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Decomposition of $L^p(\partial D_a)$ Space and Boundary Value of Holomorphic Functions*

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Abstract This paper deals with two topics mentioned in the title. First, it is proved that function f in $L^p(\partial D_a)$ can be decomposed into a sum $g+h$, where D_a is an angular domain in the complex plane, g and h are the non-tangential limits of functions in $H^p(D_a)$ and $H^p(\overline{D}_a^c)$ in the sense of $L^p(D_a)$, respectively. Second, the sufficient and necessary conditions between boundary values of holomorphic functions and distributions in n-dimensional complex space are obtained.

Keywords Hardy space, Rational function, Holomorphic function, Distribution
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1 Introduction

Concerning the decomposition, we have known that the decomposition of $L^p(\mathbb{R})$ into the sum of $H_+(\mathbb{R})$ and $H_-(\mathbb{R})$ is obtained (see [8, 11, 13]), at least for range $1<p<\infty$. That is, for any function $f(x)\in L^p(\mathbb{R})$, $1<p<\infty$, $f(x)$ can be written as the sum of two functions f_+ and f_-, where f_+ and f_- are the boundary values of a function in the Hardy spaces for the upper-half space and lower-half space, respectively. For the difficult case of $0<p<1$, Qian and Deng [5] have obtained the analogous result. Naturally, a question will be asked: Whether the decomposition theorem of $L^p(\mathbb{R})$ can be extended to spaces $L^p(D_a)$? We will show the positive answer in Section 2.

Theory of boundary value problems for analytic functions (see [3, 10]) is one of the most important branches of complex analysis (see [4, 9]). It has wide applications because many practical problems in mechanics, physics and engineering may be transformed to such problems or singular integral equations which are closely related to the boundary problems. In the third section, we will discuss the relationship between boundary values of holomorphic functions and distributions (see [1]) in n-dimensional complex space. The same conclusions were established when $n=1$ in [2]. First we introduce some notations before we state our main results.

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A measurable function f is said to belong to $L^p(\partial D_a)$, $0 < p < \infty$, if

$$\|f\|_{L^p(\partial D_a)}^p = \int_{\partial D_a} |f(z)|^p |\mathrm{d}z| = \int_0^\infty |f(r\mathrm{e}^{\mathrm{i}a})|^p \mathrm{d}r + \int_0^\infty |f(r\mathrm{e}^{-\mathrm{i}a})|^p \mathrm{d}r < \infty,$$

where $0 < a < \pi$, $D_a = \{z = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} : -a < \theta < a,\ 0 < r < \infty\}$ is an angular domain in the complex plane.

The Hardy space $H^p(D_a)$ $(0 < p < \infty)$ is defined to consist of of these functions f holomorphic in the angular domain D_a with the property that

$$\|f\|_{H^p(D_a)} = \sup_{-a<\theta<a} \int_0^\infty |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}r < \infty,$$

that is

$$H^p(D_a) = \{f : f \text{ is holomorphic in } D_a,\ \|f\|_{H^p(D_a)} < \infty\}.$$

The Hardy space $H^p(\overline{D}_a^c)$ is defined in the same way as $H^p(D_a)$, that is

$$H^p(\overline{D}_a^c) = \{f : f \text{ is holomorphic in } \overline{D}_a^c,\ \|f\|_{H^p(\overline{D}_a^c)} < \infty\},$$

where $\overline{D}_a^c$ is the complement of $\overline{D}_a$ and

$$\|f\|_{H^p(\overline{D}_a^c)} = \sup_{a<\theta<2\pi-a} \int_0^\infty |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}r < \infty.$$

We denote $L^p_{D_a}$ and $L^p_{\overline{D}_a^c}$ as the non-tangential boundary limits of Hardy space $H^p(D_a)$ and $H^p(\overline{D}_a^c)$, respectively, and those are

$$L^p_{D_a} = \{f : f \text{ is the non-tangential boundary limit of a function in } H^p(D_a)\}$$

and

$$L^p_{\overline{D}_a^c} = \{f : f \text{ is the non-tangential boundary limit of a function in } H^p(\overline{D}_a^c)\}.$$

We fix A and B as open connected subsets of $\mathbb{R}^n$. A_B is defined as

$$A_B \doteq \{x + \mathrm{i}y : x = (x_1, x_2, \cdots, x_n) \in A,\ y = (y_1, y_2, \cdots, y_n) \in B\},$$

which is a subset of

$$\mathbb{C}^n = \{z = (z_1, z_2, \cdots, z_n) : z_j = x_j + \mathrm{i}y_j,\ j = 1, 2, \cdots, n\}.$$

If $A = \mathbb{R}^n$, we define the Hardy space on tube A_B based on B as

$$H^p(T_B) \doteq \{f \in H(T_B) : \|f\|_{H^p} < \infty\},$$

where $T_B = A_B,\ \|f\|_{H^p} = \sup\limits_{y\in B}(\int_{\mathbb{R}^n} |f(x + \mathrm{i}y)|^p \mathrm{d}x)^{\frac{1}{p}}$ (see [7, 13–14]) and $H(T_B)$ consists of all the holomorphic functions on T_B. The definition above is analogy to the definition in [4, 13–14].

Definition 1.1 *Let* $S_n = \{\sigma_k : \{1, 2, \cdots, n\} \to \{+1, -1\}\}$, $\sigma_k = \{\sigma_k(1), \sigma_k(2), \cdots, \sigma_k(n)\}$, $\sigma_k(j) = \pm 1,\ 1 \le k \le 2^n$. *We define*

$$\Omega_{\sigma_k} = \{y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n : y_j \sigma_k(j) > 0,\ j = 1, 2, \cdots, n\}.$$

We say that a function f*, which is holomorphic in* $T_{\Omega_{\sigma_k}}$*, admits a boundary value in the sense of distributions if the limit*

$$\lim_{\substack{y \in \Omega_{\sigma_k} \\ y_j \sigma_k(j) \to 0_+ \\ j = 1, 2, \cdots, n}} \int_{\mathbb{R}^n} f(x + \mathrm{i}y) \varphi(x) \mathrm{d}x$$

exists for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

For example, when $n = 2$, the four quadrants of $\mathbb{R}^2$ are denoted by $\Omega_{\sigma_1},\ \Omega_{\sigma_2},\ \Omega_{\sigma_3}$ and Ω_{σ_4}, where

$$\Omega_{\sigma_1} = \{y = (y_1, y_2) \in \mathbb{R}^2 :\ y_1 > 0,\ y_2 > 0,\ \sigma_1(1) = +1,\ \sigma_1(2) = +1\},$$

$$\Omega_{\sigma_2} = \{y = (y_1, y_2) \in \mathbb{R}^2 :\ y_1 > 0,\ y_2 < 0,\ \sigma_1(1) = +1,\ \sigma_1(2) = -1\},$$

$$\Omega_{\sigma_3} = \{y = (y_1, y_2) \in \mathbb{R}^2 :\ y_1 < 0,\ y_2 > 0,\ \sigma_1(1) = -1,\ \sigma_1(2) = +1\},$$

$$\Omega_{\sigma_4} = \{y = (y_1, y_2) \in \mathbb{R}^2 :\ y_1 < 0,\ y_2 < 0,\ \sigma_1(1) = -1,\ \sigma_1(2) = -1\}.$$

Correspondingly, $\mathbb{C}^n$ can be also decomposed into 2^n tubes denoted by $T_{\Omega_{\sigma_k}},\ k = 1, 2, \cdots, 2^n$, which with the octants of $\mathbb{R}^n$ as bases, that is

$$T_{\Omega_{\sigma_k}} = \{z = x + \mathrm{i}y \in \mathbb{C}^n :\ x \in \mathbb{R}^n,\ y \in \Omega_{\sigma_k}\}.$$

If f is holomorphic in $T_\Omega = T_{\bigcup_{k=1}^{2^n} \Omega_{\sigma_k}}$, we write it as $(f_{\sigma_1}, f_{\sigma_2}, \cdots, f_{\sigma_{2^n}})$, where $f_{\sigma_k} = f\big|_{\Omega_{\sigma_k}}$. We say that f admits a boundary value in the sense of distributions if all f_{σ_k} admit boundary value in the sense of distributions.

It follows from ([12], Theorem XIII, p. 74) that the mapping which to each $\varphi \in \mathcal{D}(\mathbb{R}^n)$ (which consists of infinite differentiable functions with compact support) assigns $\lim\limits_{\substack{y \in \Omega_{\sigma_k} \\ y_j \sigma_k(j) \to 0_+}} \int_{\mathbb{R}^n} f(x + \mathrm{i}y) \varphi(x) \mathrm{d}x$ is a distribution which we denote $b_{\sigma_k}(f_{\sigma_k})$. If $f = (f_{\sigma_1}, f_{\sigma_2}, \cdots, f_{\sigma_{2^n}})$ admits boundary value, we denote the distribution

$$b(f) = \sum_{k=1}^{2^n} (-1)^{m_k} b_{\sigma_k}(f_{\sigma_k}),$$

where m_k is the number of -1 in σ_k. The distributions $b(f), b_{\sigma_k}(f_{\sigma_k})$ are called the distributional boundary values of f and f_{σ_k}, respectively.

Definition 1.2 *A function* f *holomorphic in* $T_{\Omega_{\sigma_k}}$ *is said to be slow growth if, for every compact subset* K *of* $\mathbb{R}^n$*, there exist an integer* k *and two positive constants* ϵ *and* C *such that*

$$|f(z)| \le \frac{C}{\prod\limits_{j=1}^{n} |\mathrm{Im}\, z_j|^k}, \qquad \mathrm{Im}\, z \in K,\ 0 < |\mathrm{Im}\, z_j| \le \epsilon.$$

Definition 1.3 *Let $X \subseteq \mathbb{C}^n$ be an open set. A linear form μ on the vector space $C^\infty(X)$ is called to be continuous if there is a compact set $K \subseteq X$, a constant $c \geq 0$ and a nonnegative integer N such that, for all $\phi \in C^\infty(X)$,*

$$|\langle \mu, \phi \rangle| \leq c \sum_{|\alpha| \leq N} \sup\{|D^\alpha \phi| : x \in K\}.$$

The vector space of continuous linear forms on $C^\infty(X)$ is called $\mathcal{E}'(X)$.

Remark 1.1 If $\mu \in \mathcal{E}'(X)$, then μ is a distribution of finite order.

Definition 1.4 *For a distribution $T \in \mathcal{E}'(\mathbb{R}^n)$, its Cauchy transform $C(T)$ is the distribution in $\mathbb{C}^n$ given by*

$$C(T) = \Big\langle T(\xi) \otimes \delta_0(\eta), \frac{1}{\pi^n} \prod_{j=1}^n \frac{1}{z_j - \zeta_j} \Big\rangle.$$

Note that these distributions are holomorphic in $(\mathbb{C}^n - \mathrm{supp} T)$. Moreover, since $\frac{1}{\pi z_j}$ is a fundamental solution for $\frac{\partial}{\partial \overline{z}_j}$, and $T \otimes \delta_0(y)$ has compact support, the following holds

$$\frac{\partial^n}{\partial \overline{z}_1 \cdots \partial \overline{z}_n} C(T) = T(x) \otimes \delta_0(y).$$

2 Decomposition of $L^p(\partial D_a)$ Space

As previously mentioned in introduction, we have known that the functions of $L^p(\mathbb{R})$ for all $0 < p < \infty,\ p \neq 1$, can be decomposed into a sum of two functions which are the boundary values of the functions in the Hardy spaces for the upper-half and lower-half planes respectively. That is, for a function $f(x) \in L^p(\mathbb{R})$, there exist two functions f_+ and f_-, such that

$$f = f_+ + f_-,$$

where f_+ and f_- are the boundary values of the functions in the Hardy spaces for the upper-half and lower-half planes, respectively.

In this section, we generalize the above type decomposition theorem of $L^p(\mathbb{R})$ to the spaces $L^p(\partial D_a)$. We obtain the following main theorem.

Theorem 2.1 *Suppose that $1 \leq p < \infty$ and $f \in L^p(\partial D_a)$. Then there exists a positive constant A_p and two sequences of rational functions $\{P_k(z)\}$ and $\{Q_k(z)\}$, such that*

$$P_k \in H^p(D_a), \quad Q_k \in H^p(\overline{D}_a^c).$$

Meanwhile

$$\sum_{k=1}^{\infty} (\|P_k\|_{L^p_{D_a}} + \|Q_k\|_{L^p_{\overline{D}_a^c}}) \leq A_p \|f\|_{L^p(\partial D_a)}, \tag{2.1}$$

$$\lim_{n \to \infty} \|f - \sum_{k=1}^{n} (P_k + Q_k)\|_{L^p(\partial D_a)} = 0. \tag{2.2}$$

Therefore,

$$g(z)=\sum_{k=1}^{\infty}P_k(z)\in H^p(D_a),\quad h(z)=\sum_{k=1}^{\infty}Q_k(z)\in H^p(\overline{D}_a^c). \tag{2.3}$$

Moreover, g_+, g_- *and* h_+, h_- *are the non-tangential boundary limits of functions for* $g\in H^p(D_a)$ *and* $h\in H^p(\overline{D}_a^c)$*, respectively,* $f_+=g_++h_+$*,* $f_-=g_-+h_-$ *almost everywhere, and*

$$\|f_+\|_{L^p(\partial D_a)}\le \|g_+\|_{L^p_{D_a}}+\|h_+\|_{L^p_{\overline{D}_a^c}}\le A_p\|f_+\|_{L^p(\partial D_a)}, \tag{2.4}$$

$$\|f_-\|_{L^p(\partial D_a)}\le \|g_-\|_{L^p_{D_a}}+\|h_-\|_{L^p_{\overline{D}_a^c}}\le A_p\|f_-\|_{L^p(\partial D_a)}. \tag{2.5}$$

That is, in the sense of $L^p(\partial D_a)$*,*

$$L^p(\partial D_a)=L^p_{D_a}+L^p_{\overline{D}_a^c}.$$

In order to prove the main results above we need the following lemmas.

Lemma 2.1 (see [15]) *Let* $0<p<\infty$. $D=\{z=re^{i\theta}:0<r<\infty,\ \theta_1<\theta<\theta_2\}$ *is an angular domain in the complex plane. If* $f\in L^p(\partial D)$ *and* f *is holomorphic in* D*, then* $f(z)\in H^p(D)$*.*

Lemma 2.2 (see [15]) *Let* $0<p<\infty$ *and* $f\in H^p(D_a)$*, then*

$$\|f\|_{H^p(D_a)}=\max\{\|f_a^*\|_{L^p(0,\infty)},\|f_{-a}^*\|_{L^p(0,\infty)}\},$$

where f_a^* *and* f_{-a}^* *are the non-tangential limit of* f *on* $L_a=\{z=re^{ia}:0<r<\infty\}$ *and* $L_{-a}=\{z=re^{-ia}:0<r<\infty\}$*, respectively.*

Lemma 2.3 *Let* $0<a<\pi$, $0<p<\infty$*. If* $f\in L^p(\partial D_a)$*, for any* $\epsilon>0$*, then there exist rational functions* $\{R_k(z)\}$*, the poles of* $R_k(z)$ *contained in* $\{1,-1\}$*, such that for the case* $0<p\le 1$*,*

$$\sum_{k=1}^{\infty}\|R_k(z)\|^p_{L^p(\partial D_a)}\le(1+\epsilon)\|f\|^p_{L^p(\partial D_a)},$$

$$\lim_{n\to\infty}\|f-\sum_{k=1}^{n}R_k(z)\|^p_{L^p(\partial D_a)}=0.$$

Meanwhile, for $p>1$ *there hold*

$$\sum_{k=1}^{\infty}\|R_k(z)\|_{L^p(\partial D_a)}\le(1+\epsilon)\|f\|_{L^p(\partial D_a)},$$

$$\lim_{n\to\infty}\|f-\sum_{k=1}^{n}R_k(z)\|_{L^p(\partial D_a)}=0.$$

Proof We define function $f_N(z)$ on ∂D_a as

$$f_N(z)=\begin{cases}f(z), & \dfrac{1}{N}\le|f(z)|\le N,\\ 0, & \text{eleswhere,}\end{cases} \tag{2.6}$$

then for almost everywhere $z \in \partial D_a$, we have

$$\lim_{N\to\infty} f_N(z) = f(z), \quad |f_N|^p \le |f|^p.$$

As for the case $0 < p \le 1$, for any $\epsilon_0 > 0$, there exists a positive integer N_1 such that when $N_2 > N_1$, it holds

$$\|f_{N_2} - f\|^p_{L^p(\partial D_a)} < \frac{\epsilon_0}{2^k},$$

where k is a positive integer.

By the definition of f_N, we can deduce

$$\mathrm{supp} f_N \subseteq \Big\{z \in \partial D_a : \frac{1}{N} \le |f(z)| \le N\Big\}.$$

Moreover, the following holds true

$$f_N \in L^p(A),$$

where $A \doteq \{z \in \partial D_a : \frac{1}{N} \le |f(z)| \le N\}$.

There exists a function g with compact support contained in A. For the above $\epsilon_0 > 0$, there exists $N_3 > 0$ such that for any $N > N_3$,

$$\|g - f_N\|^p_{L^p(\partial D_a)} = \|g - f_N\|^p_{L^p(A)} < \frac{\epsilon_0}{2^k}.$$

For $N > \max\{N_1, N_3\}$ we have

$$\|f - g\|^p_{L^p(\partial D_a)} \le \frac{\epsilon_0}{2^{k-1}}.$$

Denote

$$C_0(\partial D_a) = \Big\{h(z) \in C(\partial D_a) : \lim_{|z|\to\infty} h(z) = 0\Big\}$$

and

$$\mathcal{B} = \Big\{\frac{P(z)}{(z^2-1)^k} : P(z) \text{ is a polynomial, } \deg P \le 2k-1,\ z \in \partial D_a\Big\}.$$

Then $\mathcal{B}$ is a subalgebra of $C_0(\partial D_a)$. Moreover, it separates the point of $C_0(\partial D_a)$. As a consequence, $\mathcal{B}$ is dense in $C_0(\partial D_a)$. For any $g \in C_0(\partial D_a)$, one may conclude

$$g(z)(z^2-1)^{l_0} \in C_0(\partial D_a),$$

where l_0 is an integer such that $2pl_0 > 1$.

By the density of $\mathcal{B}$, there exist a class of functions $\big\{\frac{P(z)}{(z^2-1)^k}\big\}$ which belong to $\mathcal{B}$ such that

$$\Big|g(z)(z^2-1)^{l_0} - \frac{P(z)}{[z^2-1]^k}\Big| < \epsilon_0,$$

that is

$$\Big|g(z) - \frac{P(z)}{(z^2-1)^{k+l_0}}\Big| < \frac{\epsilon_0}{(z^2-1)}.$$

Therefore, one has

$$\Big\|g(z) - \frac{P(z)}{(z^2-1)^{k+l_0}}\Big\|^p_{L^p(\partial D_a)} < K\epsilon_0^p,$$

where K is a constant satisfying

$$K = \int_{\partial D_a} \frac{1}{|z^2-1|^{l_0 p}} \mathrm{d}z < \infty.$$

By direct computation we have

$$\begin{aligned}
&\left\| f(z) - \frac{P(z)}{(z^2-1)^{k+l_0}} \right\|^p_{L^p(\partial D_a)} \\
&\leq \|f-g\|^p_{L^p(\partial D_a)} + \left\| g(z) - \frac{P(z)}{(z^2-1)^{k+l_0}} \right\|^p_{L^p(\partial D_a)} \\
&\leq \frac{\epsilon_0}{2^{k-1}} + K\epsilon_0^p. \qquad (2.7)
\end{aligned}$$

Let $Q_k(z) = \frac{P(z)}{(z^2-1)^{k+l_0}}$, then we can derive that

$$\|f(z) - Q_k(z)\|^p_{L^p(\partial D_a)} \leq \frac{\epsilon_0}{2^{k-1}} + K\epsilon_0^p \leq \frac{\epsilon'}{4^k}\|f\|^p_{L^p(\partial D_a)},$$

where $\epsilon' = \frac{2^{k+1}\epsilon_0 + 4^k K\epsilon_0^p}{\|f\|^p_{L^p(\partial D_a)}}$.

Let

$$R_1(z) = Q_1(z), \quad R_2(z) = Q_2(z) - Q_1(z), \quad \cdots, \quad R_k(z) = Q_k(z) - Q_{k-1}(z).$$

Then we can show

$$Q_k(z) = \sum_{j=1}^{k} R_j(z)$$

and

$$\begin{aligned}
\sum_{k=1}^{\infty} \|R_k(z)\|^p_{L^p(\partial D_a)} &= \sum_{k=1}^{\infty} \|Q_k - Q_{k-1}\|^p_{L^p(\partial D_a)} \\
&\leq \sum_{k=1}^{\infty} (\|Q_k - f\|^p_{L^p(\partial D_a)} + \|Q_{k-1} - f\|^p_{L^p(\partial D_a)}) \\
&\leq \frac{8\epsilon'}{3}\|f\|^p_{L^p(\partial D_a)} \\
&\leq (1+\epsilon)\|f\|^p_{L^p(\partial D_a)},
\end{aligned}$$

where $\epsilon = \frac{8\epsilon'}{3}$.

In addition, we can also deduce that

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} R_k(z) \right\|^p_{L^p(\partial D_a)} = 0.$$

By applying the same argument, it is not difficult to show the following conclusions for the case $p > 1$:

$$\sum_{k=1}^{\infty} \|R_k(z)\|_{L^p(\partial D_a)} \leq (1+\epsilon)\|f\|_{L^p(\partial D_a)},$$

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} R_k(z) \right\|_{L^p(\partial D_a)} = 0.$$

Thus, the proof is completed.

Lemma 2.4 *Suppose that* $0 < p < 1$, *and* $R \in L^p(\partial D_a)$ $\left(0 < a < \frac{\pi}{2}\right)$ *is a rational function whose poles are contained in* $\{1, \mathrm{e}^{2a\mathrm{i}}\}$, *then there exist two rational functions* P *and* Q *such that*

$$R(z) = P(z) + Q(z), \quad P \in L^p(\partial D_a),\ P \in H(D_a) \text{ and } Q \in L^p(\partial D_a), \quad Q \in H(\overline{D}_a^c).$$

Proof For the case $0 < p < 1$, $R \in L^p(\partial D_a)$ and R is a rational function, whose poles are contained in $\{1, \mathrm{e}^{2a\mathrm{i}}\}$, then $R(z)$ can be written as

$$R(z) = \sum_{k=-n}^{n} c_k (\beta(z))^k, \quad \text{where } \beta(z) = \frac{1 - z^{\frac{\pi}{2a}}}{1 + z^{\frac{\pi}{2a}}}.$$

Therefore $\beta(r\mathrm{e}^{\mathrm{i}a}) = \mathrm{e}^{\mathrm{i}\theta_a(r)}$, $\beta(r\mathrm{e}^{-\mathrm{i}a}) = \mathrm{e}^{\mathrm{i}\theta_{-a}(r)}$.

For each $\varphi \in \mathbb{R}$, we define

$$P(z, \varphi) = \frac{(\beta(z))^m R(z)}{(\beta(z))^m - \mathrm{e}^{\mathrm{i}\varphi}}, \quad Q(z, \varphi) = \frac{(\beta(z))^{-m} R(z)}{(\beta(z))^{-m} - \mathrm{e}^{-\mathrm{i}\varphi}},$$

where m is any positive integer greater than the positive integer n. Now we denote

$$I_{a,p} = \int_{-\pi}^{\pi} \int_0^{\infty} |P(r\mathrm{e}^{\mathrm{i}a}, \varphi)|^p \mathrm{d}r\mathrm{d}\varphi.$$

By Fubini′s theorem and direct computation, we can conclude that

$$\begin{aligned}
I_{a,p} &= \int_{-\pi}^{\pi} \int_0^{\infty} |P(r\mathrm{e}^{\mathrm{i}a}, \varphi)|^p \mathrm{d}r\mathrm{d}\varphi \\
&= \int_{-\pi}^{\pi} \int_0^{\infty} \frac{|\beta(r\mathrm{e}^{\mathrm{i}a})|^{mp} |R(r\mathrm{e}^{\mathrm{i}a})|^p}{|(\beta(r\mathrm{e}^{\mathrm{i}a}))^m - \mathrm{e}^{\mathrm{i}\varphi}|^p} \mathrm{d}r\mathrm{d}\varphi \\
&= \int_0^{\infty} \int_{-\pi}^{\pi} \frac{|R(r\mathrm{e}^{\mathrm{i}a})|^p}{|1 - \mathrm{e}^{\mathrm{i}(\varphi - \theta_a(r))}|^p} \mathrm{d}\varphi\mathrm{d}r \\
&= \int_0^{\infty} \int_{-\pi}^{\pi} \frac{1}{|1 - \mathrm{e}^{\mathrm{i}(\varphi - \theta_a(r))}|^p} \mathrm{d}\varphi |R(r\mathrm{e}^{\mathrm{i}a})|^p \mathrm{d}r \\
&= \int_0^{\infty} \int_{-\pi}^{\pi} \frac{1}{|1 - \mathrm{e}^{\mathrm{i}\varphi}|^p} \mathrm{d}\varphi |R(r\mathrm{e}^{\mathrm{i}a})|^p \mathrm{d}r \\
&= \int_0^{\infty} \int_{-\pi}^{\pi} \frac{1}{2^p \sin^p \frac{\varphi}{2}} \mathrm{d}\varphi |R(r\mathrm{e}^{\mathrm{i}a})|^p \mathrm{d}r \\
&\le 4 \int_0^{\infty} \int_0^{\frac{\pi}{2}} \frac{1}{2^p \left(\frac{2}{\pi}\varphi\right)^p} \mathrm{d}\varphi |R(r\mathrm{e}^{\mathrm{i}a})|^p \mathrm{d}r \\
&\le \frac{2^{1-p}\pi}{1-p} \int_0^{\infty} |R(r\mathrm{e}^{\mathrm{i}a})|^p \mathrm{d}r.
\end{aligned}$$

Similarly, it results in

$$\begin{aligned}
I_{-a,P} &= \int_{-\pi}^{\pi} \int_0^{\infty} |P(r\mathrm{e}^{-\mathrm{i}a}, \varphi)|^p \mathrm{d}r\mathrm{d}\varphi \le \frac{2^{1-p}\pi}{1-p} \int_0^{\infty} |R(r\mathrm{e}^{-\mathrm{i}a})|^p \mathrm{d}r, \\
I_{a,Q} &= \int_{-\pi}^{\pi} \int_0^{\infty} |Q(r\mathrm{e}^{\mathrm{i}a}, \varphi)|^p \mathrm{d}r\mathrm{d}\varphi \le \frac{2^{1-p}\pi}{1-p} \int_0^{\infty} |R(r\mathrm{e}^{\mathrm{i}a})|^p \mathrm{d}r, \\
I_{-a,Q} &= \int_{-\pi}^{\pi} \int_0^{\infty} |Q(r\mathrm{e}^{-\mathrm{i}a}, \varphi)|^p \mathrm{d}r\mathrm{d}\varphi \le \frac{2^{1-p}\pi}{1-p} \int_0^{\infty} |R(r\mathrm{e}^{-\mathrm{i}a})|^p \mathrm{d}r.
\end{aligned}$$

Therefore, there exists a number $\varphi \in [-\pi, \pi]$ such that

$$\int_0^\infty |P(r\mathrm{e}^{\mathrm{i}a}, \varphi)|^p \mathrm{d}r \le \frac{2\pi}{1-p}\int_0^\infty |R(r\mathrm{e}^{\mathrm{i}a})|^p \mathrm{d}r,$$
$$\int_0^\infty |P(r\mathrm{e}^{-\mathrm{i}a}, \varphi)|^p \mathrm{d}r \le \frac{2\pi}{1-p}\int_0^\infty |R(r\mathrm{e}^{-\mathrm{i}a})|^p \mathrm{d}r,$$
$$\int_0^\infty |Q(r\mathrm{e}^{\mathrm{i}a}, \varphi)|^p \mathrm{d}r \le \frac{2\pi}{1-p}\int_0^\infty |R(r\mathrm{e}^{\mathrm{i}a})|^p \mathrm{d}r,$$
$$\int_0^\infty |Q(r\mathrm{e}^{-\mathrm{i}a}, \varphi)|^p \mathrm{d}r \le \frac{2\pi}{1-p}\int_0^\infty |R(r\mathrm{e}^{-\mathrm{i}a})|^p \mathrm{d}r.$$

For the specially chosen real number φ, we define

$$P(z) = P(z, \varphi), \quad Q(z) = Q(z, \varphi),$$

then

$$R(z) = P(z) + Q(z).$$

Since $m > n$, P and Q are rational functions, with the poles of P contained in $\{\mathrm{e}^{2a\mathrm{i}}\}\bigcup\{z_k : k = 0, 1, 2, \cdots, n-1\}$ and the poles of Q contained in $\{1\}\bigcup\{z_k : k = 0, 1, 2, \cdots, n-1\}$, where $z_k = \big(-\mathrm{i}\tan\big(\frac{\varphi+2k\pi}{2m}\big)\big)^{\frac{2a}{\pi}} = -\mathrm{e}^{\mathrm{i}a}\tan^{\frac{2a}{\pi}}\big(\frac{\varphi+2k\pi}{2m}\big)$. Thus, $P(z)$ is holomorphic in D_a, and $Q(z)$ is holomorphic in $\overline{D}_a^c$. Therefore, one gets

$$R(z) = P(z) + Q(z), \quad P \in L^p(\partial D_a),\ P \in H(D_a) \text{ and } Q \in L^p(\partial D_a), \quad Q \in H(\overline{D}_a^c).$$

Lemma 2.5 *Suppose that $1 < p < \infty$, and $R \in L^p(\partial D_a)$ $(0 < a < \pi)$ is a rational function whose poles are contained in $\{1, -1\}$, then there exist two rational functions P and Q such that $P \in H^p(D_a)$, $Q \in H^p(\overline{D}_a^c)$,*

$$R(z) = P(z) + Q(z).$$

Meanwhile, we have

$$\|P_*\|_{L^p_{D_a}} + \|Q_*\|_{L^p_{\overline{D}_a^c}} \le A_P\|R\|_{L^p(\partial D_a)}, \tag{2.8}$$

where P_* and Q_* are the non-tangential limit of P and Q, respectively.

Proof For the case $p > 1$, $R(z) \in L^p(\partial D_a)$ is a rational function whose poles are contained in $\{1, -1\}$. Then

$$R(z) = \frac{c_{-N,1}}{(z-1)^N} + \cdots + \frac{c_{-1,1}}{z-1} + \frac{c_{-N,2}}{(z+1)^N} + \cdots + \frac{c_{-1,2}}{z+1}.$$

Let

$$P(z) = \frac{c_{-N,2}}{(z+1)^N} + \cdots + \frac{c_{-1,2}}{z+1},$$
$$Q(z) = \frac{c_{-N,1}}{(z-1)^N} + \cdots + \frac{c_{-1,1}}{z-1},$$

where $c_{-N,1}, \cdots, c_{-1,1}, c_{-N,2}, \cdots, c_{-1,2}$ are complex constants. Since $p > 1$, we have

$$P \in L^p(\partial D_a), \quad Q \in L^p(\partial D_a).$$

It follows from Lemma 2.1 that

$$P \in H^p(D_a), \quad Q(z) \in H^p(\overline{D}_a^c).$$

By Lemma 2.2, the inequalities (2.3) holds.

Lemma 2.6 *Suppose that $R \in L^1(\partial D_a)$ is a rational function and satisfies $\int_{\partial D_a} R(z)\mathrm{d}z = 0$ whose poles are contained in $\{1, -1\}$, then there exist two rational functions $P \in H^1(D_a)$ and $Q \in H^1(\overline{D}_a^c)$ such that*

$$R(z) = P(z) + Q(z)$$

and

$$\|P_*\|_{L^1_{D_a}} + \|Q_*\|_{L^1_{\overline{D}_a^c}} \leq A\|R\|_{L^1(\partial D_a)},$$

where A is a constant, P_* and Q_* are the non-tangential limits of P and Q, respectively.

Proof Since $R(z)$ is a rational function, then it gives

$$R(z) = \frac{c_{-N,1}}{(z-1)^N} + \cdots + \frac{c_{-1,1}}{z-1} + \frac{c_{-N,2}}{(z+1)^N} + \cdots + \frac{c_{-1,2}}{z+1},$$

where $c_{-N,1}, \cdots, c_{-1,2}$ are complex numbers. Moreover, $\int_{\partial D_a} R(z)\mathrm{d}z = 0$ implies that

$$\int_{\partial D_a} \frac{1}{(z-1)^N}\mathrm{d}z = 0, \quad \int_{\partial D_a} \frac{1}{(z+1)^N}\mathrm{d}z = 0$$

for $N \geq 2$. Therefore

$$\int_{\partial D_a} \Big(\frac{c_{-1,1}}{z-1} + \frac{c_{-1,2}}{z+1}\Big)\mathrm{d}z = 0,$$

then $c_{-1,1} + c_{-1,2} = 0$.

The following fact

$$\int_{\partial D_a} \Big(\frac{c_{-1,1}}{z-1} + \frac{c_{-1,2}}{z+1}\Big)\mathrm{d}z = \int_{\partial D_a} \frac{2c_{-1,1}}{(z-1)(z+1)}\mathrm{d}z = 0$$

implies that $c_{-1,1} = 0$. Thus $c_{-1,1} = c_{-1,2} = 0$. Let

$$P(z) = \frac{c_{-N,2}}{(z+1)^N} + \cdots + \frac{c_{-1,2}}{z+1},$$
$$Q(z) = \frac{c_{-N,1}}{(z-1)^N} + \cdots + \frac{c_{-1,1}}{z-1},$$

therefore

$$R(z) = P(z) + Q(z).$$

It is easy to know that $P(z)$ is analytic in D_a, and $Q(z)$ is analytic in $\overline{D}_a^c$. Moreover,

$$\int_{\partial D_a} |P(z)|\mathrm{d}z \leq \int_{\partial D_a} \sum_{k=2}^{N} \frac{|c_{-k,2}|}{|z+1|^k}\mathrm{d}z < \infty,$$
$$\int_{\partial D_a} |Q(z)|\mathrm{d}z \leq \int_{\partial D_a} \sum_{k=2}^{N} \frac{|c_{-k,1}|}{|z-1|^k}\mathrm{d}z < \infty.$$

As a consequence,

$$P(z) \in H^1(D_a), \quad Q(z) \in H^1(\overline{D}_a^c).$$

According to Lemma 2.2, one may easily show

$$\|P_*\|_{L^1_{D_a}} + \|Q_*\|_{L^1_{\overline{D}_a^c}} \le A\|P\|_{L^1(\partial D_a)}.$$

The proof is completed.

Proof of Theorem 2.1 According to Lemma 2.3 and Lemma 2.5, there exist two sequences of rational functions $P_k(z)$ and $Q_k(z)$ such that $P_k \in H^p(D_a)$, $Q_k \in H^p(\overline{D}_a^c)$.

For the case $1 < p < \infty$, we can verify that

$$\sum_{k=1}^{\infty}(\|P_k\|_{L^p(\partial D_a)} + \|Q_k\|_{L^p(\partial D_a)}) \le A_p(1+\epsilon)\|f\|_{L^p(\partial D_a)}$$

and

$$\lim_{n\to\infty} \Big\| f - \sum_{k=1}^{n}(P_k + Q_k)\Big\|_{L^p(\partial D_a)} = 0.$$

Moreover, (1.1) implies that (1.3) holds. Therefore, the non-tangential boundary limits g_+, g_- and h_+, h_- of functions for $g \in H^p(D_a)$ and $h \in H^p(\overline{D}_a^c)$ exist almost everywhere, respectively. (1.2) shows that

$$f_+ = g_+ + h_+, \quad f_- = g_- + h_-$$

almost everywhere, and (1.4)–(1.5) hold.

An argument similar to that used in the proof of the case $1 < p < \infty$, we can prove the case for $p = 1$ by using Lemma 2.1, Lemma 2.3 and Lemma 2.6. We complete the proof.

3 Boundary Value of Holomorphic Function in the Sense of n-Dimensional Distributions

This section presents several theorems as well as a key lemma, which is the main part of this section.

First we state a lemma which will be used in the proof of theorems.

Lemma 3.1 *If a distribution of the form $T(x) \otimes \delta_0(y)$ in $\mathcal{D}'(K^o_{(-c,c)^n})$ can be written in the form $\frac{\partial^n}{\partial\overline{z}_1\cdots\partial\overline{z}_n}U$, with $\text{supp}(U)$ contained in $\mathbb{R}^n$, then $T = 0$.*

Proof Locally U can be written in a unique way as $\sum\limits_{0\le j\le n} U_j(x_1, z_2, \cdots, z_n) \otimes \frac{\partial^j}{\partial y_1^j}\delta_0(y_1, y_2, \cdots, y_n)$ (see [2, p. 260]). Hence

$$\begin{aligned}
&\frac{\partial^n}{\partial\overline{z}_1\cdots\partial\overline{z}_n}U(z_1, z_2, \cdots, z_n)\\
&= \frac{\partial^{n-1}}{\partial\overline{z}_2\cdots\partial\overline{z}_n}\Big(\frac{\partial}{\partial\overline{z}_1}U(z_1, z_2, \cdots, z_n)\Big)\\
&= \frac{\partial^{n-1}}{\partial\overline{z}_2\cdots\partial\overline{z}_n}\Big(\frac{1}{2}\sum_{0\le j\le n}\frac{\partial}{\partial x_1}U_j \otimes \frac{\partial^j}{\partial y_1^j}\delta_0 + \frac{\mathrm{i}}{2}\sum_{0\le j\le n}U_j \otimes \frac{\partial^{j+1}}{\partial y_1^{j+1}}\delta_0\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial^{n-2}}{\partial \overline{z}_3 \cdots \partial \overline{z}_n}\Big(\frac{1}{2^2} \sum_{0\le j\le n} \frac{\partial^2 U_j}{\partial x_1 \partial x_2} \otimes \delta_0 + \frac{\mathrm{i}}{2^2} \sum_{0\le j\le n} \frac{\partial U_j}{\partial x_2} \otimes \frac{\partial^{j+1}}{\partial y_1^{j+1}} \delta_0 + \frac{\mathrm{i}}{2^2} \sum_{0\le j\le n} \frac{\partial^2 U_j}{\partial x_1 \partial y_2} \otimes \delta_0 \\
&\quad + \frac{\mathrm{i}}{2^2} \sum_{0\le j\le n} \frac{\partial U_j}{\partial x_1} \otimes \frac{\partial}{\partial y_2} \delta_0 + \frac{\mathrm{i}^2}{2^2} \sum_{0\le j\le n} \frac{\partial U_j}{\partial y_2} \otimes \frac{\partial^{j+1}}{\partial y_1^{j+1}} \delta_0 + \frac{\mathrm{i}^2}{2^2} \sum_{0\le j\le n} U_j \otimes \frac{\partial^{j+2}}{\partial y_1^{j+1} \partial y_2} \delta_0 \Big) \\
&= \frac{(\mathrm{i})^n}{2^n} \sum_{0\le j\le n} U_j \otimes \frac{\partial^{j+n}}{\partial y_1^{j+1} \partial y_2 \cdots \partial y_n} \delta_0 \\
&\quad + \frac{(\mathrm{i})^{n-1}}{2^n} \sum_{i=1}^{n} \sum_{0\le j\le n} \frac{\partial}{\partial x_i} U_j \otimes \frac{\partial^{j+n-1}}{\partial y_1^{j+1} \partial y_2 \cdots \partial y_{i-1} \partial y_{i+1} \cdots \partial y_n} \delta_0 \\
&\quad + \frac{(\mathrm{i})^{n-2}}{2^n} \sum_{\substack{i,k=1 \\ i<k}}^{n} \sum_{0\le j\le n} \frac{\partial^2}{\partial x_i \partial x_k} U_j \otimes \frac{\partial^{j+n-2}}{\partial y_1^{j+1} \partial y_2 \cdots \partial y_{i-1} \partial y_{i+1} \cdots \partial y_{k-1} \partial y_{k+1} \cdots \partial y_n} \delta_0 \\
&\quad + \cdots \cdots \\
&\quad + \frac{1}{2^n} \sum_{0\le j\le n} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} U_j \otimes \frac{\partial^j}{\partial y_1^j} \delta_0 \\
&= T(x_1, x_2, \cdots, x_n) \otimes \delta_0(y_1, y_2, \cdots, y_n).
\end{aligned}
$$

In view of the uniqueness of the representation, we conclude that

$$
\begin{aligned}
&\frac{\partial^n U_0}{\partial x_1 \cdots \partial x_n} + \mathrm{i} \frac{\partial^n U_0}{\partial x_1 \cdots \partial x_{k-1} \partial y_k \partial x_{k+1} \cdots \partial x_n} + \cdots + (\mathrm{i})^{n-1} \frac{\partial^n U_0}{\partial x_1 \partial y_2 \cdots \partial y_n} = T, \\
&\frac{\partial^n}{\partial x_1 \cdots \partial x_n} U_1 + \mathrm{i} \frac{\partial^{n-1}}{\partial x_2 \cdots \partial x_n} U_0 = 0, \\
&\frac{\partial^n}{\partial x_1 \cdots \partial x_n} U_2 + \mathrm{i} \frac{\partial^{n-1}}{\partial x_2 \cdots \partial x_n} U_1 = 0, \\
&\qquad\qquad \vdots \\
&\frac{\partial^n}{\partial x_1 \cdots \partial x_n} U_n + \mathrm{i} \frac{\partial^{n-1}}{\partial x_2 \cdots \partial x_n} U_{n-1} = 0, \\
&\frac{\partial^{n-1}}{\partial x_1 \cdots \partial x_{n-1}} U_1 + \mathrm{i} \frac{\partial^{n-2}}{\partial x_2 \cdots \partial x_{n-1}} U_0 = 0, \\
&\qquad\qquad \vdots \\
&\frac{\partial^{n-1}}{\partial x_1 \cdots \partial x_{n-1}} U_n + \mathrm{i} \frac{\partial^{n-2}}{\partial x_2 \cdots \partial x_{n-1}} U_{n-1} = 0, \\
&\qquad\qquad \vdots \\
&\frac{\partial}{\partial x_1} U_1 + \mathrm{i} U_0 = 0, \quad \cdots, \quad \frac{\partial}{\partial x_1} U_n + \mathrm{i} U_{n-1} = 0, \quad \cdots, \quad U_n = 0.
\end{aligned}
$$

Therefore, $T = 0$. This completes the proof.

Theorem 3.1 *Let $\Gamma = \Omega_{\sigma_1}$ be the first octant in $\mathbb{R}^n$. If f is a holomorphic function of slow growth in T_Γ, then it admits a boundary value in the sense of distributions.*

Proof Let K be a compact subset of $\mathbb{R}^n$. For any $\epsilon > 0$, and a fixed point $z^0 = x^0 + \mathrm{i}y^0$ in T_Γ, we denote $\gamma_0, \gamma_1, \gamma_2, \cdots$ the successive primitives of f in T_Γ, vanishing at z^0,

$$\gamma_0(z_1 \cdots z_n) = \int_{z_1^0}^{z_1} \cdots \int_{z_n^0}^{z_n} f(\omega_1, \cdots, \omega_n)\mathrm{d}\omega_1 \cdots \mathrm{d}\omega_n,$$

$$\gamma_1(z_1 \cdots z_n) = \int_{z_1^0}^{z_1} \cdots \int_{z_n^0}^{z_n} \gamma_0(\omega_1, \cdots, \omega_n)\mathrm{d}\omega_1 \cdots \mathrm{d}\omega_n,$$

$$\vdots$$

One can show by recurrence that there exist some positive constants $C_0, \cdots, C_k$ and C_k' such that, for $z = x + \mathrm{i}y,\ x \in K,\ 0 < |\mathrm{Im}\, z_j| \le \epsilon$,

$$|\gamma_0(z)| \le \frac{C_0}{\prod\limits_{j=1}^n y_j^{k-1}}, \quad \cdots, \quad |\gamma_{k-2}(z)| \le \frac{C_{k-2}}{\prod\limits_{j=1}^n y_j},$$

$$|\gamma_{k-1}(z)| \le C_{k-1} \prod_{j=1}^n |\log y_j|, \quad |\gamma_k(z)| \le C_k \prod_{j=1}^n (y_j |\log y_j|) + C_k'.$$

Since γ_k is bounded, it follows that γ_{k+1} can be extended as a continuous function to $K_{[0,c)^n}$. Therefore, the family of distributions $T_y \in \mathcal{D}'(\mathrm{int}\, K)$ defined by

$$\langle T_y, \varphi \rangle = \int_K \gamma_{k+1}(x + \mathrm{i}y)\varphi(x)\mathrm{d}x$$

admits a limit T_0 in the sense of distributions when $y_j\sigma_1(j) \to 0_+$, $j = 1, 2, \cdots, n$, $0 < |\mathrm{Im}\, z_j| \le \epsilon,\ \forall j = 1, 2, \cdots, n$. According to

$$\frac{\partial^{n(k+2)}}{\partial x_1^{k+2} \cdots \partial x_n^{k+2}} \gamma_{k+1} = \frac{\partial^{n(k+2)}}{\partial {z_1}^{k+2} \cdots \partial {z_n}^{k+2}} \gamma_{k+1} = f,$$

one obtains

$$\Big\langle \frac{\partial^{n(k+2)}}{\partial x_1^{k+2} \cdots \partial x_n^{k+2}} T_y, \varphi \Big\rangle = \int_K f(x + \mathrm{i}y)\varphi(x)\mathrm{d}x,$$

which has the distribution $\frac{\partial^{n(k+2)}}{\partial x_1^{k+2} \cdots \partial x_n^{k+2}} T_0$ as a limit when $y_j\sigma_1(j) \to 0_+$, $j = 1, 2, \cdots, n$. Hence f admits the last distribution as boundary value in the sense of $\mathcal{D}'(\mathrm{int\,K})$. Since K is arbitrary and a distribution is determined by its value locally, then f admits a boundary value $b(f)$ in the sense of distributions. We complete the proof.

Under the condition of Theorem 3.1, let us denote by $\widetilde{\gamma_{k+1}}$ the function (and its associated distribution) defined in $K_{[-\infty,c)^n}$ as follows:

$$\widetilde{\gamma_{k+1}}(x + \mathrm{i}y) = \begin{cases} \gamma_{k+1}(x + \mathrm{i}y), & 0 < y_j < c,\ \forall j = 1, 2, \cdots, n, \\ \gamma_{k+1}(x + \mathrm{i}0) = \lim\limits_{y_j\sigma_1(j) \to 0_+} \gamma_{k+1}(x + \mathrm{i}y), & y = (0, 0, \cdots, 0), \\ \lim\limits_{y_j\sigma_1(j) \to 0_+} \gamma_{k+1}(x + \mathrm{i}y), & 0 \le y_j < c,\ y \ne 0,\ j = 1, 2, \cdots, n, \\ 0, & y_j < 0,\ j = 1, 2, \cdots, n, \end{cases} \tag{3.1}$$

where $0 < c \le \infty,\ [-\infty, c)^n = [-\infty, c) \times \cdots \times [-\infty, c)$.

Theorem 3.2 *Let f be a holomorphic function of slow growth in T_Γ. For a compact subset $K \subseteq \mathbb{R}^n$, we define a distribution $f_K \in \mathcal{D}'(K^o_{(-\infty,c)^n})$ as*

$$f_K = \frac{\partial^{n(k+2)}}{\partial x_1^{k+2} \cdots \partial x_n^{k+2}} \widetilde{\gamma}_{k+1}.$$

Then we have the following identity:

$$\frac{\partial^n}{\partial \overline{z}_1 \cdots \partial \overline{z}_n} f_K = \Big(\frac{\mathrm{i}}{2}\Big)^n (b(f)|_K) \otimes \delta_0(y)$$

in the sense of distributions.

Proof We keep the notation from the previous theorem and let $\varphi \in \mathcal{D}(K^o_{(-\infty,c)^n})$. Denote $\widetilde{f} = f_K$ and $\widetilde{\gamma} = \widetilde{\gamma_{k+1}}$ for simplify, therefore

$$\begin{aligned}
&\Big\langle \frac{\partial^n \widetilde{f}}{\partial \overline{z}_1 \cdots \partial \overline{z}_n}, \varphi \Big\rangle \\
&= (-1)^n \Big\langle \widetilde{f}, \frac{\partial^n \varphi}{\partial \overline{z_1} \cdots \partial \overline{z_n}} \Big\rangle \\
&= (-1)^n \Big\langle \frac{\partial^{n(k+2)}}{\partial x_1^{k+2} \cdots \partial x_n^{k+2}} \widetilde{\gamma_{k+1}}, \frac{\partial^n \varphi}{\partial \overline{z_1} \cdots \partial \overline{z_n}} \Big\rangle \\
&= (-1)^{n(k+3)} \Big\langle \widetilde{\gamma_{k+1}}, \frac{\partial^{n(k+3)}}{\partial x_1^{k+2} \cdots \partial x_n^{k+2} \partial \overline{z_1} \cdots \partial \overline{z_n}} \varphi \Big\rangle \\
&= \frac{(-1)^{n(k+3)}}{(2\mathrm{i})^n} \int_K \int_{(0,c)^n} \widetilde{\gamma_{k+1}} \frac{\partial^{n(k+3)}}{\partial x_1^{k+2} \cdots \partial x_n^{k+2} \partial \overline{z_1} \cdots \partial \overline{z_n}} \varphi \mathrm{d}\overline{z} \wedge \mathrm{d}z \\
&= \lim_{\|\epsilon\| \to 0_+} \frac{(-1)^{n(k+3)}}{(2\mathrm{i})^n} \int_K \int_{(\epsilon,c)^n} \widetilde{\gamma_{k+1}} \frac{\partial^{n(k+3)}}{\partial x_1^{k+2} \cdots \partial x_n^{k+2} \partial \overline{z_1} \cdots \partial \overline{z_n}} \varphi \mathrm{d}\overline{z} \wedge \mathrm{d}z,
\end{aligned}$$

where $\mathrm{d}\overline{z} \wedge \mathrm{d}z = (\mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1) \cdots (\mathrm{d}\overline{z}_n \wedge \mathrm{d}z_n)$. The last step used the continuity of $\widetilde{\gamma_{k+1}}$ up to the boundary. Hence, using

$$\gamma_{k+1} \frac{\partial^{k+3}}{\partial \overline{z}_j \partial x_j^{k+2}} \varphi \mathrm{d}\overline{z}_j \wedge \mathrm{d}z_j = \mathrm{d}\Big(\gamma_{k+1} \frac{\partial^{k+2}}{\partial x_j^{k+2}} \varphi \mathrm{d}z_j\Big), \quad j = 1, 2, \cdots, n,$$

and applying Stokes formula, we have

$$\begin{aligned}
&\Big\langle \frac{\partial^n \widetilde{f}}{\partial \overline{z}_1 \cdots \partial \overline{z}_n}, \varphi \Big\rangle \\
&= \frac{(-1)^{n(k+3)}}{(2\mathrm{i})^n} \int_K \gamma_{k+1}(x + \mathrm{i}0) \frac{\partial^{n(k+2)}}{\partial x_1^{k+2} \cdots \partial x_n^{k+2}} \varphi(x + \mathrm{i}0) \mathrm{d}x \\
&= \frac{(-1)^{n(k+3)}}{(2\mathrm{i})^n} \Big\langle T_0(x), \frac{\partial^{n(k+2)}}{\partial x_1^{k+2} \cdots \partial x_n^{k+2}} \varphi(x + \mathrm{i}0) \Big\rangle \\
&= \frac{(-1)^n}{(2\mathrm{i})^n} \Big\langle \frac{\partial^{n(k+2)} T_0(x)}{\partial x_1^{k+2} \cdots \partial x_n^{k+2}}, \varphi(x + \mathrm{i}0) \Big\rangle \\
&= \frac{(-1)^n}{(2\mathrm{i})^n} \langle b(f), \varphi(x + \mathrm{i}0) \rangle \\
&= \Big\langle \Big(\frac{\mathrm{i}}{2}\Big)^n b(f)\Big|_K \otimes \delta_0(y), \varphi \Big\rangle,
\end{aligned}$$

which is the desired formula.

Remark 3.1 If we start with f_{σ_k} being a holomorphic function of slow growth in $T_{\Omega_{\sigma_k}}$, for any σ_k in S_n, we obtain

$$\frac{\partial^n \widetilde{f_{\sigma_k}}}{\partial \overline{z}_1 \cdots \partial \overline{z}_n} = (-1)^{m_k} \Big(\frac{\mathrm{i}}{2}\Big)^n b_{\sigma_k}(f_{\sigma_k}) \otimes \delta_0(y),$$

where $\widetilde{f_{\sigma_k}}$ is the extension of f_{σ_k}.

Theorem 3.3 *For every $T \in \mathcal{E}'(\mathbb{R}^n)$, the Cauchy transform $C(T)$ has $(-2\mathrm{i})^n T$ as a boundary value in the sense of distributions.*

Proof Let us observe first that, T being of finite order m, $C(T)$ is slow growth. In fact, there are constants C and R such that

$$|\langle T, \varphi \rangle| \leq C \sup_{x \in B_0(R)} \sup_{0 \leq |j| \leq m} \Big| \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} \varphi(x) \Big|,$$

where $B_0(R)$ is the ball of radius R with center at the origin, $j = (j_1, \cdots, j_n)$ and $|j| = j_1 + \cdots + j_n$. Therefore, if z is not in $B_0(R)$ we have

$$|C(T)| = \Big| \Big\langle T(x), \frac{1}{\pi^n \prod\limits_{k=1}^{n} (z_k - x_k)} \Big\rangle \Big| \leq C \sup_{x \in B_0(R)} \sup_{0 \leq |j| \leq m} \Big| \prod_{k=1}^{n} \frac{j_k!}{(z_k - x_k)^{j_k+1}} \Big|.$$

It follows that, for a conveniently chosen constant $C_1 > 0$, we have

$$|C(T)| \leq \frac{C_1}{\prod\limits_{k=1}^{n} |\mathrm{Im}\, z_k|^{j_k+1}},$$

where $\mathrm{Im}\, z_k \neq 0,\ k = 1, 2, \cdots, n$.

Let us denote by f_{σ_k} the restriction $C(T)|_{\Omega_{\sigma_k}}$ and $\widetilde{f}_{\sigma_k}$ the extension to $\mathcal{D}'(\mathbb{C}^n)$ with support in $\{\overline{\Omega}_{\sigma_k}\}$. Its existence is a consequence of Theorem 3.1, and we know that

$$\frac{\partial^n \widetilde{f}_{\sigma_k}}{\partial \overline{z}_1 \cdots \partial \overline{z}_n} = (-1)^{m_k} \Big(\frac{\mathrm{i}}{2}\Big)^n b_{\sigma_k}(f_{\sigma_k}) \otimes \delta_0(y), \quad \frac{\partial^n}{\partial \overline{z}_1 \cdots \partial \overline{z}_n} C(T) = T \otimes \delta_0(y).$$

Then

$$\begin{aligned} \frac{\partial^n}{\partial \overline{z}_1 \cdots \partial \overline{z}_n} \Big(C(T) - \sum_{k=1}^{2^n} f_k \Big) &= \Big(T - \sum_{k=1}^{2^n} (-1)^{m_k} \Big(\frac{\mathrm{i}}{2}\Big)^n b_{\sigma_k}(f_{\sigma_k}) \Big) \otimes \delta_0(y) \\ &= \Big(T - \Big(\frac{\mathrm{i}}{2}\Big)^n C(T) \Big) \otimes \delta_0(y). \end{aligned}$$

By Lemma 3.1 we obtain

$$T = \Big(\frac{\mathrm{i}}{2}\Big)^n C(T).$$

The proof is completed.

Theorem 3.4 *Every distribution T in $\mathcal{D}'(\mathbb{R}^n)$ is a boundary value of a holomorphic function of slow growth in T_Ω.*

Proof Let $K_n \subseteq K_{n+1}$ be compact subsets of $\mathbb{R}^n$, and $\mathbb{R}^n = \bigcup_n K_n$. Chosen a sequence $\varphi_n \in \mathcal{D}(\mathbb{R}^n)$, $0 \le \varphi_n \le 1$, $\varphi_n \equiv 1$ in a neighborhood of K_n and $\text{supp}\varphi_n \subseteq K_{n+1}$. Let $B = [-c, c]^n$, $T_B = \{z = x + \mathrm{i}y : x \in \mathbb{R}^n, y \in B\}$, $B_n = [-c_n, c_n]^n$, $c_n \to c$ $(n \to \infty)$ and $B = \bigcup_{n=1}^{\infty} B_n$.

Define $T_1 = \varphi_1 T$, $T_n = (\varphi_n - \varphi_{n-1})T$ for $n \ge 2$. We have $T = \sum_{n\ge1} T_n$. For every $n \ge 1$ the function

$$C(T_n) = C(T_n(x) \otimes \delta_0(y))$$

is holomorphic outside $\text{supp}(T_n) + \mathrm{i}0$. In particular, for $n \ge 2$, $C(T_n)$ is holomorphic in a neighborhood of the set

$$L_{n-1} = \{z = x + \mathrm{i}y \in \mathbb{C}^n : x \in \overline{K}_{n-1},\ y \in B_{n-1}\}.$$

Since L_{n-1} is convex, it is holomorphically convex in $\mathbb{C}^n$. Therefore, we can find polynomials h_n such that

$$\sup_{z \in L_{n-1}} |C(T_n)(z) - h_n(z)| \le \frac{1}{2^n}, \quad n \ge 2.$$

Thus, the series

$$S = C(T_1) + \sum_{n\ge2} (C(T_n) - h_n)$$

converges in $\mathcal{D}'\big(T_{\bigcup_{k=1}^{2^n} \Omega_{\sigma_k}}\big)$ and defines a holomorphic function of slow growth. Furthermore,

$$\frac{\partial^n S}{\partial\overline{z}_1 \cdots \partial\overline{z}_n} = \frac{\partial^n C(T_1)}{\partial\overline{z}_1 \cdots \partial\overline{z}_n} + \sum_{n\ge2} \frac{\partial^n C(T_n)}{\partial\overline{z}_1 \cdots \partial\overline{z}_n} = \Big(\sum_{n\ge1} T_n\Big) \otimes \delta_0(y) = T(x) \otimes \delta_0(y).$$

For every compact subset K of $\mathbb{R}^n$, it leads to

$$\frac{\partial^n \widetilde{S}_{\sigma_k}}{\partial\overline{z}_1 \cdots \partial\overline{z}_n} = (-1)^{m_k} \Big(\frac{\mathrm{i}}{2}\Big)^n b_{\sigma_k}(S_{\sigma_k}) \otimes \delta_0(y),$$

where $\widetilde{S}_{\sigma_k}$ denotes the extension of $S\big|_{K_{\Omega_{\sigma_k}}}$ and $\text{supp}\widetilde{S}_{\sigma_k} \subset T_{\bigcup_{k=1}^{2^n} \overline{\Omega}_{\sigma_k}}$.

As a result, we get

$$\frac{\partial^n}{\partial\overline{z}_1 \cdots \partial\overline{z}_n} \Big(S\big|_{K_B} - \sum_{k=1}^{2^n} \widetilde{S}_{\sigma_k}\Big) = \Big(T\big|_K - \Big(\frac{\mathrm{i}}{2}\Big)^n b(S)\Big) \otimes \delta_0(y),$$

which implies that

$$T = \Big(\frac{\mathrm{i}}{2}\Big)^n b(S).$$

This completes the proof.

Theorem 3.5 *Suppose that $f \in H(T_\Gamma)$ has $T \in \mathcal{D}'(\mathbb{R}^n)$ as a boundary value. Then for every $\varphi \in \mathcal{D}(T_{(-\infty,c)^n})$, $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$, the function $g : [0, c)^n \to \mathbb{C}$ defined by*

$$g(\epsilon) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} f(x + \mathrm{i}\epsilon)\varphi(x + \mathrm{i}\epsilon)\mathrm{d}x, & 0 < \epsilon_j < c,\ j = 1, 2, \cdots, n, \\ \big\langle T, \varphi(x + \mathrm{i}0)\big\rangle, & \epsilon = (0, 0, \cdots, 0), \\ \displaystyle\lim_{\epsilon_j \sigma_1(j) \to 0_+} \int_{\mathbb{R}^n} f(x + \mathrm{i}\epsilon)\varphi(x + \mathrm{i}\epsilon)\mathrm{d}x, & 0 < \epsilon_j < c,\ \varepsilon \ne 0,\ j = 1, 2, \cdots, n \end{cases} \tag{3.2}$$

is continuous. Moreover, the integral

$$S(\varphi) = \int_{[0,c)^n} \Big(\int_{\mathbb{R}^n} f(x+\mathrm{i}\epsilon)\varphi(x+\mathrm{i}\epsilon)\mathrm{d}x \Big)\mathrm{d}\epsilon$$

exists and S defines a distribution in $T_{(-\infty,c)^n}$ with support in $(\mathbb{R}^n)_{[0,c)^n}$.

Proof The continuity of $g(\epsilon)$ follows from Definition 1.1 and the existence of $S(\varphi)$ follows from the continuity of $g(\epsilon)$. From [12, Theorem XIII, p. 74] we conclude that S is a distribution.

Theorem 3.6 *Under the same hypotheses of Theorem* 3.5*, we conclude that f is slow growth and*

$$\frac{\partial^n S}{\partial \overline{z}_1 \cdots \partial \overline{z}_n} = \Big(\frac{\mathrm{i}}{2}\Big)^n b(f) \otimes \delta_0(y) = \Big(\frac{\mathrm{i}}{2}\Big)^n T(x) \otimes \delta_0(y)$$

with S defined above.

Proof Let Ω be an open subset of $\mathbb{R}^n$, $K \subset\subset \Omega \subseteq \mathbb{R}^n$ and $0 < \epsilon < d(K, \Omega^c)$. Let ψ be a standard radial function (see [6]) in $\mathbb{C}^n$ with support in $B(0,1)$, $\psi_\delta(z) = \frac{1}{\delta^{2n}}\psi\big(\frac{z}{\delta}\big)$ $(\delta > 0)$. For fixed point $x^0 \in K$, denote by $\varphi_\delta(\zeta)$ the function $\psi_\delta(\zeta - (x^0 + \mathrm{i}\delta))$. Since S has a finite order k in the compact set $F = \{z \in \mathbb{C}^n : d(z,K) \le \epsilon\}$, there is a positive constant C such that if $\varphi \in \mathcal{D}(\mathbb{C}^n)$ has support in $(\mathrm{int\,F} \cap \Gamma)$, then

$$|\langle S, \varphi\rangle| \le \Big| \int f\varphi \mathrm{d}m \Big| \le C \sup_{\zeta \in F} \sup_{0 \le p+q \le k} \Big| \frac{\partial^{p+q}\varphi}{\partial z_1^{p_1} \cdots \partial z_n^{p_n} \partial \overline{z}_1^{q_1} \cdots \partial \overline{z}_n^{q_n}}(\zeta) \Big|,$$

where $p = p_1 + \cdots + p_n,\ q = q_1 + \cdots + q_n$.

On the other hand, it holds

$$\frac{\partial^{p+q}}{\partial z^p \partial \overline{z}^q}\varphi_\delta(\zeta) = \frac{1}{\delta^{p+q+2n}} \frac{\partial^{p+q}}{\partial z^p \partial \overline{z}^q}\psi\Big(\frac{\zeta - (x^0 + \mathrm{i}\delta)}{\delta}\Big).$$

Therefore, there is a constant $C_K > 0$ such that in the case $0 < \delta \le \epsilon$,

$$\Big| \int \varphi_\delta(\zeta) f(\zeta) \mathrm{d}m(\zeta) \Big| \le C_k \delta^{-2n-k}.$$

By the mean value property of holomorphic functions, the left-side is precisely $|f(x^0 + \mathrm{i}\delta)|$. In other words,

$$|f(x^0 + \mathrm{i}\delta)| \le C_k \delta^{-2n-k}, \quad x^0 \in K,\ 0 < \delta \le \epsilon.$$

The proof is completed.

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