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Equivalent Properties for CD Inequalities on Graphs with Unbounded Laplacians

Chao GONG[1] Yong LIN[1]

Abstract The CD inequalities are introduced to imply the gradient estimate of Laplace operator on graphs. This article is based on the unbounded Laplacians, and finally concludes some equivalent properties of the CD(K,∞) and CD(K,n).

Keywords Graph theory, CD inequality, Unbounded Laplacian
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1 Introduction

Graph theory is the basic theory of the study of graphs and networks. The spectral graph theory, which is used for describing the structure and characteristic of graphs by adjacency matrix or the spectral density of Laplacian matrix, is the classical method for studying graph (see [1]).

We have already known that we can find the curvature by some way, and there are many examples for the geometric analysis such as the famous Li-Yau gradient estimate. Moreover, we can use some data to describe the graphs and optimize it such as the Cheeger constant on graphs.

The Laplacian on graph has always been an important research topic. In fact, the Laplacian can be seen as the generator of symmetric Markov process. Laplacians always appear in the topics on the research of discrete structure for heat equations as in [2].

As for the Laplacians on graphs, the properties are different on different occasions, such as finite graphs, locally finite graphs and infinite graphs. If we assume that the graph is finite, then the properties of Laplacians are simple and good. But for some problems, the assumption of finite graph is obviously too narrow, so locally finite graphs or some infinite graphs can be a better research object. We still can get good enough properties on them. In recent years, some research topics are as follows on Laplacians on infinite graphs:

(a) Definition of the operators and essential selfadjointness.

(b) Absence of essential spectrum.

(c) Stochastic incompleteness.

The results on metric space can be seen in [3] and meanwhile, it also has a similar geometric structure as the manifold. Obviously, the graph can also be seen as a kind of metric space. We can define the distance between two vertices of the graph as the natural metric, which is

Manuscript received November 30, 2015. Revised April 17, 2017.
[1]School of Information, Renmin University of China, Beijing 100872, China.
E-mail:gongdoro@hotmail.com; linyong01@ruc.edu.cn

the number of the minimum edges connecting them. Then, we should consider whether the theories of Riemannian manifold can be extended to the graph, especially those about Ricci curvature. Many results in geometry analysis come from the Ricci curvature, especially the lower bound of Ricci curvature, such as the heat kernel estimation, Harnack inequalities and Sobolev inequalities. These conclusions have been made in [4].

On Riemannian manifolds, There exists an identical Bochner formula for any smooth function:

$$\frac{1}{2}\triangle|\bigtriangledown f|^2 = \langle\bigtriangledown f, \bigtriangledown\triangle f\rangle + \|\mathrm{Hess}f\|_2^2 + \mathrm{Ric}(\bigtriangledown f, \bigtriangledown f).$$

When its Ricci curvature has a lower bound, we can make a conclusion that for any $\eta \in TM$ there exists a $K \in \mathbb{R}$ that satisfies $\mathrm{Ric}(\eta,\eta) \geqslant K|\eta|^2$. Unfortunately in the discrete situation we can not define $\|\mathrm{Hess}f\|_2$. But on Riemannian manifolds we can make use of Cauchy-Schwarz inequalities to get an inequality $\|\mathrm{Hess}f\|_2^2 \geqslant \frac{1}{n}(\bigtriangledown f)^2$. Then the Bochner inequality can be rewritten into

$$\frac{1}{2}\triangle|\bigtriangledown f|^2 \geq \langle\bigtriangledown f, \bigtriangledown\triangle f\rangle + \frac{1}{n}(\triangle f)^2 + K|\bigtriangledown f|^2.$$

The inequality above is the so-called curvature-dimension inequality on Riemannian manifolds, and we call it CD inequality for short. Using this inequality, the "Ricci curvature" in the discrete situation can be defined. Bakery and Emery have already proved that if the chain rule is satisfied, the CD inequalities can be extended to the Markov operators on some metric space. Yet obviously the chain rule is not always true for discrete functions. Fortunately when $p = \frac{1}{2}$, u^p satisfies the chain rule even on the discrete condition. So, [4] introduced an improved CD inequality—CDE inequality. This definitely is a key for the research of the discrete geometry analysis.

This paper gives an introduction of the CD inequality and several equivalent conditions of the CD inequality for unbounded Laplacians on the graph. It is organized into three parts.

Chapter 1 gives the introduction of the graph, the Laplacians and CD inequalities on it.

Chapter 2 introduces some basic conclusions in order to get the main result, and some definitions such as the locally finite graph, the weighted graph and the domain of the operators.

Chapter 3 gives the main conclusion of this paper which includes some equivalent conditions of the CD inequalities.

2 Graphs, Laplacians and CD Inequalities

Given a graph $G = (V, E)$, for an $x \in V$, if there exists another $y \in V$ that satisfies $(x, y) \in E$, we call them neighbors, and write as $x \sim y$. If there exists an $x \in V$ satisfying $(x, x) \in E$, we call it a self-loop. In this paper we allow graphs to have self-loops.

Now we will introduce some basic definitions and theorems before we get the main results.

Definition 2.1 (Locally Finite Graph) *We call a graph G a locally finite graph if for any $x \in V$, it satisfies $\#\{y \in V | y \sim x\} < \infty$. Moreover, it is called connected if for any $x, y \in V$ there exists a sequence $\{x_i\}_{i=0}^n$ satisfying $x = x_0 \sim x_1 \sim \cdots \sim x_n = y$.*

Definition 2.2 (Weighted Graph) *Given a graph $G = (V, E)$, $\mu : E \to [0, +\infty)$ and $m : V \to [0, +\infty)$ are two mappings on it. μ is symmetric on V. For convenience, we extend μ onto $V \times V$, that is to say, for any $x, y \in V$, if $x \nsim y$, $\mu(x, y) = 0$.*

Definition 2.3 ($l^p(V,m)$ Space) *Let m be a measure defined as above. Then (V,m) is a measure space. We define an $l^p(V,m)$ $(0<p<+\infty)$ space as follows:*

$$\Big\{u : V \to \mathbb{R} : \sum_{x\in V} m(x)|u(x)|^p < \infty\Big\}$$

and write l^p_m for simplicity.

Obviously, $l^2(V,m)$ is a Hilbert space, the inner product is naturally defined as: $\langle u, v\rangle := \sum\limits_{x\in V} m(x)u(x)v(x)$, and the norm is defined as: $\|u\| := \langle u, u\rangle^{\frac{1}{2}}$.

In addition, we use $l^\infty(V)$ to define a set including all the bounded functions on V, and we can easily know that this space is not influenced by the measure m. The norm on it is defined as: $\|u\| := \sup\limits_{x\in V} |u(x)|$.

Definition 2.4 (Finitely Supported Function) *For a graph $G = (V,E)$, we define a set of finitely supported functions as: $C_0(V) := \{f : V \to \mathbb{R} | \#\{x \in V | f(x) \neq 0\} < \infty\}$.*

Let D is a dense subspace of $l^2(V,m)$. We define a symmetric nonnegative bilinear form Q on $D \times D$ to $\mathbb{R}$. D is called the domain of Q, and it is written as $D(Q)$.

In fact, this mapping is determined by its values on the diagonal line. Then if we want to define such a mapping Q, we can just define the values on the diagonal line as

$$Q(u) := \begin{cases} Q(u,u), & u \in D, \\ \infty, & u \notin D. \end{cases}$$

If Q is lower semicontinuous, we call it closed. If Q has a closed extension it is called closable and the smallest extension is called the closure of Q as defined in [5].

Definition 2.5 (Dirichlet Form) *Q is called a Dirichlet form if it is closed and for all the contractions C and $u \in l^2(V,m)$, it satisfies $Q(Cu) \leqslant Q(u)$.*

More details can be seen in [6].

On the graph we define the Dirichlet form as

$$f \mapsto Q(f) := \frac{1}{2} \sum_{x,y\in V} \mu_{xy}(f(y) - f(x))^2.$$

Then we will introduce some kinds of operators on graphs.

Definition 2.6 (Laplacians on Locally Finite Graphs) *On a locally finite graph $G = (V,E,\mu,\ m)$, the Laplacian has a form as follows*

$$\triangle f(x) = \frac{1}{m(x)} \sum_{y\in V} \mu_{xy}(f(y) - f(x)), \quad \forall f \in C_0(V).$$

Definition 2.7 (Gradient Operator Γ) *The operator Γ is defined as follows*

$$\Gamma(f,g)(x) = \frac{1}{2}(\triangle(fg) - f\triangle g - g\triangle f)(x).$$

Always we write $\Gamma(f,f)$ as $\Gamma(f)$.

Definition 2.8 (Gradient Operator Γ_2) *The operator Γ_2 is defined as follows*

$$\Gamma_2(f,g) = \frac{1}{2}(\triangle\Gamma(f,g) - \Gamma(f,\triangle g) - \Gamma(g,\Gamma f)).$$

Also we have $\Gamma_2(f) = \Gamma_2(f,f) = \frac{1}{2}\triangle\Gamma(f) - \Gamma(f,\triangle f)$.

Definition 2.9 (Nondegenerate Measure) *A measure m is called to be nondegenerate if it satisfies $\delta := \inf_{x\in V} m(x) > 0$.*

Now we can introduce some results we need.

Lemma 2.1 *For any $f \in l^p(V,m)$,$p \in [1,\infty)$, we have $P_t f \in l^p(V,m)$ and*

$$\|P_t f\|_{l^p} \leqslant \|f\|_{l^p}.$$

And for any $f \in l^2(V,m)$, we have $P_t f \in D(\triangle)$.

Lemma 2.2 *For any $f \in D(\triangle)$ we have $\triangle P_t f = P_t \triangle f$.*

Theorem 2.1 *Let m be a nondegenerate measure on V. Then for any $f \in l^p(V,m)$, $p \in [1,\infty)$,*

$$|f(x)| \leqslant \delta^{-\frac{1}{p}} \|f\|_{l^p} \quad ,\forall x \in V.$$

Moreover, for any $p < q \leqslant \infty$, $l^p(V,m) \hookrightarrow l^q(V,m)$.

The proofs of Lemmas 2.1–2.2, and Theorem 2.1 are given by Bobo Hua and Yong Lin in [7].

Now we will introduce the definition of the completeness of the graph.

Definition 2.10 (Complete Graph) *A weighted graph (V,E,μ,m) is called to be complete if there is a nondecreasing sequence of finitely supported functions $\{\eta\}_{k=1}^{\infty}$ such that*

$$\lim_{k\to\infty} \eta_k = 1 \quad \textit{and} \quad \Gamma(\eta_k) \leqslant \frac{1}{k}.$$

Next we will introduce two important lemmas as follows.

Lemma 2.3 (Green's Formula) *Let (V,E,m,μ) be a complete weighted graph. Then for any $f \in D(Q)$ and $g \in D(\triangle)$,*

$$\sum_{x\in V} f(x)\triangle g(x)m(x) = -\sum_{x\in V}\Gamma(f,g)(x)m(x).$$

Lemma 2.4 *Let (V,E,m,μ) be a complete graph. Then for any $f \in C_0(V)$ and $T > 0$, we have $\max_{[0,T]}\Gamma(P_t f) \in \ell^1_m$ and*

$$\Big\|\max_{[0,T]}\Gamma(P_t f)\Big\|_{\ell^1_m} \le C_1(T,f),$$

where $C_1(T,f)$ is a constant depending on T and f. Moveover,

$$\max_{[0,T]}\Big|\Gamma\Big(P_t f, \frac{\mathrm{d}}{\mathrm{d}t}P_t f\Big)\Big| \in \ell^1_m \quad \text{and}$$

$$\Big\|\max_{[0,T]}|\Gamma\Big(P_t f, \frac{\mathrm{d}}{\mathrm{d}t}P_t f\Big)|\Big\|_{\ell^1_m} = \Big\|\max_{[0,T]}|\Gamma(P_t f, \Delta P_t f)|\Big\|_{\ell^1_m} \le C_2(T,f).$$

These two lemmas are proved in [7].

Now we will introduce some basic CD inequalities (also see [4, 8]).

Definition 2.11 (CD(K,∞) Condition) *We say that a graph satisfies* CD(K,∞) *condition if for any $x \in V$, we have*

$$\Gamma_2(f)(x) \geqslant K\Gamma(f)(x), \quad K \in \mathbb{R}.$$

For the finite-dimensioned situation, we have the CD(K,n) condition.

Definition 2.12 (CD(K,n) Condition) *We say that a graph satisfies* CD(K,n) *condition if for any $x \in V$, we have*

$$\Gamma_2(f) \geqslant \frac{1}{n}(\triangle f)^2 + K\Gamma(f), \quad K \in \mathbb{R}.$$

Moreover, we have another condition called CDE(x,K,n).

Definition 2.13 (CDE(x,K,n) Condition) *Let $f : V \to \mathbb{R}^+$ satisfy $f(x) > 0$, $\triangle f(x) < 0$. We say that a graph satisfies* CDE(x,K,n) *condition if for any $x \in V$, we have*

$$\Gamma_2(f)(x) - \Gamma\Big(f, \frac{\Gamma(f)}{f}\Big)(x) \geqslant \frac{1}{n}(\triangle f)(x)^2 + K\Gamma(f)(x), \quad K \in \mathbb{R}.$$

Also, we denote by $P_t := \mathrm{e}^{t\triangle}$ the C_0-semigroup associated to the Dirichlet form on $l^p(V,m)$. And we let P_t be a Markov semigroup.

3 Main Results

When we look for the equivalent properties of CD inequalities, we often set a condition: $D_\mu := \max\limits_{x\in V} \frac{deg(x)}{\mu(x)} < \infty$. And the equivalent properties were proved in [9] for these bounded Laplace operator on graphs. For the unbounded Laplace operator, the following equivalent properties under the condition of nondegenerate measure were proved in [7] by Bobo Hua and Yong Lin.

Remark 3.1 Let $G = (V,E,m,\mu)$ be a complete graph and m is nondegenerate, i.e. $\inf\limits_{x\in V} m(x) > 0$. Then the following are equivalent:

(a) G satisfies CD(K,∞).

(b) For any finitely supported function f,

$$\Gamma(P_t f) \leqslant \mathrm{e}^{-2Kt} P_t(\Gamma(f)).$$

(c) For any $f \in D(Q)$,

$$\Gamma(P_t f) \leqslant \mathrm{e}^{-2Kt} P_t(\Gamma(f)).$$

In this section, similarly in [7] we will give some equivalent properties of CD(K,∞) and CD(K,n).

Theorem 3.1 *Let $G = (V,E,m,\mu)$ be a complete graph and m is nondegenerate. Then the following are equivalent:*

(a) *G satisfies* CD(K,∞).

(b) *For any finitely supported function f,*

$$\frac{e^{2Kt}-1}{K}\Gamma(P_t f) \leqslant P_t(f)^2 - (P_t f)^2 \leqslant \frac{1-e^{-2Kt}}{K} P_t(\Gamma(f)).$$

(c) *For any* $f \in D(Q)$,

$$\frac{e^{2Kt}-1}{K}\Gamma(P_t f) \leqslant P_t(f)^2 - (P_t f)^2 \leqslant \frac{1-e^{-2Kt}}{K} P_t(\Gamma(f)).$$

Proof First, for any $f, \xi \in C_0(V)$, we set

$$G(s) = \sum_{x\in V} (P_{t-s}f)^2(x) P_s \xi(x) m(x).$$

Taking formal derivative of $G(s)$ in s, we get:

$$G'(s) = \sum_{x\in V} (-2P_{t-s}f \triangle P_{t-s}f P_s(\xi(x))m(x) + (P_{t-s}f)^2(x)\triangle P_s \xi(x) m(x)).$$

Now we have to show that $G(s)$ is differentiable in s. For the first part:

$$\begin{aligned} &2\sum_{x\in V} |P_{t-s}f \triangle P_{t-s}f)||P_s(\xi(x))|m(x) \\ &\leqslant 2\|P_s(\xi(x))\|_{l^\infty} \|\triangle P_{t-s}f\|_{l^\infty} \Big(\sum_{x\in V} |P_{t-s}f|m(x)\Big). \end{aligned}$$

For $f, \xi \in C_0(V)$, from Lemma 2.1 we can get:

$$\|P_s \xi(x)\|_{l^\infty} \leqslant \|\xi\|_{l^\infty} < \infty.$$

For $f \in C_0(V)$, we know $P_{t-s}f \in D(\triangle)$ and $\|\triangle P_{t-s}f\|_{l^\infty} = \|P_{t-s}\triangle f\|_{l^\infty} \leqslant \|\triangle f\|_{l^\infty} < \infty$. So we have

$$\begin{aligned} &2\sum_{x\in V} |P_{t-s}f \triangle P_{t-s}f)||P_s(\xi(x))|m(x) \\ &\leqslant 2\|\xi\|_{l^\infty} \|\triangle f\|_{l^\infty} \|P_{t-s}f\|_{l^1_m} \\ &\leqslant 2\|\xi\|_{l^\infty} \|\triangle f\|_{l^\infty} \|f\|_{l^1_m} < \infty. \end{aligned}$$

For the second part, notice that $f, \xi \in C_0(V)$ and $\xi(x) \in D(\triangle)$,

$$\begin{aligned} &\sum_{x\in V} (P_{t-s}f)^2(x) \triangle P_s(\xi(x)) m(x) \\ &\leqslant \|\triangle P_s(\xi(x))\|_{l^\infty} \|P_{t-s}f\|^2_{l^2_m} \\ &\leqslant \|P_s \triangle \xi(x)\| \|f\|^2_{l^2_m} \\ &\leqslant \|\triangle \xi\|_{l^\infty} \|f\|^2_{l^2_m} < \infty. \end{aligned}$$

Then we know that $G(s)$ can be differentiable in s, and

$$G'(s) = \sum_{x\in V} (-2P_{t-s}f(x) \triangle P_{t-s}f(x) P_s \xi(x) m(x) + (P_{t-s}f)^2(x) \triangle P_s \xi(x) m(x)).$$

For $f \in C_0(V)$, from Lemma 2.1 and Theorem 2.1 we can easily get $(P_{t-s}f)^2 \in D(Q)$. Then from Lemma 2.3, we get:

$$G'(s) = \sum_{x\in V}(-2P_{t-s}f\triangle P_{t-s}fP_s\xi m(x) - \Gamma((P_{t-s}f)^2, P_s\xi)m(x)).$$

Now we replace $P_s\xi$ with h, where h satisfies $0 < h \in C_0(V)$, that is to say, h is a finitely supported function. Then

$$\begin{aligned}&\sum_{x\in V}(-2P_{t-s}f\triangle P_{t-s}fh(x)m(x) - \Gamma(P_{t-s}f)^2, h(x))m(x))\\ &= \sum_{x\in V}(-2P_{t-s}f\triangle P_{t-s}fh(x)m(x) + \triangle(P_{t-s}f)^2h(x)m(x))\\ &= \sum_{x\in V}2\Gamma(P_{t-s}f)h(x)m(x).\end{aligned}$$

For $0 < h \in D(Q)$, let $h_k = h\eta_k$, where η_k satisfies

$$\lim_{k\to\infty}\eta_k = 1, \quad \Gamma(\eta_k) \leqslant \frac{1}{k}, \quad k \in N.$$

We can get $0 < h_k \in C_0(V)$. Then letting $k \to \infty$, for any $0 < h \in D(Q)$, we have

$$\begin{aligned}&\sum_{x\in V}(-2P_{t-s}f\triangle P_{t-s}fh(x)m(x) - \Gamma((P_{t-s}f)^2, h(x))m(x))\\ &= \sum_{x\in V}2\Gamma(P_{t-s}f)h(x)m(x).\end{aligned}$$

For $\xi \in C_0(V)$, we easily know $P_s\xi \in D(Q)$. Then setting $h = P_s\xi$, we have

$$G'(s) = \sum_{x\in V}2\Gamma(P_{t-s}f)P_s\xi m(x).$$

Integrate the equation from 0 to t by both side

$$\begin{aligned}&\int_0^t\Big(\sum_{x\in V}2\Gamma(P_{t-s}f)P_s\xi m(x)\Big)\mathrm{d}s\\ &= \int_0^t G'(s)\mathrm{d}s = G(t) - G(0)\\ &= \sum_{x\in V}f^2(x)P_t\xi(x)m(x) - \sum_{x\in V}(P_tf)^2\xi(x)m(x).\end{aligned}$$

Since P_t is a self-adjoint operator on l_m^2, the right hand side of the equation can be changed into

$$\begin{aligned}&\int_0^t\Big(\sum_{x\in V}2\Gamma(P_{t-s}f)P_s\xi m(x)\Big)\mathrm{d}s\\ &= \int_0^t G'(s)\mathrm{d}s = \int_0^t\sum_{x\in V}2P_s\Gamma(P_{t-s}f)\xi(x)m(x)\mathrm{d}s\\ &= \sum_{x\in V}P_t(f)^2\xi(x)m(x) - \sum_{x\in V}(P_tf)^2\xi(x)m(x).\end{aligned}$$

For $\xi(x) \in C_0(V)$, let $\xi(x) = \delta_y(x)$ (when $y = x$, $\delta_y(x) = 1$, otherwise $\delta_y(x) = 0$). Then, the equation is changed into

$$P_t(f)^2(y) - (P_tf)^2(y) = 2\int_0^t P_s\Gamma(P_{t-s}f)(y)\mathrm{d}s.$$

Now notice that P_t is a Markov semigroup. Then from Remark 3.1 we now have

$$\begin{aligned}P_t(f^2) - (P_tf)^2 &\leqslant 2\int_0^t P_s(\mathrm{e}^{-2K(t-s)}P_{t-s}\Gamma(f))\mathrm{d}s\\ &= 2\mathrm{e}^{-2Kt}\int_0^t \mathrm{e}^{2Ks}\mathrm{d}s\cdot P_t\Gamma(f)\\ &= \frac{1-\mathrm{e}^{-2Kt}}{K}P_t\Gamma(f).\end{aligned}$$

Also we have

$$\begin{aligned}P_t(f^2) - (P_tf)^2 &\geqslant 2\int_0^t \mathrm{e}^{2Ks}\Gamma(P_s\circ P_{t-s}f)\mathrm{d}s\\ &= 2\int_0^t \mathrm{e}^{2Ks}\mathrm{d}s\cdot\Gamma(P_tf)\\ &= \frac{\mathrm{e}^{2Kt}-1}{K}\Gamma(P_tf).\end{aligned}$$

Now we prove the opposite.

From the definition, we have

$$P_t = e^{t\triangle} = \sum_{p=0}^{\infty}\frac{t^p\triangle^p}{p!}.$$

Then we will obtain

$$\begin{aligned}&P_t(f)^2 - (P_tf)^2\\ &= 2t\Gamma(f) + t^2\Big(\frac{1}{2}\triangle^2f^2 - (\triangle f)^2 - f\triangle^2 f\Big) + o(t^2)\\ &= 2t\Gamma(f) + t^2\Big[\Big(\frac{1}{2}\triangle^2f^2 - \triangle(f\triangle f)\Big) + (\triangle(f\triangle f) - (\triangle f)^2 - f\triangle^2 f)\Big] + o(t^2)\\ &= 2t\Gamma(f) + t^2(\triangle\Gamma(f) + 2\Gamma(f,\triangle f)) + o(t^2).\end{aligned}$$

On the other side, we have

$$\begin{aligned}\frac{1-\mathrm{e}^{-2Kt}}{K}P_t\Gamma(f) &= (2t - 2Kt^2 + o(t^2))\cdot(\Gamma(f) + t\triangle\Gamma(f) + o(t^2))\\ &= 2t\Gamma(f) + 2t^2(\triangle\Gamma(f) - K\Gamma(f)) + o(t^2)\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{e}^{2Kt}-1}{K}\Gamma(P_tf) &= (2t + 2Kt^2 + o(t^2))\cdot\Gamma(f + t\triangle f + o(t)f)\\ &= (2t + 2Kt^2 + o(t^2))\cdot(\Gamma(f) + 2t\Gamma(f,\triangle f) + o(t))\\ &= 2t\Gamma(f) + 2t^2(2\Gamma(f,\triangle f) + K\Gamma(f)) + o(t^2).\end{aligned}$$

Now we set

$$F_1(t) := P_t(f^2) - (P_t f)^2 - \frac{1 - \mathrm{e}^{-2Kt}}{K} P_t \Gamma(f)$$

and

$$F_2(t) := P_t(f^2) - (P_t f)^2 - \frac{\mathrm{e}^{2Kt} - 1}{K} \Gamma(P_t f).$$

Then we have $F_1(t) = 2t^2(K\Gamma(f) - \Gamma_2(f)) + o(t^2)$ and $F_2(t) = 2t^2(\Gamma_2(f) - K\Gamma(f)) + o(t^2)$.

Obviously $F_1(t)$ and $F_2(t)$ is differentiable. Notice that $F_1(t) \leqslant 0$, $F_2(t) \geqslant 0$ and $F_1(0) = F_2(0) = 0$. So we know $F_1'(0) \leqslant 0$ and $F_2'(0) \geqslant 0$, which equals to $\lim\limits_{t\to 0} F_1'(t) \leqslant 0$ and $\lim\limits_{t\to 0} F_2'(t) \geqslant 0$.

Notice that $t \geqslant 0$. Then we obtain $\Gamma_2(f) \geqslant K\Gamma(f)$. This is just the $\mathrm{CD}(K, \infty)$ condition.

Also we can get equivalent properties of $\mathrm{CD}(K, n)$.

Theorem 3.2 *Let $G = (V, E, m, \mu)$ be a complete graph and m is nondegenerate. Then the following are equivalent:*

(a) *G satisfies* $\mathrm{CD}(K, n)$.

(b) *For any finitely supported function f,*

$$\Gamma(P_t f) \leqslant \mathrm{e}^{-2Kt} P_t \Gamma(f) - \frac{2}{n} \int_0^t \mathrm{e}^{-2Ks} P_S (\triangle P_{t-s} f)^2 \mathrm{d}s, \quad 0 < s < t.$$

(c) *For any $f \in D(Q)$,*

$$\Gamma(P_t f) \leqslant \mathrm{e}^{-2Kt} P_t \Gamma(f) - \frac{2}{n} \int_0^t \mathrm{e}^{-2Ks} P_S (\triangle P_{t-s} f)^2 \mathrm{d}s, \quad 0 < s < t.$$

Proof First, for any $f, \xi \in C_0(V)$, we build this functional equation

$$G(s) = \mathrm{e}^{-2Ks} \sum_{x\in V} \Gamma(P_{t-s} f)(x) P_s \xi(x) m(x).$$

Taking formal derivative of $G(s)$, we define the function as A. Then

$$\begin{aligned} A = &-2K\mathrm{e}^{-2Ks} \sum_{x\in V} \Gamma(P_{t-s} f)(x) P_s \xi(x) m(x) \\ &+ \mathrm{e}^{-2Ks} \sum_{x\in V} (-2\Gamma(P_{t-s} f, \triangle P_{t-s} f)(x) P_s \xi(x) m(x) \\ &+ \mathrm{e}^{-2Ks} \sum_{x\in V} \Gamma(P_{t-s} f) \triangle P_s \xi(x) m(x). \end{aligned}$$

Now we will show that $G(s)$ is differentiable in s.

Without loss of generality, we assume that $\epsilon < s < t - \epsilon$ for some $\epsilon > 0$. For the first part, from Lemma 2.4 we have

$$\begin{aligned} &|-2K\mathrm{e}^{-2Ks} \sum_{x\in V} \Gamma(P_{t-s} f)(x) P_s \xi(x) m(x)| \\ &\leqslant |C_1| \sum_{x\in V} |\Gamma(P_{t-s} f)(x)| |P_s \xi| m(x) \\ &\leqslant |C_1| \|P_s \xi\|_{l^\infty} \|\Gamma(P_{t-s} f)\|_{l^1_m} \\ &< \infty, \end{aligned}$$

where C_1 is a constant satisfying $|-2K\mathrm{e}^{-2Ks}| \leqslant C_1$.

For the second part, from Lemma 2.4 we have

$$\begin{aligned}
&|\mathrm{e}^{-2Ks}\sum_{x\in V}(-2\Gamma(P_{t-s}f, \triangle P_{t-s}f)(x)P_s\xi(x)m(x)| \\
&\leqslant |C_2|\sum_{x\in V}|(-2\Gamma(P_{t-s}f, \triangle P_{t-s}f)(x)||P_s\xi(x)|m(x)| \\
&\leqslant |C_2|\|P_s\xi\|_{l^\infty}\|\Gamma(P_{t-s}f, \triangle P_{t-s}f)\|_{l^1_m} \\
&< \infty,
\end{aligned}$$

where C_2 is some constant satisfying $|\mathrm{e}^{-2Ks}| \leqslant C_2$.

For the last part, from Lemma 2.4 we have

$$\begin{aligned}
&|\mathrm{e}^{-2Ks}\sum_{x\in V}\Gamma(P_{t-s}f)\triangle P_s\xi(x)m(x)| \\
&\leqslant |C_2|\sum_{x\in V}|\Gamma(P_{t-s}f)||\triangle P_s\xi(x)|m(x) \\
&= |C_2|\sum_{x\in V}|\Gamma(P_{t-s}f)||P_s\triangle\xi(x)|m(x) \\
&\leqslant |C_2|\|\triangle\xi\|_{l^\infty}\|\Gamma(P_{t-s}f)\|_{l^1_m} \\
&< \infty,
\end{aligned}$$

where C_2 is defined as above.

Then we can know that $G(s)$ is differentiable in s, and

$$\begin{aligned}
G'(s) = &-2K\mathrm{e}^{-2Ks}\sum_{x\in V}\Gamma(P_{t-s}f)(x)P_s\xi(x)m(x) \\
&+ \mathrm{e}^{-2Ks}\sum_{x\in V}(-2\Gamma(P_{t-s}f, \triangle P_{t-s}f)(x)P_s\xi(x)m(x) \\
&+ \mathrm{e}^{-2Ks}\sum_{x\in V}\Gamma(P_{t-s}f)\triangle P_s\xi(x)m(x).
\end{aligned}$$

From Lemma 2.3 we get

$$\begin{aligned}
G'(s) = &-2K\mathrm{e}^{-2Ks}\sum_{x\in V}\Gamma(P_{t-s}f)(x)P_s\xi(x)m(x) \\
&+ \mathrm{e}^{-2Ks}\sum_{x\in V}(-2\Gamma(P_{t-s}f, \triangle P_{t-s}f)(x)P_s\xi(x)m(x) \\
&+ \mathrm{e}^{-2Ks}\sum_{x\in V}\Gamma(\Gamma(P_{t-s}f), P_s\xi(x)m(x).
\end{aligned}$$

Now we need to show that for all $h \in D(Q)$, we have

$$\begin{aligned}
&-2\sum_{x\in V}\Gamma(P_{t-s}f, \triangle P_{t-s}f)(x)h(x)m(x) + \sum_{x\in V}\Gamma(\Gamma(P_{t-s}f), h(x))m(x) \\
&= \sum_{x\in V}\Gamma_2(P_{t-s}f)h(x)m(x).
\end{aligned}$$

Obviously, this equation holds for all the finitely supported functions.

Now taking a series of functions $\{\eta_k\}$ in $C_0(V)$ defined as Definition 2.10. Let $h_k = h\eta_k$, obviously $h_k \in C_0(V)$, then

$$\begin{aligned}&-2\sum_{x\in V}\Gamma(P_{t-s}f,\triangle P_{t-s}f)(x)h_k(x)m(x)+\sum_{x\in V}\Gamma(\Gamma(P_{t-s}f),h_k(x))m(x)\\&=\sum_{x\in V}\Gamma_2(P_{t-s}f)h_k(x)m(x).\end{aligned}$$

Let $k\to\infty$, then for all $h\in D(Q)$, we can get

$$\begin{aligned}&-2\sum_{x\in V}\Gamma(P_{t-s}f,\triangle P_{t-s}f)(x)h(x)m(x)+\sum_{x\in V}\Gamma(\Gamma(P_{t-s}f),h(x))m(x)\\&=\sum_{x\in V}\Gamma_2(P_{t-s}f)h(x)m(x).\end{aligned}$$

For $\xi\in V$, $P_s\xi\in D(Q)$, letting $h=P_s\xi$, we get

$$\begin{aligned}&-2\sum_{x\in V}\Gamma(P_{t-s}f,\triangle P_{t-s}f)(x)P_s\xi(x)m(x)+\sum_{x\in V}\Gamma(\Gamma(P_{t-s}f),P_s\xi(x))m(x)\\&=\sum_{x\in V}\Gamma_2(P_{t-s}f)P_s\xi(x)m(x).\end{aligned}$$

Then $G'(s)$ can be rewritten as

$$G'(s)=\mathrm{e}^{-2Ks}\sum_{x\in V}(\Gamma_2(P_{t-s}f)-K\Gamma(P_{t-s}f))P_s\xi(x)m(x).$$

By use of the equivalent properties of $\mathrm{CD}(K,n)$, we get

$$G'(s)\geqslant \mathrm{e}^{-2Ks}\sum_{x\in V}\frac{2}{n}(\triangle P_{t-s}f)^2(x)P_s\xi(x)m(x).$$

Now integrate the equation from 0 to t in s by both sides, then we can get

$$\begin{aligned}&\int_0^t G'(s)=G(t)-G(0)\\&=\mathrm{e}^{-2Kt}\sum_{x\in V}\Gamma(f)(x)P_t\xi(x)m(x)-\sum_{x\in V}\Gamma(P_tf)(x)\xi(x)m(x)\\&\geqslant\frac{2}{n}\int_0^t \mathrm{e}^{-2Ks}\sum_{x\in V}(\triangle P_{t-s}f)^2P_s\xi(x)m(x)\mathrm{d}s.\end{aligned}$$

Since P_t is a self-adjoint operator on l_m^2, we can get

$$\begin{aligned}&\mathrm{e}^{-2Kt}\sum_{x\in V}P_t\Gamma(f)(x)\xi(x)m(x)-\sum_{x\in V}\Gamma(P_tf)(x)\xi(x)m(x)\\&\geqslant\frac{2}{n}\int_0^t \mathrm{e}^{-2Ks}\sum_{x\in V}P_s(\triangle P_{t-s}f)^2\xi(x)m(x)\mathrm{d}s.\end{aligned}$$

Let $\xi(x)=\delta_y(x)$, then

$$\begin{aligned}&\mathrm{e}^{-2Kt}P_t\Gamma(f)(y)m(y)-\Gamma(P_tf)(y)m(y)\\ \geqslant\ &\frac{2}{n}\int_0^t \mathrm{e}^{-2Ks}P_s(\triangle P_{t-s}f)^2m(y)\mathrm{d}s.\end{aligned}$$

For $m(y)>0$,

$$\begin{aligned}&\mathrm{e}^{-2Kt}P_t\Gamma(f)-\Gamma(P_tf))\\ \geqslant\ &\frac{2}{n}\int_0^t \mathrm{e}^{-2Ks}P_s(\triangle P_{t-s}f)^2\mathrm{d}s.\end{aligned}$$

That is to say, $\Gamma(P_tf)\leqslant \mathrm{e}^{-2Kt}P_t(\Gamma(f))-\frac{2}{n}\int_0^t \mathrm{e}^{-2Ks}P_s(\triangle P_{t-s}f)^2\mathrm{d}s$.

Now we prove the opposite.

First we set $F(t):=\Gamma(P_tf)-(\mathrm{e}^{-2Kt}P_t\Gamma(f))-\frac{2}{n}\int_0^t \mathrm{e}^{-2Ks}P_s(\triangle P_{t-s}f)^2\mathrm{d}s$. It is easy to know that $F(t)$ is differentiable and $F'(0)\leqslant 0$. So

$$F'(0)=2\Gamma(f,\triangle f)+2K\Gamma(f)-\triangle\Gamma(f)+\frac{2}{n}(\triangle f)^2\leqslant 0.$$

Then we can get

$$\Gamma_2(f)\geqslant \frac{1}{n}(\triangle f)^2+K\Gamma(f),$$

which is just the CD(K,n) condition.

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