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Liouville Type Theorem About p-Harmonic Function and p-Harmonic Map with Finite L^q-Energy*

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Abstract This paper deals with the p-harmonic function on a complete non-compact submanifold M isometrically immersed in an $(n+k)$-dimensional complete Riemannian manifold $\overline{M}$ of non-negative $(n-1)$-th Ricci curvature. The Liouville type theorem about the p-harmonic map with finite L^q-energy from complete submanifold in a partially non-negatively curved manifold to non-positively curved manifold is also obtained.

Keywords p-Harmonic map, p-Harmonic map, Kato inequality, Index, Liouville theorem
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1 Introduction

Liouville type theorem is very important in geometry and topology. In [6], Wang considered complete submanifolds in manifolds with partially non-negative curvature and studied harmonic map from complete submanifolds in manifolds of partially non-negative curvature to non-positively curved manifold. She obtained the Liouville type theorem for harmonic map. Notice that this curvature condition on the ambient manifold is interesting.

In [2], Feng and Han derived a Liouville type theorem for p-harmonic function on minimal submanifold in $\mathbb{R}^{n+m}$. About the operator $\Delta + Ak(x)$ with constant A on manifold M with $\mathrm{Ric}^M \geq -k(x)$, many results about the p-harmonic maps with finite energy from M to non-positively curved manifold has been obtained, such as [3], where they required that $A \geq \frac{4p^2}{p-1}$, $p > 2$. Changing the condition that p-harmonic map is of finite energy into the condition which is of finite L^q-energy, Pigola, Rigoli and Setti [5] investigated p-harmonic maps of finite L^q-energy into non-positively curved manifolds.

In this paper, we mainly consider the case that $A = 1$. We always assume p-harmonic map has finite L^q-energy, because for p-harmonic map with finite energy, we find that it is hard to get better estimates since the Kato inequaltiy for p-harmonic maps is not nicer than that for harmonic map, but luckily for p-harmonic function, we have a better Kato inequality than p-harmonic map. The purpose of this paper is to study p-harmonic function on submanifold in a manifold with partially non-negative curvature and p-harmonic maps from a submanifold in a manifold with partially non-negative curvature to non-positively curved manifold respectively, but with different conditions. For example, we can give the following:

(1) The following Sobolev inequality holds on M for any compactly supported function ψ

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on M:

$$c\Big(\int_M |\psi|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \leq \int_M |\nabla\psi|, \tag{1.1}$$

where c is a constant.

(2) The index of the operator $\Delta + \frac{n}{4}(S - n|H|^2)$ is zero.

(3) $\int_M (S - n|H|^2)^{\frac{n}{2}} < \big(\frac{(n-2)c}{(n-1)\sqrt{n(p-1)}}\big)^n$,

where the definitions of S, H are to be defined below. Our proof is mainly based on the methods in [2, 6].

2 Preliminary

The energy functional of p-harmonic map is defined by

$$E_p(u) = \int_M \frac{|\nabla u|^p}{2}$$

whose Euler-Lagrange equation is as follows:

$$\tau_p(u) = \mathrm{div}(|\mathrm{d}u|^{p-2}\mathrm{d}u), \tag{2.1}$$

where a map u is called a p-harmonic map if $\tau_p(u) = \mathrm{div}(|\mathrm{d}u|^{p-2}\mathrm{d}u) = 0$.

Definition 2.1 *A manifold M is said to have non-negative k-th Ricci curvature, $k \in \{1, 2, \cdots, n-1\}$, if for any $p \in M$, and any $k+1$ orthonormal basis $\eta, \eta_1, \eta_2, \cdots, \eta_k \in T_p(M)$, such that $\sum\limits_{i=1}^{k} K^M(\eta \wedge \eta_i) \geq 0$.*

It is easy to see that this notion is the generalization of section curvature and Ricci curvature.

If M^m is a submanifold in Q^{m+p}, we choose orthonormal frames $\eta_1, \eta_2, \cdots, \eta_m, \eta_{m+1}, \cdots, \eta_{m+p}$ on Q such that $\eta_1, \eta_2, \cdots, \eta_m$ is the local frames of M. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connection on Q and M respectively. The second fundamental form of the submanifold M is defined as

$$h(X, Y) = \overline{\nabla}_X Y - \nabla_X Y, \quad \forall X, Y \in T_pM.$$

The mean curvature vector of M is

$$H = \frac{1}{n}\sum_{i=1}^{m} h(\eta_i, \eta_i).$$

We denote by S the squared norm of the second fundamental form of the immersion, that is

$$S = \sum_{1 \leq i,j \leq m} (h(\eta_i, \eta_j))^2.$$

We call $\int_M |H|^n$ and $\int_M (S - n|H|^2)^{\frac{n}{2}}$ the total mean curvature and the total curvature of M, respectively.

Suppose that M is a complete manifold, we can associate the elliptic operator $L = \triangle + q(x)$ the quardratic form:

$$(\phi, -L\phi) = \int_M |\nabla\phi|^2 - q\phi^2,$$

where $q(x)$ is any continuous function on M, $\phi : M \to R$ is a piecewise smooth function with compact support, $\triangle$ is the Laplacian, $\nabla\phi$ is the gradient of ϕ. The index of the operator L is the supremum of the number of the negative eigenvalue of L with Dirichlet boundary condition.

We say a map u is of L^q-finite energy if $\int_M |\nabla u|^q < \infty$.

3 The Proof of the Main Theorems

Theorem 3.1 *Let M be an n (≥ 3)-dimensional complete non-compact submanifold isometrically immersed in an $(n+k)$-dimensional complete Riemannian manifold $\overline{M}$ of non-negative $(n-1)$-th Ricci curvature. If the index of the operator $\Delta + \frac{n}{4}(S - n|H|^2)$ is zero and $\frac{1}{n-1} \geq \big(\frac{p-2}{p-1}\big)^2$, $p \geq 2$, then any p-harmonic function with finite L^{2p-2}-energy on M is a constant.*

Proof First we recall Bochner formula for p-harmonic function (see [2, Lemma 2.1]): For any smooth function u, we have

$$\begin{aligned}\frac{1}{2}\Delta|\mathrm{d}u|^{2p-2} = {}& |\nabla|\mathrm{d}u|^{p-2}\mathrm{d}u|^2 - \langle|\mathrm{d}u|^{2p-2}\mathrm{d}u, \triangle|\mathrm{d}u|^{p-2}\mathrm{d}u\rangle \\ &+ |\mathrm{d}u|^{2p-4}\langle\mathrm{d}u(\mathrm{Ric}^M(e_k)), \mathrm{d}u(e_k)\rangle,\end{aligned} \tag{3.1}$$

and the following Kato inequality for p-harmonic function (see [2, Lemma 2.4]):

$$|\nabla u|^2 \geq \frac{n}{n-1}|\nabla|u||^2. \tag{3.2}$$

Since M be an n (≥ 3)-dimensional complete non-compact isometrically immersed in an $(n+k)$-dimensional complete Riemannian manifold $\overline{M}$ of non-negative $(n-1)$-th Ricci curvature. From the proof of Theorem 1.1 in [6], we see that

$$\mathrm{Ric}^M \geq -\frac{n}{4}(S - n|H|^2). \tag{3.3}$$

If we set $f = |\mathrm{d}u|^{p-1}$, we have

$$\frac{1}{2}\Delta f^2 + \frac{n}{4}(S - n|H|^2)f^2 \geq \frac{n}{n-1}|\nabla f|^2 - \langle|\mathrm{d}u|^{2p-2}\mathrm{d}u, \delta d(|\mathrm{d}u|^{2p-2}\mathrm{d}u)\rangle. \tag{3.4}$$

Now we choose a cut-off function on M, which satisfies that

$$\begin{cases} 0 \leq \phi(x) \leq 1, & x \in M, \\ \phi(x) = 1, & x \in B_r(x_0), \\ \phi(x) = 0, & x \in M - B_{3r}(x_0), \\ |\nabla\phi(x)| \leq \dfrac{C}{r}, & x \in M. \end{cases}$$

Integrating on M after multiplying both sides of (3.4) by ϕ^2, we get

$$\begin{aligned}&\int_M \frac{1}{2}\phi^2\Delta f^2 + \int_M \frac{n}{4}(S - n|H|^2)f^2\phi^2 \\ &\geq \frac{n}{n-1}\int_M \phi^2|\nabla f|^2 - \int_M \phi^2\langle|\mathrm{d}u|^{2p-2}\mathrm{d}u, \delta d(|\mathrm{d}u|^{2p-2}\mathrm{d}u)\rangle.\end{aligned} \tag{3.5}$$

From the proof of Theorem 1.1 in [6], we know that

$$\int_M \frac{n}{4}(S-n|H|^2)f^2\phi^2 \le \int_M f^2|\nabla\phi|^2 + \int_M \phi^2|\nabla f|^2 - \int_M \frac{1}{2}\phi^2\Delta f^2. \tag{3.6}$$

For any map $u: M \to N$, and any function f, from Lemma 13 in [5], we have

$$|\mathrm{d}(f\mathrm{d}u)| \le |\mathrm{d}f||\mathrm{d}u|. \tag{3.7}$$

Thus we can cite the following inequality in the proof of Theorem 1.1 in [2],

$$\int_M \phi^2\langle|\mathrm{d}u|^{2p-2}\mathrm{d}u, \delta d(|\mathrm{d}u|^{2p-2}\mathrm{d}u)\rangle \le 2\frac{p-2}{p-1}\int_M \phi|\nabla\phi||\nabla f||f| + \Big(\frac{p-2}{p-1}\Big)^2\int_M \phi^2|\nabla f|^2. \tag{3.8}$$

Thus combining (3.5)–(3.6), (3.8), it yields that

$$\begin{aligned}
&\int_M \frac{1}{2}\phi^2\Delta f^2 + \int_M f^2|\nabla\phi|^2 + \int_M \phi^2|\nabla f|^2 - \int_M \frac{1}{2}\phi^2\Delta f^2 \\
&\ge -2\frac{p-2}{p-1}\int_M \phi|\nabla\phi||\nabla f||f| - \Big(\frac{p-2}{p-1}\Big)^2\int_M \phi^2|\nabla f|^2 + \frac{n}{n-1}\int_M \phi^2|\nabla f|^2 \\
&\ge -2\varepsilon\int_M \phi^2|\nabla f|^2 - \frac{1}{2\epsilon}\Big(\frac{p-2}{p-1}\Big)^2\int_M f^2|\nabla\phi|^2 \\
&\quad - \Big(\frac{p-2}{p-1}\Big)^2\int_M \phi^2|\nabla f|^2 + \frac{n}{n-1}\int_M \phi^2|\nabla f|^2,
\end{aligned} \tag{3.9}$$

where the constant $p \ge 2$. It implies that

$$\Big(\frac{1}{n-1} - \Big(\frac{p-2}{p-1}\Big)^2 - 2\epsilon\Big)\int_M \phi^2|\nabla f|^2 \le \Big(1 + \frac{1}{2\epsilon}\Big(\frac{p-2}{p-1}\Big)^2\Big)\int_M f^2|\nabla\phi|^2.$$

If we require that $\frac{1}{n-1} - \big(\frac{p-2}{p-1}\big)^2 > 0$, we can choose sufficiently small ϵ, such that $\frac{1}{n-1} - \big(\frac{p-2}{p-1}\big)^2 - \epsilon > 0$, then we can easily deduce that f is a constant. If $f \ne 0$, the formula (3.6) implies that

$$\int_M \frac{n}{4}(S-n|H|^2)\phi^2 \le \int_M f^2|\nabla\phi|^2.$$

So we have $S - n|H|^2 = 0$. From (3.3), we have $\mathrm{Ric}^M \ge 0$, thus M has infinity volume (see [7]). This is a contradiction since $\int_M f^2 < \infty$. Consequently, $f = 0$ and u is a constant.

Theorem 3.2 *Let M be an n (≥ 3)-dimensional complete non-compact isometrically immersed in an $(n+k)$-dimensional complete Riemannian manifold $\overline{M}$ of non-negative $(n-1)$-th Ricci curvature, and let the Sobolev inequality* (1.1) *hold on M. Furthermore, we suppose*

$$\int_M (S-n|H|^2)^{\frac{n}{2}} < \Big(\frac{(n-2)c}{(n-1)\sqrt{n(p-1)}}\Big)^n, \tag{3.10}$$

where the constant c appearing in (3.10) *is the same as that from the Sobolev inequality* (1.1). *Then any p-harmonic map with finite L^{2p-2}-energy from M to a complete manifold with non-positive curvature is a constant.*

Proof For p-harmonic maps, we have the Bochner formula (see Lemma 12 in [5])

$$\begin{aligned}
\frac{1}{2}\Delta|\mathrm{d}u|^{2p-2} &= |\nabla|\mathrm{d}u|^{p-2}\mathrm{d}u|^2 - \langle|\mathrm{d}u|^{2p-2}\mathrm{d}u, \triangle|\mathrm{d}u|^{p-2}\mathrm{d}u\rangle \\
&\quad + |\mathrm{d}u|^{2p-4}\langle \mathrm{d}u(\mathrm{Ric}^M(e_k)), \mathrm{d}u(e_k))\rangle \\
&\quad - |\mathrm{d}u|^{2p-4}\mathbb{R}^N(\mathrm{d}u(e_i), \mathrm{d}u(e_k), \mathrm{d}u(e_i), \mathrm{d}u(e_k)).
\end{aligned} \tag{3.11}$$

If we set $f = |\mathrm{d}u|^{p-1}$, since $\mathrm{Ric}^M \geq -\frac{n}{4}(S - n|H|^2)$, and the sectional curvture of N is non-positive, we have

$$\Delta f^2 \geq |\nabla f|^2 - \langle |\mathrm{d}u|^{2p-2}\mathrm{d}u, \triangle |\mathrm{d}u|^{p-2}\mathrm{d}u\rangle - \frac{n}{4}(S - n|H|^2)f^2,$$

where we have used the following Kato inequality for differential forms (see [1])

$$|\nabla |\mathrm{d}u|^{p-2}\mathrm{d}u|^2 \geq |\nabla |\mathrm{d}u|^{p-1}|^2.$$

It follows that

$$\begin{aligned} &\int_M \frac{1}{2}\phi^2\Delta f^2 + \int_M \frac{n}{4}(S - n|H|^2)f^2\phi^2 \\ &\geq -2\frac{p-2}{p-1}\int_M \phi|\nabla\phi||\nabla f||f| - \Big(\frac{p-2}{p-1}\Big)^2\int_M \phi^2|\nabla f|^2 + \int_M \phi^2|\nabla f|^2. \end{aligned} \tag{3.12}$$

From the inequality (3.11) in the proof of Theorem 1.3 in [6], we have

$$\begin{aligned} \Big(\int_M |\psi|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} &\leq \frac{4(n-1)^2}{(n-2)^2c^2}\int_M |\nabla\psi|^2 \\ &= c'\int_M |\nabla\psi|^2, \end{aligned} \tag{3.13}$$

where the constant $c' = \frac{4(n-1)^2}{(n-2)^2c^2}$.

Let ϕ be the same cut-off function as in the proof of Theorem 2.1. It follows from the divergence theorem that

$$\int_M \frac{1}{2}\phi^2\Delta f^2 = -2\int_M \phi|f|\langle\nabla\phi, \nabla f\rangle. \tag{3.14}$$

On the other hand, set $A_0 = \int_M (S - n|H|^2)^{\frac{n}{2}})^{\frac{2}{n}}$, we have

$$\begin{aligned} \int_M (S - n|H|^2)f^2\phi^2 &\leq \Big(\int_M (S - n|H|^2)^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big(\int_M (f\phi)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \\ &\leq \Big(\int_M (S - n|H|^2)^{\frac{n}{2}}\Big)^{\frac{2}{n}} c'\int_M |\nabla(\phi f)|^2 \\ &= A_0c'\int_M f^2|\nabla\phi|^2 + \phi^2|\nabla f|^2 + 2\phi f\langle\nabla f, \nabla\phi\rangle. \end{aligned} \tag{3.15}$$

Combining (3.12), (3.14)–(3.15), we deduce that

$$\begin{aligned} &-2\frac{p-2}{p-1}\int_M \phi|\nabla\phi||\nabla f||f| - \Big(\frac{p-2}{p-1}\Big)^2\int_M \phi^2|\nabla f|^2 + \int_M \phi^2|\nabla f|^2 \\ &\leq \frac{n}{4}A_0c'\Big(\int_M \phi^2|\nabla f|^2 + f^2|\nabla\phi|^2 + 2\phi f\langle\nabla f, \nabla\phi\rangle\Big) - 2\int_M \phi|f|\langle\nabla\phi, \nabla f\rangle. \end{aligned} \tag{3.16}$$

Using Young's inequality , it is easy to see that

$$\begin{aligned} &\Big(1 - \Big(\frac{p-2}{p-1}\Big)^2 - \frac{nc'A_0}{4}\Big)\int_M \phi^2|\nabla f|^2 \\ &\leq \frac{nc'A_0}{4}\int_M f^2|\nabla\phi|^2 + 2\Big(\frac{nc'A_0}{4} - 1 + \frac{p-2}{p-1}\Big)\phi f\langle\nabla f, \nabla\phi\rangle \\ &\leq \frac{nc'A_0}{4}\int_M f^2|\nabla\phi|^2 + \Big(1 - \frac{nc'A_0}{4} - \frac{p-2}{p-1}\Big)\int_M \phi^2|\nabla f|^2 + \phi^2|\nabla f|^2. \end{aligned}$$

So it yields

$$\Big(\frac{p-2}{p-1}-\Big(\frac{p-2}{p-1}\Big)^2\Big)\int_M \phi^2|\nabla f|^2 \le \Big(1-\frac{p-2}{p-1}\Big)\int_M f^2|\nabla\phi|^2.$$

This implies that f is constant. Since the Sobolev inequality (1.1) holds on M, it is well known that M has infinite volume. Hence, we conclude from $\int_M f^2 < \infty$ that $\mathrm{d}u = 0$, i.e, u is constant.

Corollary 3.1 *Let M be an m (≥ 3)-dimensional complete non-compact isometrically immersed in an $(m+p)$-dimensional complete Riemannian manifold $\mathbb{R}^{m+p}$, there exists a sufficiently small constant ε, and a constant $C_2(n,p) = \big(\frac{(n-2)c}{(n-1)\sqrt{n(p-1)}}\big)^n$ such that*

$$\int_M |H|^n < \epsilon, \quad \int_M (S - n|H^2|)^{\frac{n}{2}} < C_2(n,p). \tag{3.17}$$

Then any p-harmonic map with finite L^{2p-2}-energy from M to a complete manifold with non-positive curvature is a constant.

Proof From the proof of Corollary 3.3 in [6], we see that the Sobolev inequality holds on M. So it follows from Theorem 3.2.

Remark 3.1 Combining the skills used in the proof of Theorem 2 in [3] and Theorem 1.1 in [6], we can also prove the Liouville theorem about p-harmonic map under the similar curvature condition on domain manifold, codomain manifold and index condition imposed on some operator. We should note that the similarity between the index condition and the stable condition for submanifold.

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