Chin. Ann. Math.
38B(4), 2017, 1019–1036
DOI: 10.1007/s11401-017-1109-2

Chinese Annals of Mathematics, Series B

L-Factor of Irreducible $\chi_1 \times \chi_2 \rtimes \sigma$

Yusuf DANISMAN[1]

Abstract The L-factor of irreducible $\chi_1 \times \chi_2 \rtimes \sigma$ defined by Piatetski-Shapiro is computed by using non-split Bessel functional.

Keywords Bessel model, L-factor, GSp(4), Regular pole
2000 MR Subject Classification 11F70, 11F85

1 Introduction

The local Langlands conjecture for $\mathrm{GSp}_4(k)$ was proved by Gan and Takeda in [6] for a non-Archimedean local field k of characteristic zero. As a definition of the local factors, Gan and Takeda use the one defined by Shahidi in [11] for generic representations and they extend this definition to all nongeneric nonsupercuspidal representations by Langlands classification and multiplicativity. Therefore, the definition of these local factors are not valid for nongeneric supercuspidal representations. There is an alternative construction of local factors given by Piatetski-Shapiro [7] by using Bessel models. Bessel models are defined for all representations except one dimensional ones whereas Whittaker models are defined only for generic representations. Therefore the definition of local factors in the construction of Piatetski-Shapiro is valid for all infinite dimensional irreducible representations. Completing local factor part of the local Langlands conjecture by Bessel model, improves the version of Gan and Takeda [6] as equality of the local factors in both sides as in the conjecture for $\mathrm{GL}(n)$. Hence, the parametrization of the conjecture will be formulized in terms of the local factors of Galois and representation theoric sides.

There are two types of Bessel model called split and non-split defined in Section 2. As given in the table of Theorem 6.2.2 of [10], an infinite dimensional irreducible representation of $\mathrm{GSp}_4(k)$ might have both kinds of Bessel models or only just one of them. Since there are representations which have only split or non-split Bessel model, local factors in these two cases should be computed. For the representations which have both kinds of Bessel models one needs to show that the local factors obtained in these two cases agree with each other.

In [5] we computed the local factors provided by Piatetski-Shapiro of the nongeneric supercuspidal representations of $\mathrm{GSp}_4(k)$. By using the same construction in [3] and [4] we also computed the regular poles of the L-factors of the spinor (degree 4) L-functions for the representations which have Jacquet module length three or less. As a consequence, by [7, Theorem 4.3], we also obtained the L-factors of the generic ones. Therefore by [9, Table A.3] only the

Manuscript received October 27, 2016.
[1]Basic Sciences Department, University of Dammam, PO BOX 1982, Dammam 31441, Kingdom of Saudi Arabia. E-mail: yusufdanisman@gmail.com

computation of regular poles of irreducible $\chi_1 \times \chi_2 \rtimes \sigma$, which has a Jacquet module length four, is remained. Since this representation is generic, computing regular poles is equivalent to finding the L-factor. In this paper, we compute the L-factor of irreducible $\chi_1 \times \chi_2 \rtimes \sigma$ by using the construction of Piatetski-Shapiro. We show that, our results agree with the results of [6], [12] and the local Langlands conjecture.

Let us briefly explain the construction of Piatetski-Shapiro. Let S be the unipotent radical of the Siegel parabolic subgroup of $\mathrm{GSp}_4(k)$ and ψ be any nondegenerate character of S. Let $M := \left\{ \left(\begin{smallmatrix} A & 0_2 \\ 0_2 & D \end{smallmatrix}\right) \in \mathrm{GSp}_4(k) \right\}$ and T be the connected component of $\mathrm{Stab}_M \psi$. Then T consists of the units of a quadratic extension of k or $k \oplus k$. Since $R = TS$ is a subgroup of $\mathrm{GSp}_4(k)$, for any character Λ of T, a character $\alpha_{\Lambda,\psi}$ of R can be defined as $\alpha_{\Lambda,\psi}(r) := \Lambda(t)\psi(s)$, where $r = st$ and $r \in R,\ s \in S,\ t \in T$.

Let (Π, V_Π) be an infinite dimensional, irreducible, and admissible representation of $\mathrm{GSp}_4(k)$. If Π is isomorphic to a subspace of $\mathrm{Ind}_R^{\mathrm{GSp}_4(k)} \alpha_{\Lambda,\psi}$, then its image is unique (see [7]) and called Bessel model.

Piatetski-Shapiro defined the L-factors by using the local integrals:

$$L(s; W_u, \Phi, \mu) = \int_{N\backslash G} W_u(g)\Phi[(0,1)g]\mu(\det \mathrm{g})|\det g|_k^{s+\frac{1}{2}}\mathrm{d}g,$$

where $\Phi \in C_c^\infty(K^2)$ and K is either a quadratic extension of k or direct sum of two copies of k, μ is a character of k^*, $u \in V_\Pi$, W_u is an element of the Bessel model and N, G are some subgroups of $\mathrm{GSp}_4(k)$ defined in the next section.

The integral family $\{L(s; W_u, \Phi, \mu) : u \in V_\Pi,\ \Phi \in C_c^\infty(K^2)\}$ admits a greatest common denominator for all its elements. Hence there exists a function $L(s, \Pi, \mu)$ called L-factor such that $L(s; W_u, \Phi, \mu)/L(s, \Pi, \mu)$ is entire for all $u \in V_\Pi$ and $\Phi \in C_c^\infty(K^2)$. The poles of the L-factor, coming from an integral with a Schwartz function vanishing at zero, are called regular poles.

Let $\varphi_u(x) = W_u \left(\begin{smallmatrix} x\mathrm{I}_2 & \\ & \mathrm{I}_2 \end{smallmatrix}\right)$. By [3, Proposition 2.5], the regular poles of the L-factors are the poles of the meromorphic continuation of the integrals

$$\int_{k*} \varphi_u(x)\mu(x)|x|^{s-\frac{3}{2}}\mathrm{d}^*x.$$

Hence, the regular poles depend only on the asymptotic behavior of $\varphi_u(x)$. Determining asymptotic behavior of $\varphi_u(x)$ is the main subject of this paper and this is done in Theorem 4.1.

In this paper, we follow the similar steps of [3] and [4] but by extending their results about action of k^* and asymptotic behavior of Bessel model. This paper is organized as follows. In Section 2, we give the basics regarding subgroups of $\mathrm{GSp}_4(k)$, Bessel model, local L-factors, and regular poles. In Section 3, we obtain the Jacquet module structure of $\chi_1 \times \chi_2 \rtimes \sigma$. In Section 4, we determine the representations of k^* involved in some exact sequences. In Section 5, we obtain the asymptotic behavior of φ_u and possible regular poles. In Section 6, we show that existence of homomorphisms from the constituents of the Jacquet module to the character Λ of T affects the asymptotic behavior of φ_u.

Throughout this paper, k always denotes a non-Archimedean local field of odd characteristic, v_k its valuation, ν the absolute value, $\mathcal{O}$ its ring of integers, $\mathcal{P}$ the unique maximal prime ideal of $\mathcal{O}$, ϖ a fixed generator of $\mathcal{P}$ and q is the cardinality of the residue field. Let ψ be a nontrivial additive character of k with conductor $\mathcal{O}$ and let $\mathrm{d}x = \mathrm{d}_\psi x$ be the self-dual Haar measure on k. For a representation ξ, let V_ξ, $\widetilde{\xi}$ and ω_ξ denote its space, contragradient and the central character, respectively. Also $\mathbb{1}$ denotes the trivial representation of k^*.

1.1 $\mathbf{GSp_4}(k)$ and its subgroups

In this section we provide the definitions of $\mathrm{GSp}_4(k)$ and its important subgroups. Let $w = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} & w \\ -w & \end{smallmatrix}\right)$ and $X' = w(X^t)w$ for $X \in \mathrm{GL}_2(k)$. Then we define

$$\mathrm{GSp}_4(k) = \{g \in \mathrm{GL}_4(k) : g^t J g = \lambda(g) J \text{ for some } \lambda(g) \in k^*\},$$

$$P = \left\{ g \in \mathrm{GSp}_4(k) : g = \begin{pmatrix} A & B \\ & D \end{pmatrix},\ A, B, D \in M_2(k) \right\},$$

$$M = \left\{ \begin{pmatrix} A & \\ & \lambda(A')^{-1} \end{pmatrix} : A \in \mathrm{GL}_2(k),\ \lambda \in k^* \right\},$$

$$S = \left\{ \begin{pmatrix} \mathrm{I}_2 & Y \\ & \mathrm{I}_2 \end{pmatrix} : Y = Y' \right\}.$$

All characters of S are in the form of

$$\psi_\beta \begin{pmatrix} \mathrm{I}_2 & Y \\ & \mathrm{I}_2 \end{pmatrix} = \psi[\mathrm{tr}(\beta Y)]$$

for some $\beta = \beta'$. ψ_β is called nondegenerate if $\beta \in \mathrm{GL}_2(k)$.

Let ψ_β be a nondegenerate character of S and let T be the connected component of $\mathrm{Stab}_M \psi_\beta$. Then $T = K^*$ where $K = k(\sqrt{\rho})$ for some $\rho \notin (k^*)^2$ and T is called non-split or $K = k \oplus k$ and T is called split.

Since Theorem 4.9 of [3] is proved only for non-split case, in this paper we only consider the non-split case and let $K = k(\sqrt{\rho})$. The group

$$G = \{g \in \mathrm{GL}_2(K) : \det g \in k^*\}$$

can be realized in $\mathrm{GSp}_4(k)$ by the embedding

$$\begin{pmatrix} a + b\sqrt{\rho} & c + d\sqrt{\rho} \\ e + f\sqrt{\rho} & m + n\sqrt{\rho} \end{pmatrix} \hookrightarrow \left(\begin{array}{cc|cc} a & b & c & d \\ b\rho & a & d\rho & c \\ \hline e & f & m & n \\ f\rho & e & n\rho & m \end{array}\right).$$

The group

$$N = \left\{ \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} : n \in K \right\}$$

is a subgroup of G and can be realized as a subgroup of S.

In the non-split case,

$$T = \left\{ \begin{pmatrix} a + b\sqrt{\rho} & \\ & a - b\sqrt{\rho} \end{pmatrix} : a + b\sqrt{\rho} \in K^* \right\},$$

which is a subgroup of G and can be realized as a subgroup of M. The center of $\mathrm{GSp}_4(k)$ is

$$Z = \left\{ a\mathrm{I}_4 = \begin{pmatrix} a\mathrm{I}_2 & \\ & a^2((a\mathrm{I}_2)')^{-1} \end{pmatrix} : a \in k^* \right\}$$

and define the group

$$H := \left\{ \begin{pmatrix} x\mathrm{I}_2 & \\ & \mathrm{I}_2 \end{pmatrix} = \begin{pmatrix} x\mathrm{I}_2 & \\ & x((x\mathrm{I}_2)')^{-1} \end{pmatrix} : x \in k^* \right\}.$$

1.2 Bessel model, L-factor and regular poles

Let Λ be a character of T and ψ be a nondegenerate character of S. $\alpha_{\Lambda,\psi}(r) := \Lambda(t)\psi(s)$ is a character of R for $r = ts \in R$. By [7, Theorem 3.1] and the following remarks, if Π is an infinite dimensional, smooth, admissible, and preunitary representation of $\mathrm{GSp}_4(k)$, then $\dim[\mathrm{Hom}_R(\Pi, \alpha_{\Lambda,\psi})] = 1$ for some choice of Λ, ψ, and R (or equivalently K).

A linear functional $l : V_\Pi \to \mathbb{C}$ such that $l(\Pi(r)u) = \alpha_{\Lambda,\psi}(r)l(u)$ is called Bessel functional. For $u \in V_\Pi$ define W_u on $\mathrm{GSp}_4(k)$ by $W_u(g) := l(\Pi(g)u)$. The space $\mathbf{W}^{\Lambda,\psi} := \{W_u : u \in V_\Pi\}$ is called the Bessel model of Π.

A representation of $\mathrm{GSp}_4(k)$ can be defined on $\mathbf{W}^{\Lambda,\psi}$ by right translation and $\Pi \cong \mathbf{W}^{\Lambda,\psi}$. For $h_x := \left(\begin{smallmatrix} x\mathrm{I}_2 & \\ & \mathrm{I}_2 \end{smallmatrix}\right) \in H$ define $\varphi_u(x) := W_u(h_x)$.

From the third section of [7], for $s \in \mathbb{C}$, $\Phi \in C_c^\infty(K^2)$ and $u \in V_\Pi$ the integral

$$L(s; W_u, \Phi, \mu) = \int_{N\backslash G} W_u(g)\Phi[(0,1)g]\mu(\det g)|\det g|_k^{s+\frac{1}{2}}\mathrm{d}g$$

converges absolutely for $\mathrm{Re}(s)$ large enough and has a meromorphic continuation to the whole plane. The set $\{L(s; W_u, \Phi, \mu) : \Phi \in C_c^\infty(K^2),\ u \in V_\Pi\}$ forms a fractional ideal of the ring $\mathbb{C}[q^s, q^{-s}]$ of the form $L(s;\Pi,\mu)\mathbb{C}[q^s, q^{-s}]$. The factor $L(s;\Pi,\mu)$ is of the form $P(q^{-s})^{-1}$, where $P(X) \in \mathbb{C}[X]$, $P(0) = 1$ and is called the L-factor of Π twisted by μ.

A pole of $L(s;\Pi,\mu)$ is called a regular pole if it is a pole of some $L(s; W_u, \Phi, \mu)$ with $\Phi(0,0) = 0$. Any other pole is called an exceptional pole. Regular poles will be expressed as poles of the Tate L-functions

$$L(s,\chi) = \begin{cases} 1, & \text{if } \chi \text{ is ramified,} \\ (1 - \chi(\varpi)q^{-s})^{-1}, & \text{if } \chi \text{ is unramified,} \end{cases}$$

where χ is a character of k^*.

By [3, Proposition 2.5], regular poles of $L(s;\Pi,\mu)$ are the poles of the integrals

$$\int_{k*} \varphi_u(x)\mu(x)|x|^{s-\frac{3}{2}}\mathrm{d}^*x, \quad v \in V_\Pi.$$

Therefore, the regular poles depend only on the asymptotic behavior of $\varphi_u(x)$. Also since $\chi_1 \times \chi_2 \rtimes \sigma$ is generic, by [7, Theorem 4.3] there is no exceptional pole. Therefore regular poles determine the L-factor.

1.3 Parabolic induction and the Jacquet module

Let (τ, V_τ) be a representation of M and let δ_P be the modular character of P. If $p = \left(\begin{smallmatrix} A & * \\ & \lambda(A')^{-1} \end{smallmatrix}\right) \in P$ for $A \in \mathrm{GL}_2(k)$, then $\delta_P(p) = |\det(A)^3\lambda^{-3}|$. The space of functions $f : \mathrm{GSp}_4(k) \to V_\tau$ which satisfy

$$f(msg) = \delta_P(m)^{\frac{1}{2}}\tau(m)f(g) \quad \text{for } m \in M,\ s \in S \text{ and } g \in \mathrm{GSp}_4(k)$$

is called normalized parabolic induction from P to $\mathrm{GSp}_4(k)$ and denoted by $\mathrm{ind}_P^{\mathrm{GSp}_4}\tau$. An action of $\mathrm{GSp}_4(k)$ on $\mathrm{ind}_P^{\mathrm{GSp}_4}\tau$ can be defined by right translation.

For $V_S(\Pi) := \mathrm{span}\{v - \Pi(s)v : s \in S, v \in V_\Pi\}$ the space $(\Pi_S, V_\Pi/V_S(\Pi))$ is called the Jacquet module and $R_S(\Pi) = \Pi_S \otimes \delta_P^{-\frac{1}{2}}$ is the normalized Jacquet module.

Now we will define parabolic induction similarly for $\mathrm{GL}_2(k)$. Let B denotes the Borel subgroup of $\mathrm{GL}_2(k)$. The modular character of B is $\delta_B(\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right)) = \left|\frac{a}{d}\right|$. Let χ_1 and χ_2 be characters of k^*. The space of functions $f : \mathrm{GL}_2(k) \to \mathbb{C}$ denoted by $\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2)$ and which satisfy

$$f\Big(\begin{pmatrix} a & b \\ & d \end{pmatrix} g\Big) = \delta_B\Big(\begin{pmatrix} a & b \\ & d \end{pmatrix}\Big)^{\frac{1}{2}} \chi_1(a)\chi_2(d)f(g)$$

is called the normalized induction from B to $\mathrm{GL}_2(k)$.

The Jacquet module with respect to $N_{\mathrm{GL}_2(k)} = \{\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)\}$ of a representation τ of $\mathrm{GL}_2(k)$ is defined as

$$J(\tau) = V_\tau / \{\tau(n)u - u : n \in N_{\mathrm{GL}_2(k)},\, u \in V_\tau\}.$$

It is a representation of the diagonal subgroup of $\mathrm{GL}_2(k)$. $\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\nu^{\frac{1}{2}}, \nu^{-\frac{1}{2}})$ has two constituents by [1, Theorem 4.5.1]. Its infinite dimensional subrepresentation is denoted by $\mathrm{St}_{\mathrm{GL}_2(k)}$ and its one dimensional quotient is denoted by $1_{\mathrm{GL}_2(k)}$. By [1, Theorem 4.5.4]

$$J(\mathrm{St}_{\mathrm{GL}_2(k)})\begin{pmatrix} a & \\ & d \end{pmatrix} = \left|\frac{a}{d}\right|, \quad J(1_{\mathrm{GL}_2(k)}) = \mathbb{1}.$$

Let σ be a character of k^*. $\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2) \otimes \sigma$ is a representation of $\mathrm{GL}_2(k) \times \mathrm{GL}_1(k) \cong M$. This representation can be extended trivially on S and becomes a representation of P. $\chi_1 \times \chi_2 \rtimes \sigma = \sigma = \mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2) \rtimes \sigma$ denotes the representation

$$\mathrm{ind}_P^{\mathrm{GSp}_4(k)} \mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2) \otimes \sigma.$$

This representation is irreducible if and only if $\chi_1 \neq \nu^{\mp 1}$, $\chi_2 \neq \nu^{\mp 1}$ and $\chi_1 \neq \nu^{\mp 1}\chi_2^{\mp 1}$ (see [9]). Throughout this paper, Π denotes the representation $\chi_1 \times \chi_2 \rtimes \sigma$.

By [8, Proposition 2.3], for the existence of the Bessel model for the representation $\chi_1 \times \chi_2 \rtimes \sigma$, the relation $\omega_{\chi_1 \times \chi_2 \rtimes \sigma} = \Lambda|_{k^*}$ should be satisfied and therefore $\chi_1\chi_2\sigma^2 = \Lambda|_{k^*}$.

2 The Jacquet Module Structure

In this section we give the Jacquet module structures of $\Pi = \mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2) \rtimes \sigma$ due to 6.3 of [2]. For a representation θ of M define

$$\Pi_n = \Big\{ f \in \mathrm{ind}_P^G \theta : \mathrm{supp}(f) \subset G_n = \bigcup_{\dim(P/PwP) \geq n} PwP \Big\}$$

as in [4].

Proposition 2.1 *For* Π *we have*

$$0 \underbrace{\subset}_{\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1^{-1}, \chi_2^{-1}) \otimes \chi_1\chi_2\sigma} (\Pi_3)_S \underbrace{\subset}_{\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_2, \chi_1^{-1}) \otimes \chi_1\sigma} (\Pi_2')_S$$
$$\underbrace{\subset}_{\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2^{-1}) \otimes \chi_2\sigma} (\Pi_2)_S \underbrace{\subset}_{\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2) \otimes \sigma} \Pi_S,$$

where Π_2' *is a subrepresentation of* Π_2.

Proof First constituent By [4, Proposition 3.4(i)], the first constituent is

$$\widetilde{\mathrm{ind}_B^{\mathrm{GL}_2(k)}}(\chi_1,\chi_2)\otimes\omega_{\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1,\chi_2)}\sigma=\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1^{-1},\chi_2^{-1})\otimes\chi_1\chi_2\sigma.$$

Second constituent By [4, Proposition 3.4(ii)], we need to consider

$$J(\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1,\chi_2))\begin{pmatrix}a & \\ & \frac{\lambda}{d}\end{pmatrix}|ad|^{-\frac{1}{2}}\otimes\sigma(\lambda)|\lambda|^{\frac{1}{2}}.$$

Since

$$0\to\delta_B^{\frac{1}{2}}(\chi_2,\chi_1)\to J(\mathrm{ind}_B^{\mathrm{GL}(2)}(\chi_1,\chi_2))\to\delta_B^{\frac{1}{2}}(\chi_1,\chi_2)\to 0$$

and

$$\begin{aligned}&\delta_B^{\frac{1}{2}}(\chi_2,\chi_1)\begin{pmatrix}a & \\ & \frac{\lambda}{d}\end{pmatrix}|d|^{-1}\otimes\sigma(\lambda)|\lambda|^{\frac{1}{2}}\\ &=\Big|\frac{a}{\frac{\lambda}{d}}\Big|^{\frac{1}{2}}\chi_2(a)\chi_1\Big(\frac{\lambda}{d}\Big)|ad|^{-\frac{1}{2}}\otimes\sigma(\lambda)|\lambda|^{\frac{1}{2}}\\ &=(\chi_2,\chi_1^{-1})\otimes\chi_1\sigma,\end{aligned}$$

$\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_2,\chi_1^{-1})\otimes\chi_1\sigma$ is the subrepresentation of the second constituent. Since

$$\begin{aligned}&\delta_B^{\frac{1}{2}}(\chi_1,\chi_2)\begin{pmatrix}a & \\ & \frac{\lambda}{d}\end{pmatrix}|ad|^{-\frac{1}{2}}\otimes\sigma(\lambda)|\lambda|^{\frac{1}{2}}\\ &=\Big|\frac{a}{\frac{\lambda}{d}}\Big|^{\frac{1}{2}}\chi_1(a)\chi_2\Big(\frac{\lambda}{d}\Big)|ad|^{-\frac{1}{2}}\otimes\sigma(\lambda)|\lambda|^{\frac{1}{2}}\\ &=(\chi_1,\chi_2^{-1})\otimes\chi_2\sigma,\end{aligned}$$

$\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1,\chi_2^{-1})\otimes\chi_2\sigma$ is the quotient of the second constituent.

Third constituent By [4, Proposition 3.4(iii)], $\Pi_S/(\Pi_2)_S$ is $\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1,\chi_2)\otimes\sigma$.

3 Representations of k^*

In this section, we determine the splittings of some exact sequences of representations of k^* that we will need in the following section.

Proposition 3.1 *Let χ_1,χ_2 and χ_3 be characters of k^*, (ρ,U) and (ρ,U_1) be representations of k^* such that*

$$0\to U_1\to U\to\oplus\chi_3\to 0$$

and

$$0\to\oplus\chi_1\to U_1\to\oplus\chi_2\to 0,$$

where $\oplus\chi_i$ is a vector space on which k^* acts as χ_i for $i=1,2$.

(i) *If $\chi_1=\chi_2$ and $\chi_1\neq\chi_3$ then $U=U_1\bigoplus\oplus\chi_3$. Hence for all $u\in U$ there exist $u_1\in U_1$ and $u_3\in\oplus\chi_3$ such that $u=u_1+u_3$ and for some $u_1'\in\oplus\chi_1$ we have*

$$\rho(x)u=\chi_1(x)u_1+\chi_1(x)v_k(x)u_1'+\chi_3(x)u_3.$$

(ii) *If* $\chi_2 = \chi_3$ *and* $\chi_1 \neq \chi_2$ *then* $U = U_1' \bigoplus \oplus\chi_1$ *where* $0 \to \oplus\chi_2 \to U_1' \to \oplus\chi_2 \to 0$. *Hence for all* $u \in U$ *there exist* $u_1 \in \oplus\chi_1$ *and* $u_2 \in U_1'$ *such that* $u = u_1 + u_2$ *and for some* $u_2' \in \oplus\chi_2$ *we have*

$$\rho(x)u = \chi_1(x)u_1 + \chi_2(x)u_2 + \chi_2(x)v_k(x)u_2'.$$

(iii) *If* $\chi_1 = \chi_2 = \chi_3 = \chi$ *and* $u \in U$ *then for some* $w_1 \in U_1$ *and* $w_2 \in \oplus\chi$ *we have*

$$\rho(x)u = \chi(x)u + \chi(x)v_k(x)w_1 + \chi(x)v_k^2(x)w_2.$$

Proof Since the proofs are similar, we give only the proof of the last one which is also the most complicated one. For $u \in U$ there is a $w(x) \in U_1$ such that

$$\rho(x)u = \chi(x)u + w(x). \tag{3.1}$$

By [3, Lemma 5.10(ii)], there exists $t(x) \in \oplus\chi$ such that

$$\rho(y)w(x) = \chi(y)w(x) + \chi(y)v_k(y)t(x). \tag{3.2}$$

From (3.1),

$$\rho(xy)u = \chi(xy)u + w(xy). \tag{3.3}$$

By (3.1)–(3.2),

$$\begin{aligned}
\rho(xy)u &= \rho(y)[\rho(x)u] \\
&= \rho(y)[\chi(x)u + w(x)] \\
&= \chi(x)\rho(y)u + \rho(y)w(x) \\
&= \chi(x)[\chi(y)u + w(y)] + [\chi(y)w(x) + \chi(y)v_k(y)t(x)] \\
&= \chi(xy)u + \chi(x)w(y) + \chi(y)w(x) + \chi(y)v_k(y)t(x).
\end{aligned} \tag{3.4}$$

Hence by (3.3)–(3.4) we have

$$w(xy) = \chi(x)w(y) + \chi(y)w(x) + \chi(y)v_k(y)t(x). \tag{3.5}$$

By symmetry we also have

$$w(xy) = \chi(y)w(x) + \chi(x)w(y) + \chi(x)v_k(x)t(y). \tag{3.6}$$

By (3.5)–(3.6) we have

$$\chi(x)v_k(x)t(y) = \chi(y)v_k(y)t(x).$$

Hence for $y = \varpi$ we have

$$\frac{t(\varpi)}{\chi(\varpi)}\chi(x)v_k(x) = t(x)$$

and by (3.5),

$$w(xy) = \chi(x)w(y) + \chi(y)w(x) + \frac{t(\varpi)}{\chi(\varpi)}\chi(y)v_k(y)\chi(x)v_k(x).$$

So we have

$$\frac{w(xy)}{\chi(xy)} = \frac{w(y)}{\chi(y)} + \frac{w(x)}{\chi(x)} + \frac{t(\varpi)}{\chi(\varpi)}v_k(y)v_k(x). \tag{3.7}$$

If we restrict $\frac{w}{\chi}$ to $\mathcal{O}^*$ we get a homomorphism from $\mathcal{O}^*$ to U_1. Since $\frac{w}{\chi}$ is a locally constant function the image of compact subgroup $\mathcal{O}^*$ is a finite subgroup of U_1. The only finite subgroup of U_1 is $\{0\}$. Hence the restriction of $\frac{w}{\chi}$ to $\mathcal{O}^*$ is zero. Hence for $x' \in \mathcal{O}^*$ we have

$$\frac{w(\varpi^i x')}{\chi(\varpi^i x')} = \frac{w(x')}{\chi(x')} + \frac{w(\varpi^i)}{\chi(\varpi^i)} = \frac{w(\varpi^i)}{\chi(\varpi^i)}.$$

By (3.7) and induction, we get

$$\frac{w(x)}{\chi(x)} = \frac{w(\varpi)}{\chi(\varpi)} v_k(x) + \frac{t(\varpi)}{\chi(\varpi)} \sum_{l=1}^{v_k(x)-1} l.$$

Hence

$$\begin{aligned}\frac{w(x)}{\chi(x)} &= \frac{w(\varpi)}{\chi(\varpi)} v_k(x) + \frac{t(\varpi)}{\chi(\varpi)} \frac{(v_k(x)-1)v_k(x)}{2} \\ &= \Big[\frac{w(\varpi)}{\chi(\varpi)} - \frac{t(\varpi)}{2\chi(\varpi)}\Big] v_k(x) + \frac{t(\varpi)}{2\chi(\varpi)} v_k^2(x).\end{aligned}$$

Hence by (3.1) we have

$$\rho(x)u = \chi(x)u + \Big[\frac{w(\varpi)}{\chi(\varpi)} - \frac{t(\varpi)}{2\chi(\varpi)}\Big] v_k(x)\chi(x) + \frac{t(\varpi)}{2\chi(\varpi)} v_k^2(x)\chi(x).$$

Proposition 3.2 *Let χ_1, χ_2, χ_3 and χ_4 be characters of k^*, and let (ρ, U), (ρ, U_1) and (ρ, U_2) be representations of k^* such that*

$$\begin{aligned} &0 \to U_1 \to U \to \oplus\chi_4 \to 0, \\ &0 \to U_2 \to U_1 \to \oplus\chi_3 \to 0, \\ &0 \to \oplus\chi_1 \to U_2 \to \oplus\chi_2 \to 0. \end{aligned} \tag{3.8}$$

(i) *If χ_1, χ_2, χ_3 and χ_4 are all different then $U = \oplus\chi_1 \bigoplus \oplus\chi_2 \bigoplus \oplus\chi_3 \bigoplus \oplus\chi_4$. Hence for all $u \in U$ there exist $u_i \in \oplus\chi_i$ for $i = 1, 2, 3, 4$ such that $u = u_1 + u_2 + u_3 + u_4$ and*

$$\rho(x)u = \chi_1(x)u_1 + \chi_2(x)u_2 + \chi_3(x)u_3 + \chi_4(x)u_4.$$

(ii) *If $\chi_1 = \chi_2$, $\chi_3 = \chi_4$ and $\chi_1 \neq \chi_3$ then $U = U_2 \oplus U_2'$ where $0 \to \oplus\chi_3 \to U_2' \to \oplus\chi_3 \to 0$. Hence for all $u \in U$ there exist $u_1 \in U_2$ and $u_2 \in U_2'$ such that $u = u_1 + u_2$ and for some $u_1' \in \oplus\chi_1$ and $u_2' \in \oplus\chi_3$ we have*

$$\rho(x)u = \chi_1(x)u_1 + \chi_1(x)v_k(x)u_1' + \chi_3(x)u_2 + \chi_3(x)v_k(x)u_2'.$$

(iii) *If $\chi_1 = \chi_3$, $\chi_2 = \chi_4$ and $\chi_1 \neq \chi_2$ then $U = U_2' \oplus U_2''$ where $0 \to \oplus\chi_1 \to U_2' \to \oplus\chi_1 \to 0$ and $0 \to \oplus\chi_2 \to U_2'' \to \oplus\chi_2 \to 0$. Hence for all $u \in U$ there exist $u_1 \in U_2'$ and $u_2 \in U_2''$ such that $u = u_1 + u_2$ and for some $u_1' \in \oplus\chi_1$ and $u_2' \in \oplus\chi_2$ we have*

$$\rho(x)u = \chi_1(x)u_1 + \chi_1(x)v_k(x)u_1' + \chi_2(x)u_2 + \chi_2(x)v_k(x)u_2'.$$

(iv) *If* $\chi_1 = \chi_4$, $\chi_2 = \chi_3$ *and* $\chi_1 \neq \chi_2$ *then* $U = U_2' \oplus U_2''$ *where* $0 \to \oplus\chi_1 \to U_2' \to \oplus\chi_1 \to 0$ *and* $0 \to \oplus\chi_2 \to U_2'' \to \oplus\chi_2 \to 0$. *Hence for all* $u \in U$ *there exist* $u_1 \in U_2'$ *and* $u_2 \in U_2''$ *such that* $u = u_1 + u_2$ *and for some* $u_1' \in \oplus\chi_1$ *and* $u_2' \in \oplus\chi_2$ *we have*

$$\rho(x)u = \chi_1(x)u_1 + \chi_1(x)v_k(x)u_1' + \chi_2(x)u_2 + \chi_2(x)v_k(x)u_2'.$$

(v) *If* $\chi_1 = \chi_2 = \chi_3 = \chi_4 = \chi$ *and* $u \in U$ *then for some* $u_1 \in U_1$, $u_2 \in U_2$ *and* $u_3 \in \oplus\chi$ *we have*

$$\rho(x)u = \chi(x)u + \chi(x)v_k(x)u_1 + \chi(x)v_k^2(x)u_2 + \chi(x)v_k^3(x)u_3.$$

(vi) *If* $\chi_1 = \chi_4$ *and* χ_1, χ_2, χ_3 *are different then* $U = U_2' \bigoplus \oplus\chi_2 \oplus \chi_3$ *where* $0 \to \oplus\chi_1 \to U_2' \to \oplus\chi_1 \to 0$. *Hence for all* $u \in U$ *there exist* $u_1 \in U_2'$, $u_2 \in \oplus\chi_2$ *and* $u_3 \in \oplus\chi_3$ *such that* $u = u_1 + u_2 + u_3$ *and for some* $u_1' \in \oplus\chi_1$ *we have*

$$\rho(x)u = \chi_1(x)u_1 + \chi_1(x)v_k(x)u_1' + \chi_2(x)u_2 + \chi_3(x)u_3.$$

(vii) *If* $\chi_2 = \chi_3$ *and* χ_1, χ_2, χ_4 *are different then* $U = U_2' \bigoplus \oplus\chi_1 \oplus \chi_4$ *where* $0 \to \oplus\chi_2 \to U_2' \to \oplus\chi_2 \to 0$. *Hence for all* $u \in U$ *there exist* $u_1 \in U_2'$, $u_2 \in \oplus\chi_1$ *and* $u_3 \in \oplus\chi_4$ *such that* $u = u_1 + u_2 + u_3$ *and for some* $u_1' \in \oplus\chi_2$ *we have*

$$\rho(x)u = \chi_2(x)u_1 + \chi_2(x)v_k(x)u_1' + \chi_1(x)u_2 + \chi_4(x)u_3.$$

Proof Since the proofs are similar we give only the proof of the fifth one which is also the most complicated one. For $u \in U$ there is a $w(x) \in U_1$ such that

$$\rho(x)u = \chi(x)u + w(x). \tag{3.9}$$

By Proposition 3.1(iii) there exist $t(x) \in U_2$ and $h(x) \in \oplus\chi$ such that

$$\rho(y)w(x) = \chi(y)w(x) + \chi(y)v_k(y)t(x) + \chi(y)v_k^2(y)h(x). \tag{3.10}$$

From (3.9),

$$\rho(xy)u = \chi(xy)u + w(xy). \tag{3.11}$$

By (3.9)–(3.10),

$$\begin{aligned}
\rho(xy)u &= \rho(y)[\rho(x)u] \\
&= \rho(y)[\chi(x)u + w(x)] \\
&= \chi(x)\rho(y)u + \rho(y)w(x) \\
&= \chi(x)[\chi(y)u + w(y)] + [\chi(y)w(x) + \chi(y)v_k(y)t(x) + \chi(y)v_k^2(y)h(x)] \\
&= \chi(xy)u + \chi(x)w(y) + \chi(y)w(x) + \chi(y)v_k(y)t(x) + \chi(y)v_k^2(y)h(x). \qquad (3.12)
\end{aligned}$$

Hence by (3.11)–(3.12) we have

$$w(xy) = \chi(x)w(y) + \chi(y)w(x) + \chi(y)v_k(y)t(x) + \chi(y)v_k^2(y)h(x). \tag{3.13}$$

By symmetry we also have

$$w(xy) = \chi(y)w(x) + \chi(x)w(y) + \chi(x)v_k(x)t(y) + \chi(x)v_k^2(x)h(y).$$

Hence

$$\chi(y)v_k(y)t(x) + \chi(y)v_k^2(y)h(x) = \chi(x)v_k(x)t(y) + \chi(x)v_k^2(x)h(y).$$

If $y = \varpi$ then

$$\chi(\varpi)t(x) + \chi(\varpi)h(x) = \chi(x)v_k(x)t(\varpi) + \chi(x)v_k^2(x)h(\varpi), \tag{3.14}$$

and if $y = \varpi^2$ then

$$2\chi(\varpi)^2 t(x) + 4\chi(\varpi)^2 h(x) = \chi(x)v_k(x)t(\varpi^2) + \chi(x)v_k^2(x)h(\varpi^2). \tag{3.15}$$

By (3.14)–(3.15) we have

$$\begin{aligned} h(x) &= A_1\chi(x)v_k(x) + A_2\chi(x)v_k^2(x), \\ t(x) &= B_1\chi(x)v_k(x) + B_2\chi(x)v_k^2(x), \end{aligned}$$

where

$$A_1 = \frac{2\chi(\varpi)t(\varpi) - t(\varpi^2)}{-2\chi(\varpi)^2}, \quad A_2 = \frac{2\chi(\varpi)h(\varpi) - h(\varpi^2)}{-2\chi(\varpi)^2}$$

and

$$B_1 = \frac{-4\chi(\varpi)t(\varpi) + t(\varpi^2)}{-2\chi(\varpi)^2}, \quad B_2 = \frac{h(\varpi^2) - 4\chi(\varpi)h(\varpi)}{-2\chi(\varpi)^2}.$$

By (3.13),

$$\begin{aligned} w(xy) = \chi(x)w(y) + \chi(y)w(x) + \chi(xy)v_k(x)v_k^2(y)A_1 + \chi(xy)v_k^2(x)v_k^2(y)A_2 \\ + \chi(xy)v_k(x)v_k(y)B_1 + \chi(xy)v_k^2(x)v_k(y)B_2. \end{aligned} \tag{3.16}$$

If we restrict $\frac{w}{\chi}$ to $\mathcal{O}^*$ we get a homomorphism from $\mathcal{O}^*$ to U_1. Since $\frac{w}{\chi}$ is a locally constant function the image of compact subgroup $\mathcal{O}^*$ is a finite subgroup of U_1. The only finite subgroup of U_1 is $\{0\}$. Hence the restriction of $\frac{w}{\chi}$ to $\mathcal{O}^*$ is zero. Hence for $x' \in \mathcal{O}^*$ we have

$$\frac{w(\varpi^i x')}{\chi(\varpi^i x')} = \frac{w(x')}{\chi(x')} + \frac{w(\varpi^i)}{\chi(\varpi^i)} = \frac{w(\varpi^i)}{\chi(\varpi^i)}.$$

By (3.16) and induction, we get

$$\frac{w(x)}{\chi(x)} = \frac{w(\varpi)}{\chi(\varpi)}v_k(x) + (A_1 + B_1)\sum_{l=1}^{v_k(x)-1} l + (A_2 + B_2)\sum_{l=1}^{v_k(x)-1} l^2.$$

Hence

$$\begin{aligned} \frac{w(x)}{\chi(x)} &= \frac{w(\varpi)}{\chi(\varpi)}v_k(x) + (A_1 + B_1)\frac{(v_k(x)-1)v_k(x)}{2} \\ &\quad + (A_2 + B_2)\frac{(v_k(x)-1)v_k(x)(2v_k(x)-1)}{6} \\ &= \Big[\frac{w(\varpi)}{\chi(\varpi)} - \frac{A_1 + B_1}{2} + \frac{A_2 + B_2}{6}\Big]v_k(x) \\ &\quad + \Big[\frac{A_1 + B_1}{2} - \frac{A_2 + B_2}{2}\Big]v_k^2(x) + \Big[\frac{A_2 + B_2}{3}\Big]v_k^3(x). \end{aligned}$$

Hence by (3.9) we have

$$\begin{aligned} \rho(x)u = \chi(x)u + \Big[\frac{w(\varpi)}{\chi(\varpi)} - \frac{A_1 + B_1}{2} + \frac{A_2 + B_2}{6}\Big]\chi(x)v_k(x) \\ + \Big[\frac{A_1 + B_1}{2} - \frac{A_2 + B_2}{2}\Big]\chi(x)v_k^2(x) + \Big[\frac{A_2 + B_2}{3}\Big]\chi(x)v_k^3(x). \end{aligned}$$

4 Asymptotic Behavior of φ_u

In this section, we determine the asymptotic behavior of $\varphi_u(x)$ for small enough $|x|$ and compute the possible poles of the L-factor.

Proposition 4.1 *Let $u, w_1, w_1', w_2 \in V_\Pi$. If*

$$\begin{aligned}
&\Pi(h_x)u - \chi(x)u - \chi(x)v_k(x)w_1 - \chi(x)v_k^2(x)w_2 \in V_S(\Pi),\\
&\Pi(h_x)w_1 - \chi(x)w_1 - \chi(x)v_k(x)w_1' \in V_S(\Pi),\\
&\Pi(h_x)w_1' - \chi(x)w_1' \in V_S(\Pi),\\
&\Pi(h_x)w_2 - \chi(x)w_2 \in V_S(\Pi),
\end{aligned}$$

then for sufficiently small $|x|$ and constants C_1, C_2, C_3 we have

$$\varphi_u(x) = C_1\chi(x) + C_2\chi(x)v_k(x) + C_3\chi(x)v_k^2(x).$$

Proof It is an easier version of Proposition 4.3.

Proposition 4.2 *Let $0 \to U_1 \to U \to \oplus\mathbb{1} \to 0$, $0 \to \oplus\mathbb{1} \to U_1 \to \oplus\mathbb{1} \to 0$, U be a subrepresentation of $R_S(\Pi)$ as an H module for an appropriate choice of χ_1, χ_2 and σ and $u \in V_\Pi$ such that the image of u in $R_S(\Pi)$ is $\overline{u}$ and $\overline{u} \in U$. Then for small enough $|x|$ and constants C_1, C_2 and C_3, we have*

$$\varphi_u(x) = C_1|x|^{\frac{3}{2}} + C_2|x|^{\frac{3}{2}}v_k(x) + C_3|x|^{\frac{3}{2}}v_k^2(x).$$

Proof By Proposition 3.1(iii), we have

$$\delta_P^{-\frac{1}{2}}\Pi_S(h_x)\overline{u} = \overline{u} + v_k(x)\overline{w}_1 + v_k^2(x)\overline{w}_2$$

for some $w_1, w_2 \in V_\Pi$ such that $\overline{w}_1 \in U_1$, $\overline{w}_2 \in \oplus\mathbb{1}$. Note that $\delta_P(h_x) = |x|^3$. Hence

$$\Pi_S(h_x)u - |x|^{\frac{3}{2}}u - |x|^{\frac{3}{2}}v_k(x)w_1 - |x|^{\frac{3}{2}}v_k^2(x)w_2 \in V_S(\Pi).$$

Now use Proposition 4.1 for which parts related to w_1 and w_2 follows from Propositions 3.2 and 3.5 of [3].

Lemma 4.1 *If $\varphi_u(x) = C|x|^{\frac{3}{2}}\chi(x)$ for some character χ of k^* and $|x| \le q^{-j_0}$, then the pole of $\int_{k^*}\varphi_u(x)\mu(x)|x|^{s-\frac{3}{2}}\,\mathrm{d}^*x$ is the pole of $CL(s,\mu\chi)$.*

Proof Similar to the proof of [3, Lemma 3.4].

Lemma 4.2 *If $\varphi_u(x) = C_1|x|^{\frac{3}{2}}\chi(x) + C_2|x|^{\frac{3}{2}}\chi(x)v_k(x)$ for some character χ of k^* and $|x| \le q^{-j_0}$, then the poles of $\int_{k^*}\varphi_u(x)\mu(x)|x|^{s-\frac{3}{2}}\,\mathrm{d}^*x$ are the poles of the least common multiple of $C_1L(s,\mu\chi)$ and $C_2L(s,\mu\chi)^2$.*

Proof Similar to the proof of [3, Lemma 3.7].

Lemma 4.3 *If $\varphi_u(x) = C_1|x|^{\frac{3}{2}}\chi(x) + C_2|x|^{\frac{3}{2}}\chi(x)v_k(x) + C_3|x|^{\frac{3}{2}}\chi(x)v_k^2(x)$ for some character χ of k^* and $|x| \le q^{-j_0}$, then the poles of $\int_{k^*}\varphi_u(x)\mu(x)|x|^{s-\frac{3}{2}}\,\mathrm{d}^*x$ are the poles of the least common multiple of $C_1L(s,\mu\chi)$, $C_2L(s,\mu\chi)^2$ and $C_3L(s,\mu\chi)^3$.*

Proof It is an easier version of Lemma 4.5.

Proposition 4.3 *Let $u, w_1, w_1', w_{11}', w_1'', w_2, w_3 \in V_\Pi$. If*

$$\begin{aligned}
&\Pi(h_x)u - \chi(x)u - \chi(x)v_k(x)w_1 - \chi(x)v_k^2(x)w_2 - \chi(x)v_k^3(x)w_3 \in V_S(\Pi),\\
&\Pi(h_x)w_1 - \chi(x)w_1 - \chi(x)v_k(x)w_1' - \chi(x)v_k^2(x)w_1'' \in V_S(\Pi),\\
&\Pi(h_x)w_1' - \chi(x)w_1' - \chi(x)v_k(x)w_{11}' \in V_S(\Pi),\\
&\Pi(h_x)w_{11}' - \chi(x)w_{11}' \in V_S(\Pi),\\
&\Pi(h_x)w_1'' - \chi(x)w_1'' \in V_S(\Pi),\\
&\Pi(h_x)w_2 - \chi(x)w_2 - \chi(x)v_k(x)w_2' \in V_S(\Pi),\\
&\Pi(h_x)w_2' - \chi(x)w_2' \in V_S(\Pi),\\
&\Pi(h_x)w_3 - \chi(x)w_3 \in V_S(\Pi),
\end{aligned}$$

then for sufficiently small $|x|$ and constants D_1, D_2, D_3, D_4, we have

$$\varphi_u(x) = D_1\chi(x) + D_2\chi(x)v_k(x) + D_3\chi(x)v_k^2(x) + D_4\chi(x)v_k^3(x).$$

Proof By [3, Proposition 3.2], $\varphi_{w_3}(x) = B\chi(x)$, by [3, Proposition 3.5], $\varphi_{w_2}(x) = A_1\chi(x) + A_2v_k(x)\chi(x)$ and by Proposition 4.1, $\varphi_{w_1}(x) = C_1\chi(x) + C_2v_k(x)\chi(x) + C_3v_k^2(x)\chi(x)$ for small enough $|x|$. Let $x_0 \in \varpi\mathcal{O}^*$. Then we have

$$\Pi(h_{x_0})u - \chi(x_0)u - \chi(x_0)v_k(x_0)w_1 - \chi(x_0)v_k^2(x_0)w_2 - \chi(x_0)v_k^3(x_0)w_3 \in V_S(\Pi).$$

By [3, Proposition 3.1],

$$\varphi_{\Pi(h_{x_0})u - \chi(x_0)u - \chi(x_0)v_k(x_0)w_1 - \chi(x_0)v_k^2(x_0)w_2 - \chi(x_0)v_k^3(x_0)w_3}$$

vanishes near zero. So there exists a constant $\epsilon(x_0)$ such that for $x = x_0$ and $|t| \le \epsilon(x_0)$ such that

$$\begin{aligned}
0 &= \varphi_{\Pi(h_x)u - \chi(x)u - \chi(x)v_k(x)w_1 - \chi(x)v_k^2(x)w_2 - \chi(x)v_k^3(x)w_3}(t)\\
&= \varphi_u(xt) - \chi(x)\varphi_u(t) - \chi(x)v_k(x)\varphi_{w_1}(t) - \chi(x)v_k^2(x)\varphi_{w_2}(t)\\
&\quad - \chi(x)v_k^3(x)\varphi_{w_3}(t)\\
&= \varphi_u(xt) - \chi(x)\varphi_u(t) - \chi(x)v_k(x)[C_1\chi(t) + C_2\chi(t)v_k(t) + C_3\chi(t)v_k^2(t)]\\
&\quad - \chi(x)v_k^2(x)[A_1\chi(t) + A_2v_k(t)\chi(t)] - \chi(x)v_k^3(x)B\chi(t)\\
&= \varphi_u(xt) - \chi(x)\varphi_u(t) - C_1\chi(xt)v_k(x) - C_2\chi(xt)v_k(x)v_k(t)\\
&\quad - C_3\chi(xt)v_k(x)v_k^2(t) - A_1\chi(xt)v_k^2(x) - A_2\chi(xt)v_k^2(x)v_k(t)\\
&\quad - B\chi(xt)v_k^3(x).
\end{aligned}$$

Hence

$$\begin{aligned}
&\varphi_u(xt)\\
&= \chi(x)\varphi_u(t) + C_1\chi(xt)v_k(x) + C_2\chi(xt)v_k(x)v_k(t) + C_3\chi(xt)v_k(x)v_k^2(t)\\
&\quad + A_1\chi(xt)v_k^2(x) + A_2\chi(xt)v_k^2(x)v_k(t) + B\chi(xt)v_k^3(x). \qquad (4.1)
\end{aligned}$$

Since Π and χ are smooth this is also valid when x is near x_0 and $|t| \le \epsilon(x_0)$, so by compactness of $\varpi\mathcal{O}^*$, this is also true for $x \in \varpi\mathcal{O}^*$ and $|t| \le \epsilon = q^{-j_0}$ for some constant j_0.

Lemma 4.4

$$\varphi_u(\varpi^i z) = \chi(\varpi^i z)\Big[\chi(\varpi^{-j_0})\varphi_u(\varpi^{j_0}) + (i - j_0)(C_1 + A_1 + B) + (C_2 + A_2)\sum_{l=j_0}^{i-1} l + C_3 \sum_{l=j_0}^{i-1} l^2\Big]$$

for $i \geq j_0 + 1$ *and* $z \in \mathcal{O}^*$.

Proof Proof is by induction. The base case follows from (4.1). Now assume that the result holds for some $i \geq j_0 + 1$ and prove it for $i + 1$:

$$\begin{aligned}
&\varphi_u(\varpi^{i+1} z)\\
&= \varphi_u(\varpi z \varpi^i)\\
&= \chi(\varpi z)\varphi_u(\varpi^i) + \chi(\varpi^{i+1} z)[C_1 + C_2 i + C_3 i^2 + A_1 + A_2 i + B]\\
&= \chi(\varpi^{i+1} z)\Big[\chi(\varpi^{-j_0})\varphi_u(\varpi^{j_0}) + (i - j_0)(C_1 + A_1 + B)\\
&\quad + (C_2 + A_2)\sum_{l=j_0}^{i-1} l + C_3 \sum_{l=j_0}^{i-1} l^2\Big]\\
&\quad + \chi(\varpi^{i+1} z)[C_1 + C_2 i + C_3 i^2 + A_1 + A_2 i + B]\\
&= \chi(\varpi^{i+1} z)\Big[\chi(\varpi^{-j_0})\varphi_u(\varpi^{j_0}) + (i + 1 - j_0)(C_1 + A_1 + B)\\
&\quad + (C_2 + A_2)\sum_{l=j_0}^{i} l + C_3 \sum_{l=j_0}^{i} l^2\Big].
\end{aligned}$$

Let $D_1 = \chi(\varpi^{-j_0})\varphi_u(\varpi^{j_0}) - j_0(C_1 + A_1 + B) - \frac{(C_2+A_2)(j_0-1)j_0}{2} - C_3\frac{(j_0-1)j_0(2j_0-1)}{6}$, $D_2 = (C_1 + A_1 + B) - \frac{C_2+A_2}{2} + \frac{C_3}{6}$, $D_3 = \frac{C_2+A_2}{2} - \frac{C_3}{2}$, $D_4 = \frac{C_3}{3}$ and $|x| \leq q^{-j_0}$. Then $x = \varpi^i z$ for some $i \geq j_0 + 1$ and $z \in \mathcal{O}^*$, so the proposition follows from the previous lemma.

Proposition 4.4 *Let* $0 \to U_1 \to U \to \oplus\mathbb{1} \to 0$, $0 \to U_2 \to U_1 \to \oplus\mathbb{1} \to 0$, $0 \to \oplus\mathbb{1} \to U_2 \to \oplus\mathbb{1} \to 0$, U *be a subrepresentation of* $R_S(\Pi)$ *as an* H *module and* $u \in V_\Pi$ *such that the image of* u *in* $R_S(\Pi)$ *is* $\overline{u}$ *and* $\overline{u} \in U$. *Then for small enough* $|x|$ *and constants* D_1, D_2, D_3 *and* D_4 *we have*

$$\varphi_u(x) = D_1|x|^{\frac{3}{2}} + D_2|x|^{\frac{3}{2}}v_k(x) + D_3|x|^{\frac{3}{2}}v_k^2(x) + D_4|x|^{\frac{3}{2}}v_k^3(x).$$

Proof By Proposition 3.2(iii), we have

$$\delta_P^{-\frac{1}{2}}\Pi_S(h_x)\overline{u} = \overline{u} + v_k(x)\overline{w}_1 + v_k^2(x)\overline{w}_2 + v_k^3(x)\overline{w}_3$$

for some $w_1, w_2, w_3 \in V_\Pi$ and $\overline{w}_1 \in U_1$, $\overline{w}_2 \in U_2$, $\overline{w}_3 \in \oplus\mathbb{1}$. Hence

$$\Pi_S(h_x)u - |x|^{\frac{3}{2}}u - |x|^{\frac{3}{2}}v_k(x)w_1 - |x|^{\frac{3}{2}}v_k^2(x)w_2 \in V_S(\Pi).$$

Now use Proposition 4.3. Parts related to w_1, w_2, w_3 follows from Proposition 4.2 and Propositions 3.2 and 3.5 of [3].

Lemma 4.5 *If*

$$\varphi_u(x) = \chi(x)[D_1|x|^{\frac{3}{2}} + D_2|x|^{\frac{3}{2}}v_k(x) + D_3|x|^{\frac{3}{2}}v_k^2(x) + D_4|x|^{\frac{3}{2}}v_k^3(x)]$$

for some character χ of k^* and $|x| \leq q^{-j_0}$, then the poles of $\int_{k^*} \varphi_u(x)\mu(x)|x|^{s-\frac{3}{2}}\mathrm{d}^*x$ are the poles of the least common multiple of $D_1L(s,\mu\chi)$, $D_2L(s,\mu\chi)^2$, $D_3L(s,\mu\chi)^3$ and $D_4L(s,\mu\chi)^4$.

Proof

$$\begin{aligned}
&\int_{|x|\leq q^{-j_0}} D_4|x|^{\frac{3}{2}}\mu(x)\chi(x)v_k^3(x)|x|^{s-\frac{3}{2}}\mathrm{d}^*x\\
&= D_4\int_{|x|\leq q^{-j_0}} \mu(x)\chi(x)v_k^3(x)|x|^{s}\mathrm{d}^*x\\
&= D_4\sum_{i=j_0}^{\infty}\int_{|x|=q^{-i}} \mu(x)\chi(x)v_k^3(x)|x|^{s}\mathrm{d}^*x\\
&= D_4\sum_{i=j_0}^{\infty} i^3q^{-is}\mu(\varpi^i)\chi(\varpi^i)\int_{\mathcal{O}^*}\mu(u)\chi(u)\mathrm{d}u\\
&= \begin{cases} 0, & \mu\chi\text{: ramified,}\\ D_4\frac{\alpha^{j_0}[j_0^3-(3j_0^3-3j_0^2-3j_0-1)\alpha+(3j_0^3-6j_0^2+4)\alpha^2-(j_0-1)^3\alpha^3]}{(1-\alpha)^4}(1-\frac{1}{q}), & \text{otherwise,}\end{cases}
\end{aligned}$$

where $\alpha = q^{-s}\mu(\varpi)\chi(\varpi)$. Now the result follows from Lemmas 3.4 and 3.7 of [3] and Lemma 4.3.

Theorem 4.1 *Let $u \in V_\Pi$ and $|x|$ small enough. Then the asymptotic behavior of Bessel model of Π is*

(i) *if $\chi_1 = \mathbb{1}$, $\chi_2 \neq \mathbb{1}$, then*

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2\chi_2\sigma(x) + D_3v_k(x)\sigma(x) + D_4v_k(x)\chi_2\sigma(x)],$$

(ii) *if $\chi_2 = \mathbb{1}$, $\chi_1 \neq \mathbb{1}$, then*

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2v_k(x)\sigma(x) + D_3\chi_1\sigma(x) + D_4v_k(x)\chi_1\sigma(x)],$$

(iii) *if $\chi_1 = \chi_2 = \chi$, $\chi^2 = \mathbb{1}$, $\chi \neq \mathbb{1}$, then*

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2\chi\sigma(x) + D_3v_k(x)\chi\sigma(x) + D_4v_k(x)\sigma(x)],$$

(iv) *if $\chi_1 = \chi_2 = \chi$, $\chi^2 \neq \mathbb{1}$, then*

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2\chi\sigma(x) + D_3v_k(x)\chi\sigma(x) + D_4\chi^2\sigma(x)],$$

(v) *if $\chi_1 = \chi_2^{-1} = \chi$, $\chi^2 \neq \mathbb{1}$, then*

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2\chi^{-1}\sigma(x) + D_3\chi\sigma(x) + D_4v_k(x)\sigma(x)],$$

(vi) *if $\chi_1 = \chi_2 = \mathbb{1}$, then*

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2v_k(x)\sigma(x) + D_3v_k^2(x)\sigma(x) + D_4v_k^3(x)\sigma(x)],$$

(vii) *otherwise,*

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2\chi_2\sigma(x) + D_3\chi_1\sigma(x) + D_4\chi_1\chi_2\sigma(x)].$$

Proof By Proposition 2.1, the constituents of the Jacquet module of Π are

$$\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1^{-1}, \chi_2^{-1}) \otimes \chi_1\chi_2\sigma, \quad \mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1^{-1}, \chi_2) \otimes \chi_1\sigma,$$

$$\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2^{-1}) \otimes \chi_2\sigma, \quad \mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2) \otimes \sigma.$$

As a representation of H, the constituents are $\oplus\sigma, \oplus\chi_2\sigma, \oplus\chi_1\sigma, \oplus\chi_1\chi_2\sigma$.

(i) If $\chi_1 = \mathbb{1},\ \chi_2 \neq \mathbb{1}$, then the constituents are $\oplus\sigma, \oplus\chi_2\sigma, \oplus\sigma, \oplus\chi_2\sigma$ and we are in the case of Proposition 3.2(iii). Hence the result follows from [3, Proposition 3.5].

(ii) If $\chi_2 = \mathbb{1},\ \chi_1 \neq \mathbb{1}$, then the constituents are $\oplus\sigma, \oplus\sigma, \oplus\chi_1\sigma, \oplus\chi_1\sigma$ and we are in the case of Proposition 3.2(ii). Hence the result follows from [3, Proposition 3.5].

(iii) If $\chi_1 = \chi_2 = \chi,\ \chi^2 = \mathbb{1}, \chi \neq \mathbb{1}$, then the constituents are $\oplus\sigma, \oplus\chi\sigma, \oplus\chi\sigma, \oplus\sigma$ and we are in the case of Proposition 3.2(iv). Hence the result follows from [3, Proposition 3.5].

(iv) If $\chi_1 = \chi_2 = \chi,\ \chi^2 \neq \mathbb{1}$, then the constituents are $\oplus\sigma, \oplus\chi\sigma, \oplus\chi\sigma, \oplus\chi^2\sigma$. So we are in the case of Proposition 3.2(vii). Hence the result follows from [3, Propositions 3.2 and 3.5].

(v) If $\chi_1 = \chi_2^{-1} = \chi,\ \chi^2 \neq \mathbb{1}$, then the constituents are $\oplus\sigma, \oplus\chi^{-1}\sigma, \oplus\chi\sigma, \oplus\sigma$. So we are in the case of Propositions 3.2(vi). Hence the result follows from [3, Proposition 3.2 and 3.5].

(vi) If $\chi_1 = \chi_2 = \mathbb{1}$ then the constituents are four σ. So we are in the case of Proposition 3.2(v). Hence the result follows from Proposition 4.4.

(vii) Otherwise the constituents $\oplus\sigma, \oplus\chi_2\sigma, \oplus\chi_1\sigma, \oplus\chi_1\chi_2\sigma$ are all different. So we are in the case of Proposition 3.2(i). Hence the result follows from [3, Proposition 3.2].

5 Computation of *L*-factor

In this section we determine whether constants D_i for $i = 1, 2, 3, 4$ in Theorem 4.1 are nonzero or not.

Let

$$\overline{V}_T(\Lambda, \Pi) := \{\Pi(t)\overline{v} - \Lambda(t)\overline{v} : \overline{v} \in V_\Pi / V_S(\Pi)\}.$$

The representation $\Pi = \chi_1 \times \chi_2 \rtimes \sigma$ has Jacquet module length 4. By Proposition 2.1 as a representation of H we have

$$0 \underbrace{\subset}_{\oplus\sigma} (\Pi_3)_S \underbrace{\subset}_{\oplus\chi_2\sigma} (\Pi_2')_S \underbrace{\subset}_{\oplus\chi_1\sigma} (\Pi_2)_S \underbrace{\subset}_{\oplus\chi_1\chi_2\sigma} \Pi_S.$$

Case 1 If $\chi_1 = \mathbb{1},\ \chi_2 \neq \mathbb{1}$, then by Theorem 4.1(i) for every $u \in V_\Pi$ we have

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2\chi_2(x)\sigma(x) + D_3 v_k(x)\sigma(x) + D_4 v_k(x)\chi_2(x)\sigma(x)].$$

Proposition 5.1 *For some choice of u, the constants D_3 and D_4 are nonzero.*

Proof By Proposition 3.2(iii) $\Pi_S = U_2' \oplus U_2''$ where U_2' is an extension of two $\oplus\sigma$ and U_2'' is an extension of two $\oplus\chi_2\sigma$. Also $(\Pi_2)_S = U_2' \bigoplus \oplus\chi_2\sigma$. If $D_3 = 0$, then for every $\overline{u} \in U_2'$ there exists a $\overline{u}_1 \in \oplus\sigma$ such that $\overline{u} - \overline{u}_1 \in \overline{V}_T(\Lambda, \Pi)$. Hence

$$\overline{u} - \overline{u}_1 = \sum_{i=1}^{N_1} a_i[\Pi_S(t_i)\overline{u}_1^i - \Lambda(t_i)\overline{u}_1^i] + \sum_{j=1}^{N_2} b_j[\Pi_S(t_j)\overline{u}_2^j - \Lambda(t_j)\overline{u}_2^j],$$

where $a_i, b_j \in k,\ t_i, t_j \in T$ and $\overline{u}_1^i \in U_2', \overline{u}_2^j \in U_2''$. Note that

$$\sum_{i=1}^{N_1} a_i[\Pi_S(t_i)\overline{u}_1^i - \Lambda(t_i)\overline{u}_1^i] \in U_2',$$
$$\sum_{j=1}^{N_2} b_j[\Pi_S(t_j)\overline{u}_2^j - \Lambda(t_j)\overline{u}_2^j] \in U_2''.$$

Since we have a direct sum $\overline{u} - \overline{u}_1 - \sum_{i=1}^{N_1} a_i[\Pi_S(t_i)\overline{u}_1^i - \Lambda(t_i)\overline{u}_1^i] = 0$. Hence

$$\overline{u} = \overline{u}_1 + \sum_{i=1}^{N_1} a_i[\Pi_S(t_i)\overline{u}_1^i - \Lambda(t_i)\overline{u}_1^i]$$

and

$$\begin{aligned} 0 &= \mathrm{Hom}_T\Big[U_2' \bigoplus \oplus\chi_2\sigma / \oplus\sigma \bigoplus \oplus\chi_2\sigma, \Lambda\Big] \\ &= \mathrm{Hom}_T[(\Pi_2)_S/(\Pi_2')_S, \Lambda] \\ &= \mathrm{Hom}_T[\mathrm{Hom}_T[\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2^{-1}) \otimes \chi_2\sigma, \Lambda], \end{aligned}$$

which is a contradiction by [13, Proposition 1.6].

If $D_4 = 0$, then for all $\overline{u} \in \Pi_S$ there exists $\overline{u}_2 \in (\Pi_2)_S$ such that $\overline{u} - \overline{u}_2 \in \overline{V}_T(\Lambda, \Pi)$. Hence

$$\begin{aligned} 0 &= \mathrm{Hom}_T[\Pi_S/(\Pi_2)_S, \Lambda] \\ &= \mathrm{Hom}_T[\mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2) \otimes \sigma, \Lambda] \\ &= \mathrm{Hom}_T[\sigma \mathrm{ind}_B^{\mathrm{GL}_2(k)}(\chi_1, \chi_2), \Lambda], \end{aligned}$$

which is a contradiction by [13, Proposition 1.6].

Case 2 If $\chi_1 \neq \mathbb{1},\ \chi_2 = \mathbb{1}$, then by Theorem 4.1(ii) for every $u \in V_\Pi$ we have

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2 v_k(x)\sigma(x) + D_3\chi_1(x)\sigma(x) + D_4 v_k(x)\chi_1(x)\sigma(x)].$$

Proposition 5.2 *For some choice of u, the constants D_2 and D_4 are nonzero.*

Proof It is similar to the proof of Proposition 5.1.

Case 3 If $\chi_1 = \chi_2 = \chi,\ \chi^2 = \mathbb{1},\ \chi \neq \mathbb{1}$, then by Theorem 4.1(iii) for every $u \in V_\Pi$ we have

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2\chi(x)\sigma(x) + D_3 v_k(x)\chi(x)\sigma(x) + D_4 v_k(x)\sigma(x)].$$

Proposition 5.3 *For some choice of u, the constants D_3 and D_4 are nonzero.*

Proof It is similar to the proof of Proposition 5.1.

Case 4 If $\chi_1 = \chi_2 = \chi,\ \chi^2 \neq \mathbb{1}$, then by Theorem 4.1(iv) for every $u \in V_\Pi$ we have

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2\chi\sigma(x) + D_3 v_k(x)\chi(x)\sigma(x) + D_4\chi^2(x)\sigma(x)].$$

Proposition 5.4 *For some choice of u, the constants D_1, D_3 and D_4 are nonzero.*

Proof It is similar to the proof of Proposition 5.1.

Case 5 If $\chi_1 = \chi_2^{-1} = \chi,\ \chi^2 \neq \mathbb{1}$, then by Theorem 4.1(v) for every $u \in V_\Pi$ we have

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2\chi^{-1}(x)\sigma(x) + D_3\chi(x)\sigma(x) + D_4 v_k(x)\sigma(x)].$$

Proposition 5.5 *For some choice of u, the constants* D_2, D_3 *and* D_4 *are nonzero.*

Proof It is similar to the proof of Proposition 5.1.

Case 6 If $\chi_1 = \chi_2$, then by Theorem 4.1(vi) for every $u \in V_\Pi$ we have

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\sigma(x) + D_2 v_k(x)\sigma(x) + D_3 v_k(x)^2\sigma(x) + D_4 v_k(x)^3\sigma(x)].$$

Proposition 5.6 *For some choice of u,* D_4 *is nonzero.*

Proof The proof is similar to the proof of Proposition 5.1.

Case 7 If $\sigma, \chi_1\sigma, \chi_2\sigma$ and $\chi_1\chi_2\sigma$ are all different, then by Theorem 4.1(vii) for every $u \in V_\Pi$ we have

$$\varphi_u(x) = |x|^{\frac{3}{2}}[D_1\chi_1\chi_2\sigma(x) + D_2\sigma(x) + D_3\chi_1\sigma(x) + D_4\chi_2\sigma(x)].$$

Proposition 5.7 *For some choice of u, the constants* D_i*'s for* $i = 1, 2, 3, 4$ *are all nonzero.*

Proof The proof is similar to the proof of Proposition 6.5 of [4].

Theorem 5.1 *L-factor of* Π *is*
(i) *if* $\chi_1 = \mathbb{1}$ *and* $\chi_2 \neq \mathbb{1}$*, then* $L(s,\mu\sigma)^2 L(s,\mu\chi_2\sigma)^2$,
(ii) *if* $\chi_1 \neq \mathbb{1}$ *and* $\chi_2 = \mathbb{1}$*, then* $L(s,\mu\sigma)^2 L(s,\mu\chi_1\sigma)^2$,
(iii) *if* $\chi_1 = \chi_2 = \chi,\ \chi^2 = \mathbb{1}$ *and* $\chi \neq \mathbb{1}$*, then* $L(s,\mu\sigma)^2 L(s,\mu\chi\sigma)^2$,
(iv) *if* $\chi_1 = \chi_2 = \chi,\ \chi^2 \neq \mathbb{1}$*, then* $L(s,\mu\sigma)L(s,\mu\chi\sigma)^2 L(s,\mu\chi^2\sigma)$,
(v) *if* $\chi_1 = \chi_2^{-1} = \chi,\ \chi^2 \neq \mathbb{1}$*, then* $L(s,\mu\sigma)^2 L(s,\mu\chi^{-1}\sigma)L(s,\mu\chi\sigma)$,
(vi) *if* $\chi_1 = \chi_2$*, then* $L(s,\mu\sigma)^4$,
(vii) *if* $\sigma, \chi_1\sigma, \chi_2\sigma$ *and* $\chi_1\chi_2\sigma$ *are all different, then*

$$L(s,\mu\sigma)L(s,\mu\chi_2\sigma)L(s,\mu\chi_1\sigma)L(s,\mu\chi_1\chi_2\sigma).$$

Proof The result follows respectively from
(i) Proposition 5.1, Lemma 4.2,
(ii) Proposition 5.2, Lemma 4.2,
(iii) Proposition 5.3, Lemma 4.2,
(iv) Proposition 5.4, Lemma 4.1, Lemma 4.2,
(v) Proposition 5.5, Lemma 4.1, Lemma 4.2,
(vi) Proposition 5.6, Lemma 4.5,
(vii) Proposition 5.7, Lemma 4.1.

Theorem 5.2 *L-factor of* $\Pi = \chi_1 \times \chi_2 \rtimes \sigma$ *is*

$$L(s,\mu\sigma)L(s,\mu\chi_2\sigma)L(s,\mu\chi_1\sigma)L(s,\mu\chi_1\chi_2\sigma).$$

Proof This is the restatement of the previous theorem.

Acknowledgements The author would like to thank the referee for the careful review and the valuable comments, which provided insights that helped improve the paper. The author would especially like to thank his kids, Hamza Ali and Betul Meryem, for their love during the time that this paper was being prepared. It is them to whom this work is dedicated.

References

[1] Bump, D., Automorphic Forms and Representations, Cambridge Studies in Advanced Mathematics, New York, NY, USA, 1998.

[2] Casselman, W., Introductions to the theory of admissible representations of reductive p-adic groups, preprint.

[3] Danisman, Y., Regular poles for the p-adic group GSp_4, *Turk. J. Math.*, **38**, 2014, 587–613.

[4] Danisman, Y., Regular poles for the p-adic group GSp_4-II, *Turk. J. Math.*, **39**, 2015, 369–394.

[5] Danisman, Y., Local factors of nongeneric supercuspidal representations of GSp_4, *Math. Ann.*, **361**, 2015, 1073–1121.

[6] Gan, W. and Takeda, S., The local Langlands conjecture for GSp(4), *Ann. Math.*, **173**, 2011, 1841–1882.

[7] Piatetski-Shapiro, I., L-functions for GSp_4, *Pacific J. Math.*, Olga Taussky Todd Memorial Issue, 1997, 259–275.

[8] Prasad, D. and Takloo-Bighash, R., Bessel models for GSp(4), *J. Reine und Angew Math.* (*Crelles J.*), **655**, 2011, 189–243.

[9] Roberts, B. and Schmidt, R., New Forms for GSp(4), Springer Lecture Notes in Mathematics, **1918**, Springer-Verlag, Heidelberg, Germany, 2007.

[10] Roberts, B. and Schmidt, R., Some remarks on Bessel functionals for GSp(4), *Documenta Math.*, **21**, 2016, 467–553.

[11] Shahidi, F., A proof of Langlands conjecture on plancherel measures; complementary series for p-adic groups, *Ann. Math.*, **132**, 1990, 273–330.

[12] Takloo-Bighash, R., L-functions for the p-adic group GSp(4), *Amer. J. Math.*, **122**, 2000, 1085–1120.

[13] Tunnell, J., Local L-factors and characters of GL(2), *Amer. J. Math.*, **105**, 1983, 1277–1307.