nabla}_{\overline{\overline{j}}}\ (X^j E_{\overline{\overline{j}}}) = (E_{\overline{\overline{j}}} X^j)E_{\overline{\overline{j}}} + X^j \widetilde{\nabla}_{E_{\overline{\overline{j}}}} E_{\overline{\overline{j}}} = 0. \end{aligned}$$

Thus, we have the proposition below.

Proposition 3.3 *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$.*

(i) *The 0-lift ${}^0X$ of a vector field X on M is an incompressible vector field on $(T^2M, \widetilde{g})$ if and only if X is an incompressible vector field on (M, g).*

(ii) *The I-lift ${}^IX$ and II-lift ${}^{II}X$ of a vector field X on M are both incompressible vector fields on $(T^2M, \widetilde{g})$.*

Let $\widetilde{X} = \widetilde{X}^\varepsilon E_\epsilon$ and $\widetilde{Y} = \widetilde{Y}^\varepsilon E_\varepsilon$ be vector fields on T^2M. Then the covariant derivative $\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}$ has components

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}^\alpha = \widetilde{X}^\varepsilon E_\varepsilon \widetilde{Y}^\alpha + \widetilde{\Gamma}^\alpha_{\varepsilon\beta}\widetilde{Y}^\beta \widetilde{X}^\varepsilon$$

with respect to the adapted frame $\{E_\beta\}$, where $\widetilde{\Gamma}^\alpha_{\varepsilon\beta}$ are components of the Levi-Civita connection $\widetilde{\nabla}$ of the metric $\widetilde{g}$. Then, using Proposition 3.2 and (2.5), we can make the following proposition for pairs $\widetilde{X}$ and $\widetilde{Y}$.

Proposition 3.4 *The Levi-Civita connection* $\widetilde{\nabla}$ *of* $\widetilde{g}$ *on* T^2M *is given by the following conditions:*

$$\begin{cases} \widetilde{\nabla}_{^0X}\ {}^0Y = {}^0(\nabla_X Y) + \frac{1}{2}{}^I(R(Y,X)y) + \frac{1}{2}{}^{II}(R(Y,X)\omega), \\ \widetilde{\nabla}_{^IX}\ {}^0Y = \frac{1}{2}{}^0(R(y,X)Y), \quad \widetilde{\nabla}_{^0X}\ {}^IY = \frac{1}{2}{}^0(R(y,Y)X) + {}^I(\nabla_X Y), \\ \widetilde{\nabla}_{^{II}X}\ {}^0Y = \frac{1}{2}{}^0(R(\omega,X)Y), \quad \widetilde{\nabla}_{^0X}\ {}^{II}Y = \frac{1}{2}{}^0(R(\omega,Y)X) + {}^{II}(\nabla_X Y), \\ \widetilde{\nabla}_{^IX}\ {}^IY = 0, \quad \widetilde{\nabla}_{^{II}X}\ {}^IY = 0, \quad \widetilde{\nabla}_{^{II}X}\ {}^{II}Y = 0, \quad \widetilde{\nabla}_{^IX}\ {}^{II}Y = 0 \end{cases}$$

for all vector fields X, Y *on* M *(see also* [5]*).*

4 Riemannian Almost Product Structures on the Second-Order Tangent Bundle

A Riemannian almost product manifold (M, J, g) is a manifold M with a $(1,1)$-tensor field J and a Riemannian metric g such that

$$\begin{aligned} &J^2 = \mathrm{id}, \quad J \neq \pm\mathrm{id}, \\ &g(JX, Y) = g(X, JY) \end{aligned} \tag{4.1}$$

for all vector fields X and Y on M. (4.1) is referred to as the condition for g to be pure with respect to J. An integrable Riemannian almost product manifold whose the Nijenhuis tensor N_J, determined by

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y]$$

for all vector fields X and Y on M, is zero is usually called a locally Riemannian product manifold. The Riemannian almost product manifold (M, J, g) is a locally decomposable Riemannian manifold if and only if $\nabla J = 0$, where ∇ is the Levi-Civita connection of g. It is proved that the condition $\nabla J = 0$ is equivalent to decomposability of the pure metric g (see [11]), i.e., $\Phi_J g = 0$, where Φ_J is the Tachibana operator (see [14–15]): $(\Phi_J g)(X, Y, Z) = (JX)(g(Y, Z)) - X(g(JY, Z)) + g((L_Y J)X, Z) + g(Y, (L_Z\ J)X)$.

Let $\widetilde{J}$ be a $(1,1)$-tensor field on T^2M which satisfies the conditions

$$\begin{cases} \widetilde{J}\ {}^0X = -\ {}^0X, \\ \widetilde{J}\ {}^\lambda X = {}^\lambda X, \quad \lambda = I, II \end{cases} \tag{4.2}$$

for any vector field X on M. The tensor field $\widetilde{J}$ is an almost product structure on T^2M. In fact,

$$\begin{aligned} &\widetilde{J}^2({}^0X) = \widetilde{J}(\widetilde{J}({}^0X)) = \widetilde{J}(-\ {}^0X) = {}^0X, \\ &\widetilde{J}^2({}^\lambda X) = \widetilde{J}(\widetilde{J}({}^\lambda X)) = \widetilde{J}({}^\lambda X) = {}^\lambda X, \quad \lambda = I, II, \end{aligned}$$

i.e., $\widetilde{J}$ satisfies $\widetilde{J}^2 = I_{T^2M}$. In the adapted frame $\{E_\beta\}$, the almost product structure $\widetilde{J}$ has the components

$$\widetilde{J} = \begin{pmatrix} -\delta^i_j & 0 & 0 \\ 0 & \delta^i_j & 0 \\ 0 & 0 & \delta^i_j \end{pmatrix}.$$

Now let us compute

$$P(\widetilde{X},\widetilde{Y})=\widetilde{g}(\widetilde{J}\widetilde{X},\widetilde{Y})-\widetilde{g}(\widetilde{X},\widetilde{J}\widetilde{Y})$$

for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ on T^2M. By the definitions of the metric $\widetilde{g}$ and the almost product structure $\widetilde{J}$, we obtain $P(\widetilde{X},\widetilde{Y})=0$ for any vector fields $\widetilde{X}={}^0X,\ {}^IX,\ {}^{II}X$ and $\widetilde{Y}={}^0Y,\ {}^IY,\ {}^{II}Y$, i.e., $\widetilde{g}$ is pure with respect to $\widetilde{J}$. We then calculate

$$(\Phi_{\widetilde{J}}\widetilde{g})(\widetilde{X},\widetilde{Y},\widetilde{Z})=(\widetilde{J}\widetilde{X})(\widetilde{g}(\widetilde{Y},\widetilde{Z}))-\widetilde{X}(\widetilde{g}(\widetilde{J}\widetilde{Y},\widetilde{Z}))+\widetilde{g}((L_{\widetilde{Y}}\ \widetilde{J})\widetilde{X},\widetilde{Z})+\widetilde{g}(\widetilde{Y},(L_{\widetilde{Z}}\ \widetilde{J})\widetilde{X})$$

for all vector fields $\widetilde{X},\widetilde{Y},\widetilde{Z}$ on T^2M. For any vector fields $\widetilde{X}={}^0X,\ {}^IX,\ {}^{II}X$, $\widetilde{Y}={}^0Y,\ {}^IY,\ {}^{II}Y$ and $\widetilde{Z}={}^0Z,\ {}^IZ,\ {}^{II}Z$, we then get

$$\begin{aligned}(\Phi_{\widetilde{J}}\widetilde{g})({}^0X,{}^0Y,{}^IZ)&=-2\widetilde{g}({}^I(R(X,Y)y),\ {}^IZ),\\(\Phi_{\widetilde{J}}\widetilde{g})({}^0X,{}^0Y,{}^{II}Z)&=-2\widetilde{g}({}^{II}(R(X,Y)\omega),\ {}^{II}Z),\\(\Phi_{\widetilde{J}}\widetilde{g})({}^0X,{}^IY,{}^0Z)&=-2\widetilde{g}({}^IY,\ {}^I(R(X,Z)y)),\\(\Phi_{\widetilde{J}}\widetilde{g})({}^0X,{}^{II}Y,{}^0Z)&=-2\widetilde{g}({}^{II}Y,\ {}^{II}(R(X,Z)\omega)),\\\text{otherwise}&=0.\end{aligned}\tag{4.3}$$

Therefore, we have the following theorem.

Theorem 4.1 *Let (M,g) be a Riemannian manifold and T^2M be its second-order bundle equipped with the metric $\widetilde{g}$ and the almost product structure $\widetilde{J}$. The triple $(T^2M,\widetilde{J},\widetilde{g})$ is a locally decomposable Riemannian manifold if and only if M is flat.*

Let (M,J,g) be a non-integrable Riemannian almost product manifold. The non-integrable Riemannian almost product manifold (M,J,g) is called a Riemannian almost product $\mathcal{W}_3$-manifold if $\underset{X,Y,Z}{\sigma}g((\nabla_XJ)Y,Z)=0$, where σ is the cyclic sum by X,Y,Z (see [13]). In [12], it is proved that

$$\underset{X,Y,Z}{\sigma}g((\nabla_XJ)Y,Z)=0$$

is equivalent to

$$(\Phi_Jg)(X,Y,Z)+(\Phi_Jg)(Y,Z,X)+(\Phi_Jg)(Z,X,Y)=0.$$

If we compute

$$A(\widetilde{X},\widetilde{Y},\widetilde{Z})=(\Phi_{\widetilde{J}}\widetilde{g})(\widetilde{X},\widetilde{Y},\widetilde{Z})+(\Phi_{\widetilde{J}}\widetilde{g})(\widetilde{Y},\widetilde{Z},\widetilde{X})+(\Phi_{\widetilde{J}}\widetilde{g})(\widetilde{Z},\widetilde{X},\widetilde{Y})$$

for all vector fields $\widetilde{X},\widetilde{Y},\widetilde{Z}$ on T^2M, from (4.3) we have

$$\begin{aligned}A({}^0X,{}^0Y,{}^IZ)&=(\Phi_{\widetilde{J}}\widetilde{g})({}^0X,{}^0Y,{}^IZ)+(\Phi_{\widetilde{J}}\widetilde{g})({}^0Y,{}^IZ,\ {}^0X)+(\Phi_{\widetilde{J}}\widetilde{g})({}^IZ,\ {}^0X,\ {}^0Y)\\&=-2g(R(X,Y)y,Z)-2g(Z,R(Y,X)y)=0,\\A({}^0X,{}^IY,{}^0Z)&=(\Phi_{\widetilde{J}}\widetilde{g})({}^0X,{}^IY,{}^0Z)+(\Phi_{\widetilde{J}}\widetilde{g})({}^IY,{}^0Z,{}^0X)+(\Phi_{\widetilde{J}}\widetilde{g})({}^0Z,\ {}^0X,\ {}^IY)\\&=-2g(Y,R(X,Z)y)-2g(R(Z,X)y,Y)=0,\\A({}^IX,{}^0Y,{}^0Z)&=(\Phi_{\widetilde{J}}\widetilde{g})({}^IX,{}^0Y,{}^0Z)+(\Phi_{\widetilde{J}}\widetilde{g})({}^0Y,{}^0Z,\ {}^IX)+(\Phi_{\widetilde{J}}\widetilde{g})({}^0Z,\ {}^IX,\ {}^0Y)\\&=-2g(R(Y,Z)y,X)-2g(X,R(Z,Y)y)=0,\\A({}^0X,{}^0Y,{}^{II}Z)&=(\Phi_{\widetilde{J}}\widetilde{g})({}^0X,{}^0Y,{}^{II}Z)+(\Phi_{\widetilde{J}}\widetilde{g})({}^0Y,{}^{II}Z,{}^0X)+(\Phi_{\widetilde{J}}\widetilde{g})({}^{II}Z,\ {}^0X,\ {}^0Y)\end{aligned}$$

$$
\begin{aligned}
&= -2g(R(X,Y)\omega, Z) - 2g(Z, R(Y,X)\omega) = 0,\\
A({}^0X, {}^{II}Y, {}^0Z) &= (\Phi_{\widetilde{J}}\widetilde{g})({}^0X, {}^{II}Y, {}^0Z) + (\Phi_{\widetilde{J}}\widetilde{g})({}^{II}Y, {}^0Z, {}^0X) + (\Phi_{\widetilde{J}}\widetilde{g})({}^0Z, {}^0X, {}^{II}Y)\\
&= -2g(Y, R(X,Z)\omega) - 2g(R(Z,X)\omega, Y) = 0,\\
A({}^{II}X, {}^0Y, {}^0Z) &= (\Phi_{\widetilde{J}}\widetilde{g})({}^{II}X, {}^0Y, {}^0Z) + (\Phi_{\widetilde{J}}\widetilde{g})({}^0Y, {}^0Z, {}^{II}X) + (\Phi_{\widetilde{J}}\widetilde{g})({}^0Z, {}^{II}X, {}^0Y)\\
&= -2g(R(Y,Z)\omega, X) - 2g(Y, R(Z,Y)\omega) = 0.
\end{aligned}
$$

All other $A(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$ are automatically zero. Hence we state the following theorem.

Theorem 4.2 *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$ and the almost product structure $\widetilde{J}$. The triple $(T^2M, \widetilde{J}, \widetilde{g})$ is a Riemannian almost product $\mathcal{W}_3$-manifold.*

Remark 4.1 Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$. Another almost product structure on T^2M is defined by the conditions

$$
\begin{aligned}
&\widetilde{\varphi}\ {}^0X = {}^0X,\\
&\widetilde{\varphi}\ {}^\lambda X = -\ {}^\lambda X, \quad \lambda = I, II
\end{aligned}
$$

for any vector field X on M (see [1]). Similarly, the triple $(T^2M, \widetilde{\varphi}, \widetilde{g})$ is another Riemannian almost product $\mathcal{W}_3$-manifold.

The almost product structure $\widetilde{J}$ on $(T^2M, \widetilde{g})$ is harmonic if and only if

$$
\mathrm{trace}(\widetilde{\nabla}\widetilde{J}) = 0, \tag{4.4}
$$

where "trace" is taken with respect to the metric $\widetilde{g}$ (see [8]). The harmonicity condition (4.4) of the almost product structure $\widetilde{J}$ on T^2M with respect to the metric $\widetilde{g}$ has the following form in the adapted frame $\{E_\beta\}$:

$$
g^{ij}(\widetilde{\nabla}_{E_i}\widetilde{J})E_j = 0, \quad g^{ij}(\widetilde{\nabla}_{E_{\overline{i}}}\widetilde{J})E_{\overline{j}} = 0, \quad g^{ij}(\widetilde{\nabla}_{E_{\overline{\overline{i}}}}\widetilde{J})E_{\overline{\overline{j}}} = 0.
$$

Due to (4.4) and Proposition 3.2, we have

$$
\begin{aligned}
g^{ij}(\widetilde{\nabla}_{E_i}\widetilde{J})E_j &= g^{ij}\{(\widetilde{\nabla}_{E_i}\widetilde{J}E_j) - \widetilde{J}(\widetilde{\nabla}_{E_i}E_j)\} = g^{ij}\{-(\widetilde{\nabla}_{E_i}E_j) - \widetilde{J}(\widetilde{\nabla}_{E_i}E_j)\}\\
&= g^{ij}\Big\{-\Gamma^k_{ij}E_k - \frac{1}{2}y^pR_{jip}{}^kE_{\overline{k}} - \frac{1}{2}\omega^sR_{jis}{}^kE_{\overline{\overline{k}}} + \Gamma^k_{ij}E_k\\
&\quad - \frac{1}{2}y^pR_{jip}{}^kE_{\overline{k}} - \frac{1}{2}\omega^sR_{jis}{}^kE_{\overline{\overline{k}}}\Big\}\\
&= -g^{ij}y^pR_{jip}{}^kE_{\overline{k}} - \omega^sg^{ij}R_{jis}{}^kE_{\overline{\overline{k}}} = 0,\\
g^{ij}(\widetilde{\nabla}_{E_{\overline{i}}}\widetilde{J})E_{\overline{j}} &= g^{ij}\{(\widetilde{\nabla}_{E_{\overline{i}}}\widetilde{J}E_{\overline{j}}) - \widetilde{J}(\widetilde{\nabla}_{E_{\overline{i}}}E_{\overline{j}})\} = 0,\\
g^{ij}(\widetilde{\nabla}_{E_{\overline{\overline{i}}}}\widetilde{J})E_{\overline{\overline{j}}} &= g^{ij}\{(\widetilde{\nabla}_{E_{\overline{\overline{i}}}}\widetilde{J}E_{\overline{\overline{j}}}) - \widetilde{J}(\widetilde{\nabla}_{E_{\overline{\overline{i}}}}E_{\overline{\overline{j}}})\} = 0,
\end{aligned}
$$

from which the below theorem follows.

Theorem 4.3 *Let (M, g) be a Riemannian manifold and let T^2M be its second-order bundle equipped with the metric $\widetilde{g}$ and the almost product structure $\widetilde{J}$. Then the almost product structure $\widetilde{J}$ on $(T^2M, \widetilde{g})$ is harmonic.*

5 The Riemannian Curvature Tensors

In this section, we compute the Riemann curvature and the scalar curvature of T^2M with the metric $\widetilde{g}$. The expression of the Riemann curvature $\widetilde{R}$ with respect to the adapted frame $\{E_\beta\}$ is given by

$$\widetilde{R}_{\delta\gamma\beta}{}^{\alpha} = E_\delta\widetilde{\Gamma}^{\alpha}_{\gamma\beta} - E_\gamma\widetilde{\Gamma}^{\alpha}_{\delta\beta} + \widetilde{\Gamma}^{\alpha}_{\delta\varepsilon}\widetilde{\Gamma}^{\varepsilon}_{\gamma\beta} - \widetilde{\Gamma}^{\alpha}_{\gamma\varepsilon}\widetilde{\Gamma}^{\varepsilon}_{\delta\beta} - \Omega_{\delta\gamma}{}^{\varepsilon}\widetilde{\Gamma}^{\alpha}_{\varepsilon\beta}.$$

Using Proposition 3.2 and (3.2), and writing $R(E_\delta, E_\gamma)E_\beta = \widetilde{R}_{\delta\gamma\beta}{}^{\alpha}E_\alpha$, standard calculations yield the proposition below.

Proposition 5.1 *The curvature tensor $\widetilde{R}$ of the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ on T^2M is given by the following formulas:*

$$\begin{aligned}
\text{(i) } \widetilde{R}(E_h, E_i)E_j &= \Big\{R_{hij}{}^{k} + \frac{1}{4}y^p y^r(R_{plh}{}^{k}R_{jir}{}^{l} - R_{pli}{}^{k}R_{jhr}{}^{l} - 2R_{ihp}{}^{l}R_{rlj}{}^{k}) \\
&\quad + \frac{1}{4}\omega^s\omega^t(R_{slh}{}^{k}R_{jit}{}^{l} - R_{sli}{}^{k}R_{jht}{}^{l} - 2R_{ihs}{}^{l}R_{tlj}{}^{k})\Big\}E_k \\
&\quad + \Big\{\frac{1}{2}y^p(\nabla_h R_{jip}{}^{k} - \nabla_i R_{jhp}{}^{k})\Big\}E_{\overline{k}} \\
&\quad + \Big\{\frac{1}{2}\omega^s(\nabla_h R_{jis}{}^{k} - \nabla_i R_{jhs}{}^{k})\Big\}E_{\overline{\overline{k}}}, \\
\text{(ii) } \widetilde{R}(E_{\overline{h}}, E_i)E_j &= \Big\{-\frac{1}{2}y^p\nabla_i R_{phj}{}^{k}\Big\}E_k + \Big\{\frac{1}{2}R_{jih}{}^{k} - \frac{1}{4}y^p y^r R_{lip}{}^{k}R_{rhj}{}^{l}\Big\}E_{\overline{k}} \\
&\quad + \Big\{-\frac{1}{4}y^p\omega^s R_{phj}{}^{l}R_{lis}{}^{k}\Big\}E_{\overline{\overline{k}}}, \\
\text{(iii) } \widetilde{R}(E_h, E_i)E_{\overline{j}} &= \Big\{\frac{1}{2}y^p(\nabla_h R_{pji}{}^{k} - \nabla_i R_{pjh}{}^{k})\Big\}E_k \\
&\quad + \Big\{R_{hij}{}^{k} + \frac{1}{4}y^p y^r(R_{lhp}{}^{k}R_{rji}{}^{l} - R_{lip}{}^{k}R_{rjh}{}^{l})\Big\}E_{\overline{k}} \\
&\quad + \Big\{\frac{1}{4}y^p\omega^s(R_{lhs}{}^{k}R_{pji}{}^{l} - R_{lis}{}^{k}R_{pjh}{}^{l})\Big\}E_{\overline{\overline{k}}}, \\
\text{(iv) } \widetilde{R}(E_{\overline{h}}, E_{\overline{i}})E_j &= \Big\{R_{hij}{}^{k} + \frac{1}{4}y^p y^r(R_{plh}{}^{k}R_{rij}{}^{l} - R_{pil}{}^{k}R_{rhj}{}^{l})\Big\}E_k, \\
\text{(v) } \widetilde{R}(E_{\overline{h}}, E_i)E_{\overline{j}} &= \Big\{\frac{1}{2}R_{hji}{}^{k} + \frac{1}{4}y^p y^r R_{phl}{}^{k}R_{rji}{}^{l}\Big\}E_k, \\
\text{(vi) } \widetilde{R}(E_{\overline{\overline{h}}}, E_i)E_j &= \Big\{-\frac{1}{2}\omega^s\nabla_i R_{shj}{}^{k}\Big\}E_k + \Big\{-\frac{1}{4}y^p\omega^s R_{lip}{}^{k}R_{shj}{}^{l}\Big\}E_{\overline{k}} \\
&\quad + \Big\{\frac{1}{2}R_{jih}{}^{k} - \frac{1}{4}\omega^s\omega^t R_{lis}{}^{k}R_{thj}{}^{l}\Big\}E_{\overline{\overline{k}}}, \\
\text{(vii) } \widetilde{R}(E_h, E_i)E_{\overline{\overline{j}}} &= \Big\{\frac{1}{2}\omega^s(\nabla_h R_{sji}{}^{k} - \nabla_i R_{sjh}{}^{k})\Big\}E_k \\
&\quad + \Big\{\frac{1}{4}y^p\omega^s(R_{lhp}{}^{k}R_{sji}{}^{l} - R_{lip}{}^{k}R_{sjh}{}^{l})\Big\}E_{\overline{k}} \\
&\quad + \Big\{R_{hij}{}^{k} + \frac{1}{4}\omega^s\omega^t(R_{lhs}{}^{k}R_{tji}{}^{l} - R_{lis}{}^{k}R_{tjh}{}^{l})\Big\}E_{\overline{\overline{k}}}, \\
\text{(viii) } \widetilde{R}(E_{\overline{\overline{h}}}, E_{\overline{\overline{i}}})E_j &= \Big\{R_{hij}{}^{k} + \frac{1}{4}\omega^s\omega^t(R_{shl}{}^{k}R_{tij}{}^{l} - R_{sil}{}^{k}R_{thj}{}^{l})\Big\}E_k, \\
\text{(ix) } \widetilde{R}(E_{\overline{\overline{h}}}, E_i)E_{\overline{\overline{j}}} &= \Big\{\frac{1}{2}R_{hji}{}^{k} + \frac{1}{4}\omega^s\omega^t R_{shl}{}^{k}R_{tji}{}^{l}\Big\}E_k, \\
\text{(x) } \widetilde{R}(E_{\overline{h}}, E_i)E_{\overline{\overline{j}}} &= \Big\{\frac{1}{4}y^p\omega^s R_{phl}{}^{k}R_{sji}{}^{l}\Big\}E_k,
\end{aligned}$$

$$\text{(xi)}\ \widetilde{R}(E_{\overline{h}},E_{\overline{\overline{i}}})E_j=\Big\{\frac{1}{4}y^p\omega^s(R_{phl}{}^kR_{sij}{}^l-R_{sil}{}^kR_{phj}{}^l)\Big\}E_k,$$

$$\text{(xii)}\ \widetilde{R}(E_h,E_{\overline{\overline{i}}})E_{\overline{j}}=\Big\{-\frac{1}{4}y^p\omega^sR_{sil}{}^kR_{pjh}{}^l\Big\}E_k,$$

$$\text{(xiii)}\ \widetilde{R}(E_{\overline{\overline{h}}},E_i)E_{\overline{j}}=\Big\{\frac{1}{4}y^p\omega^sR_{shl}{}^kR_{pji}{}^l\Big\}E_k,$$

$$\text{(xiv)}\ \widetilde{R}(E_{\overline{h}},E_{\overline{i}})E_{\overline{j}}=0,\quad \widetilde{R}(E_{\overline{\overline{h}}},E_{\overline{\overline{i}}})E_{\overline{\overline{j}}}=0,\quad \widetilde{R}(E_{\overline{h}},E_{\overline{\overline{i}}})E_{\overline{\overline{j}}}=0,\quad \widetilde{R}(E_{\overline{\overline{h}}},E_{\overline{\overline{i}}})E_{\overline{j}}=0,$$
$$\widetilde{R}(E_{\overline{h}},E_{\overline{i}})E_{\overline{\overline{j}}}=0,\quad \widetilde{R}(E_{\overline{h}},E_{\overline{\overline{i}}})E_{\overline{j}}=0$$

with respect to the adapted frame $\{E_\beta\}$.

Next, we give the following theorem related to the condition that T^2M with the metric $\widetilde{g}$ is locally flat.

Theorem 5.1 *Let (M,g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$. $(T^2M,\widetilde{g})$ is locally flat if and only if (M,g) is locally flat.*

Proof From Proposition 5.1, it is clear that if (M,g) is locally flat, then $(T^2M,\widetilde{g})$ is also locally flat. In contrast, under the assumption $\widetilde{R}=0$, we evaluate (i) of Proposition 5.1 at an arbitrary point $(x^i,y^i,z^i)=(x^i,0,0)$ in the zero section of T^2M, then we have

$$[\widetilde{R}(E_h,E_i)E_j]_{(x^i,0,0)}=R_{hij}{}^k=0.$$

Hence (M,g) is locally flat.

The scalar curvature $\widetilde{S}$ of T^2M with the metric $\widetilde{g}$ is defined by

$$\widetilde{S}=\widetilde{g}^{\gamma\beta}\ \widetilde{R}_{\gamma\beta},$$

where $\widetilde{g}^{\gamma\beta}$ are the components of the inverse matrix of $\widetilde{g}_{\gamma\beta}$, and $\widetilde{R}_{\gamma\beta}$ are the components of the Ricci tensor of $(T^2M,\widetilde{g})$ denoted by $\widetilde{R}_{\gamma\beta}=\widetilde{R}_{\alpha\gamma\beta}{}^{\alpha}$. Using Proposition 5.1 and (3.1), we calculate

$$\begin{aligned}\widetilde{S}&=\widetilde{g}^{\gamma\beta}\ \widetilde{R}_{\gamma\beta}=\widetilde{g}^{ij}\widetilde{R}_{ij}+\widetilde{g}^{\overline{i}\overline{j}}\widetilde{R}_{\overline{i}\overline{j}}+\widetilde{g}^{\overline{\overline{i}}\overline{\overline{j}}}\widetilde{R}_{\overline{\overline{i}}\overline{\overline{j}}}\\
&=g^{ij}\Big\{R_{ij}+\frac{1}{4}y^ry^p(-R_{pli}{}^hR_{jhr}{}^l-2R_{ihp}{}^lR_{rlj}{}^h-R_{lip}{}^hR_{rhj}{}^l)\\
&\quad+\frac{1}{4}\omega^s\omega^t(-R_{sli}{}^hR_{jht}{}^l-2R_{ihs}{}^lR_{tlj}{}^h-R_{lis}{}^hR_{thj}{}^l)\Big\}\\
&\quad+g^{ij}\Big\{-\frac{1}{4}y^py^rR_{pil}{}^hR_{rjh}{}^l\Big\}+g^{ij}\Big\{-\frac{1}{4}\omega^s\omega^tR_{sil}{}^hR_{tjh}{}^l\Big\}\\
&=S-\frac{1}{4}y^py^rg^{ij}g^{hm}g^{nl}(R_{plim}R_{rnjh}+2R_{pnih}R_{rljm}-R_{pmli}R_{rhnj}-R_{pilm}R_{rjnh})\\
&\quad-\frac{1}{4}\omega^s\omega^tg^{ij}g^{hm}g^{nl}(R_{slim}R_{tnjh}+2R_{snih}R_{tljm}-R_{smli}R_{thnj}-R_{silm}R_{tjnh})\\
&=S-\frac{1}{4}y^py^rg^{ij}g^{hm}g^{nl}R_{plim}R_{rnjh}-\frac{1}{4}\omega^s\omega^tg^{ij}g^{hm}g^{nl}R_{slim}R_{tnjh}\\
&=S-\frac{1}{4}\|yR\|^2-\frac{1}{4}\|\omega R\|^2.\end{aligned}$$

Thus we have the following proposition.

Proposition 5.2 *Let (M,g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$. Denote the scalar curvatures of (M,g) and $(T^2M,\widetilde{g})$*

by S and $\widetilde{S}$ respectively. The dependence between the scalar curvatures S and $\widetilde{S}$ is described by the rule

$$\widetilde{S} = S - \frac{1}{4}\|yR\|^2 - \frac{1}{4}\|\omega R\|^2, \tag{5.1}$$

where $\|yR\|^2 = y^p y^r g^{ij} g^{hm} g^{nl} R_{pli\ m} R_{rnjh}$ and $\|\omega R\|^2 = \omega^s \omega^t g^{ij} g^{hm} g^{nl} R_{sli\ m} R_{tnjh}$.

We can now compare the scalar curvatures on (M, g) and $(T^2M, \widetilde{g})$. We state following theorem.

Theorem 5.2 *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$. $(T^2M, \widetilde{g})$ is of constant scalar curvature if and only if (M, g) is flat.*

Proof It is a direct consequence of (5.1) that $R = 0$ implies $S = 0$ and hence $\widetilde{S} = 0$. Conversely, suppose that $\widetilde{S} = \overline{S}_0 = \text{const}$. If we restrict the relation (5.1) to the zero section of T^2M, then we get $S = \overline{S}_0$, and hence (5.1) reduces to

$$0 = \frac{1}{4}\|yR\|^2 + \frac{1}{4}\|\omega R\|^2.$$

The last equation directly gives $\|yR\| = 0$ and $\|\omega R\| = 0$. Consequently, we obtain $R = 0$, which completes the proof.

Remark 5.1 The results in Theorems 5.1–5.2 can be found in [5]. For these results, we present a detailed proof by using different method.

6 Metric Connections with Nonvanishing Torsion on the Second-Order Tangent Bundle

Let ∇ be a linear connection on a Riemannian manifold (M, g). If the torsion tensor of ∇ is zero, then ∇ is symmetric; otherwise ∇ is non-symmetric. Also, the connection ∇ is a metric connection if it satisfies $\nabla g = 0$, otherwise ∇ is non-metric. As is well-known, a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. The goal of this section is to discuss some metric connections with nonvanishing torsion on T^2M with respect to the metric $\widetilde{g}$.

6.1 *H*-lift of linear connections on the second-order tangent bundle

Given a linear connection ∇ on M, there is a unique linear connection ${}^H\nabla$ on T^2M such that

$$\begin{cases} {}^H\nabla_{{}^0X}\ {}^\lambda Y = {}^\lambda(\nabla_X Y), \\ {}^H\nabla_{{}^\mu X}\ {}^\lambda Y = 0 \end{cases} \tag{6.1}$$

for all vector fields X, Y on M, where $\mu = I, II,\ \lambda = 0, I, II$ (see [6]). The unique linear connection ${}^H\nabla$ on T^2M is called the H-lift of ∇ to T^2M.

Let ${}^HT$ be the torsion tensor of ${}^H\nabla$. Then, from (2.4) and (6.1), it follows that

$$\begin{aligned} &{}^HT({}^\mu X, {}^\lambda Y) = 0, \\ &{}^HT({}^0X, {}^0Y) = {}^0(T(X, Y)) + {}^I(R(X, Y)y) + {}^{II}(R(X, Y)\omega) \end{aligned}$$

for all vector fields X, Y on M, where $\mu, \lambda = 0, I, II$ and either $\mu \neq 0$ or $\lambda \neq 0$ (see [6]). Moreover, note that ${}^H\nabla$ is non-symmetric even though ∇ is the Levi-Civita connection of g.

As an application, we compute the covariant derivative of the metric $\widetilde{g}$ with respect to ${}^H\nabla$. Using the definition of the metric $\widetilde{g}$ and (6.1), a straightforward computation leads to the following formulas:

$$\begin{cases} ({}^H\nabla_{{}^0X}\widetilde{g})({}^\lambda Y, {}^\mu Z) = (\nabla_X g)(Y, Z) & \text{for } \lambda = \mu, \\ ({}^H\nabla_{{}^0X}\widetilde{g})({}^\lambda Y, {}^\mu Z) = 0 & \text{for } \lambda \neq \mu, \\ ({}^H\nabla_{{}^IX}\widetilde{g})({}^\lambda Y, {}^\mu Z) = ({}^H\nabla_{{}^{II}X}\widetilde{g})({}^\lambda Y, {}^\mu Z) = 0 & \text{for all } \lambda, \mu \end{cases} \tag{6.2}$$

for all vector fields X, Y, Z on M, where $\lambda, \mu = 0, I, II$. From (6.2), it follows that ${}^H\nabla_{\widetilde{X}}\widetilde{g} = 0$ if and only if $\nabla_X g = 0$. Thus, we can say that the H-lift ${}^H\nabla$ of the Levi-Civita connection ∇ of g is a metric connection with nonvanishing torsion with respect to the metric $\widetilde{g}$.

The H-lift ${}^H\nabla$ of the Levi-Civita connection ∇ of g is given by

$$\begin{cases} {}^H\nabla_{E_i}E_j = \Gamma_{ij}^k E_k, \\ {}^H\nabla_{E_i}E_{\overline{j}} = \Gamma_{ij}^k E_{\overline{k}}, \\ {}^H\nabla_{E_i}E_{\overline{\overline{j}}} = \Gamma_{ij}^k E_{\overline{\overline{k}}}, \\ \text{otherwise} = 0 \end{cases}$$

with respect to the adapted frame $\{E_\beta\}$. The curvature tensor ${}^HR$ of ${}^H\nabla$ is given by

$$\begin{cases} {}^HR(E_h, E_i)E_j = R_{hij}{}^k E_k, \\ {}^HR(E_h, E_i)E_{\overline{j}} = R_{hij}{}^k E_{\overline{k}}, \\ {}^HR(E_h, E_i)E_{\overline{\overline{j}}} = R_{hij}{}^k E_{\overline{\overline{k}}}, \\ \text{otherwise} = 0 \end{cases}$$

with respect to the adapted frame $\{E_\beta\}$. For the scalar curvature ${}^HS$ of ${}^H\nabla$ with respect to the metric $\widetilde{g}$, we get

$${}^HS = \widetilde{g}^{\gamma\beta}\ {}^HR_{\gamma\beta} = g^{ij}R_{ij} = S,$$

which gives the following result.

Theorem 6.1 *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$. Then, the scalar curvature of T^2M with the metric connection ${}^H\nabla$ with respect to the metric $\widetilde{g}$ is zero if and only if the scalar curvature of M is zero, where ∇ is the Levi-Civita connection of g.*

6.2 Product conjugate connection on the second-order tangent bundle

Let (M, J) be an almost product manifold. Given a linear connection ∇ on (M, J), the product conjugate connection ${}^{(J)}\nabla$ of ∇ is defined by

$${}^{(J)}\nabla_X Y = J(\nabla_X JY)$$

for all vector fields X, Y on M. If (M, J, g) is a Riemannian almost product manifold, then $({}^{(J)}\nabla_X g)(JY, JZ) = (\nabla_X g)(Y, Z)$, i.e., ∇ is a metric connection with respect to g if and only if ${}^{(J)}\nabla$ is a metric connection with respect to g. From this, we can say that if ∇ is the Levi-Civita connection of g, then ${}^{(J)}\nabla$ is a metric connection with respect to g (see [2]).

The metric connection ${}^{(\widetilde{J})}\widetilde{\nabla}$ of the metric $\widetilde{g}$ is given as follows:

$$ {}^{(\widetilde{J})}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \widetilde{J}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{J}\widetilde{Y}) $$

for all vector fields $\widetilde{X}, \widetilde{Y}$ on T^2M. By Proposition 3.4 and (4.2), we state the following result.

Proposition 6.1 *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$ and the almost product structure $\widetilde{J}$. Then the metric connection ${}^{(\widetilde{J})}\widetilde{\nabla}$ of $\widetilde{g}$ satisfies*

$$ \begin{cases} {}^{(\widetilde{J})}\widetilde{\nabla}_{{}^0X}\, {}^0Y = {}^0(\nabla_X Y) - \dfrac{1}{2}\,{}^I(R(Y,X)y) - \dfrac{1}{2}\,{}^{II}(R(Y,X)\omega), \\ {}^{(\widetilde{J})}\widetilde{\nabla}_{{}^IX}\, {}^0Y = \dfrac{1}{2}\,{}^0(R(y,X)Y), \quad {}^{(\widetilde{J})}\widetilde{\nabla}_{{}^0X}\, {}^IY = -\dfrac{1}{2}\,{}^0(R(y,Y)X) + {}^I(\nabla_X Y), \\ {}^{(\widetilde{J})}\widetilde{\nabla}_{{}^{II}X}\, {}^0Y = \dfrac{1}{2}\,{}^0(R(\omega,X)Y), \quad {}^{(\widetilde{J})}\widetilde{\nabla}_{{}^0X}\, {}^{II}Y = -\dfrac{1}{2}\,{}^0(R(\omega,Y)X) + {}^{II}(\nabla_X Y), \\ {}^{(\widetilde{J})}\widetilde{\nabla}_{{}^IX}\, {}^IY = 0, \quad {}^{(\widetilde{J})}\widetilde{\nabla}_{{}^{II}X}\, {}^IY = 0, \quad {}^{(\widetilde{J})}\widetilde{\nabla}_{{}^{II}X}\, {}^{II}Y = 0, \quad {}^{(\widetilde{J})}\widetilde{\nabla}_{{}^IX}\, {}^{II}Y = 0 \end{cases} $$

for all vector fields X, Y on M.

The torsion tensor $\widetilde{T}_{{}^{(\widetilde{J})}\widetilde{\nabla}}$ of the metric connection ${}^{(\widetilde{J})}\widetilde{\nabla}$ of the metric $\widetilde{g}$ has the following properties:

$$ \begin{aligned} \widetilde{T}_{{}^{(\widetilde{J})}\widetilde{\nabla}}({}^0X, {}^0Y) &= {}^I(R(X,Y)y) + {}^{II}(R(X,Y)\omega), \\ \widetilde{T}_{{}^{(\widetilde{J})}\widetilde{\nabla}}({}^0X, {}^IY) &= -{}^0(R(y,Y)X), \\ \widetilde{T}_{{}^{(\widetilde{J})}\widetilde{\nabla}}({}^IX, {}^0Y) &= {}^0(R(y,X)Y), \\ \widetilde{T}_{{}^{(\widetilde{J})}\widetilde{\nabla}}({}^0X, {}^{II}Y) &= -{}^0(R(\omega,Y)X), \\ \widetilde{T}_{{}^{(\widetilde{J})}\widetilde{\nabla}}({}^{II}X, {}^0Y) &= {}^0(R(\omega,X)Y), \\ \text{otherwise} &= 0 \end{aligned} $$

for all vector fields X, Y on M. These equations lead to the following result.

Theorem 6.2 *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$. The metric connection ${}^{(\widetilde{J})}\widetilde{\nabla}$ is symmetric if and only if M is flat.*

The relationship between curvature tensors R_∇ and $R_{{}^{(\widetilde{J})}\nabla}$ of the connections ∇ and ${}^{(\widetilde{J})}\nabla$ respectively is as follows: $R_{{}^{(\widetilde{J})}\nabla}(X, Y, Z) = J(R_\nabla(X, Y)JZ)$ for all vector fields X, Y, Z on M (see [2]). Using Proposition 5.1 and (4.2), by $\widetilde{R}_{{}^{(\widetilde{J})}\widetilde{\nabla}}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = \widetilde{J}(\widetilde{R}_{\widetilde{\nabla}}(\widetilde{X}, \widetilde{Y})\widetilde{J}\widetilde{Z})$, the curvature tensor $\widetilde{R}_{{}^{(\widetilde{J})}\widetilde{\nabla}}$ of the metric connection ${}^{(\widetilde{J})}\widetilde{\nabla}$ can be easily written. The scalar curvature $\widetilde{S}_{{}^{(\widetilde{J})}\widetilde{\nabla}}$ of ${}^{(\widetilde{J})}\widetilde{\nabla}$ with respect to the metric $\widetilde{g}$ is in the following form:

$$ \widetilde{S}_{{}^{(\widetilde{J})}\widetilde{\nabla}} = S - \frac{5}{4}\|yR\|^2 - \frac{5}{4}\|\omega R\|^2. $$

Thus we have the following theorem.

Theorem 6.3 *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$. Then the scalar curvature of T^2M with the metric connecton ${}^{(\widetilde{J})}\widetilde{\nabla}$ with respect to the metric $\widetilde{g}$ is constant if and only if (M, g) is flat.*

Proof The proof is similar to that of Theorem 5.2.

Comparing the metric connections $\widetilde{\nabla}$, ${}^{H}\nabla$ and ${}^{(\widetilde{J})}\widetilde{\nabla}$ of the metric $\widetilde{g}$, we have the last theorem given below.

Theorem 6.4 *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$. Then $\widetilde{\nabla} ={}^{H} \nabla ={}^{(\widetilde{J})} \widetilde{\nabla}$ if and only if (M, g) is flat.*

Proof The statement follows directly from (6.1), and Propositions 3.4 and 6.1.

A Riemann-Cartan manifold is a triple $(M, g, \overline{\nabla})$, where (M, g) is an n-dimensional $(n \geq 2)$ Riemannian manifold with a linear connection $\overline{\nabla}$ having non-zero torsion such that $\overline{\nabla}g = 0$. The Riemann-Cartan manifold was introduced in [3]. The paper ends the following result.

Proposition 6.2 *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the metric $\widetilde{g}$. $(T^2M, \widetilde{g}, {}^{H}\nabla)$ and $(T^2M, \widetilde{g}, {}^{(\widetilde{J})}\widetilde{\nabla})$ are both Riemann-Cartan manifolds.*

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