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Some Weak Specification Properties and Strongly Mixing*

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Abstract In this paper, the authors first construct a dynamical system which is strongly mixing but has no weak specification property. Then the authors introduce two new concepts which are called the quasi-weak specification property and the semi-weak specification property in this paper, respectively, and the authors prove the equivalence of quasi-weak specification property, semi-weak specification property and strongly mixing.

Keywords Weak specification property, Mixing property, Symbolic dynamics
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1 Introduction

By a topological dynamical system (X, f) (a dynamical system for short), we mean that X is a compact metric space with metric d and $f : X \to X$ is continuous.

The specification property has turned out to be an important notion in the study of dynamical systems. It was firstly introduced by Bowen in [2] (see also [1, 3, 6] for some examples with the specification property and some basic properties). Nowadays, many authors have given their attention to the study of the specification property and raised several kinds of specification properties, such as the strong specification property, the periodic specification property, the almost specification property, the weak specification property, etc. (see [4–5, 7]). In this article, we will follow the terminology of [4].

Definition 1.1 (see [4]) *We say that a surjective continuous map $f : X \to X$ has the weak specification property* (*briefly WSP*), *if for any $\delta > 0$, there is a positive integer N_δ such that for any two points y_1, y_2 and any sequence $0 = j_1 \le k_1 < j_2 \le k_2$ with $j_2 - k_1 \ge N_\delta$ there is a point $x \in X$ such that, for each positive integer $m = 1, 2$ and all integers i with $j_m \le i \le k_m$, the following condition holds:*

$$d(f^i(x), f^i(y_m)) < \delta.$$

WSP is one of the weakest forms of specification property. And it is known that a map with WSP is strongly mixing. This result is strongly dependent on the assumption that f is surjective, since this result may not be true if all the other conditions of WSP but the surjective property are satisfied. See the following example for details.

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Example 1.1 Let $X = \{\frac{1}{n} : n = 1, 2, 3, \cdots\} \cup \{0\}$ and $f(0) = 0$, $f\big(\frac{1}{n}\big) = \frac{1}{n+1}$ for any $n \geq 1$. Under the usual metric of $\mathbb{R}$ (the real space), (X, f) is a dynamical system. Then (X, f) satisfies all the other conditions of WSP except the surjective property.

Proof Clearly, f is not surjective. For any $\delta > 0$, there is $N_\delta > 0$ such that $\frac{\delta}{2} > \frac{1}{N_\delta}$, that is, for all $n > N_\delta$, $\frac{1}{n} < \frac{\delta}{2}$. It is evident that for any $z \in X$, $\lim\limits_{k\to\infty} f^k(z) = 0$. So for any two points $y_1, y_2 \in X$ and any sequence $0 = j_1 \leq k_1 < j_2 \leq k_2$ with $j_2 - k_1 \geq N_\delta$, choose $x = y_1$, then we have

$$|f^i(x) - f^i(y_1)| = |f^i(y_1) - f^i(y_1)| = 0 < \delta, \qquad \forall j_1 \leq i \leq k_1,$$
$$|f^j(x) - f^j(y_2)| \leq |f^j(x) - 0| + |0 - f^j(y_2)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta, \quad \forall j_2 \leq j \leq k_2.$$

Therefore, (X, f) satisfies all the other conditions of WSP except the surjective property.

Conversely, a natural question appears: Does the strongly mixing property imply WSP? In this paper, we first show by an example that the strongly mixing property is not enough to imply WSP; the concrete example will be given in Section 3. Furthermore, we introduce two weaker concepts of specification property than WSP, which are called the quasi-weak specification property and the semi-weak specification property in this article, respectively. See the following definitions in more details.

Definition 1.2 *We say that a surjective continuous map $f : X \to X$ has the quasi-weak specification property (briefly QWSP), if for any $\delta > 0$, there is a positive integer N_δ such that for any two points y_1, y_2 and any $n \geq N_\delta$ there is a point $x \in X$ such that $d(x, y_1) < \delta$ and $d(f^n(x), f^n(y_2)) < \delta$.*

Definition 1.3 *We say that a surjective continuous map $f : X \to X$ has the semi-weak specification property (briefly SWSP), if for any $\delta > 0$, there is a positive integer N_δ such that for any two points y_1, y_2 and any sequence $0 \leq j_1 < k_1 < j_2 < k_2$ with $j_2 - k_1 \geq N_\delta$ there is a point $x \in X$ and for each positive integer $m = 1, 2$ there exists an integer i with $j_m \leq i \leq k_m$ such that*

$$d(f^i(x), f^i(y_m)) < \delta.$$

On the basis of these concepts, we show the equivalence of the quasi-weak specification property, the semi-weak specification property and strongly mixing. If we note the example given in Section 3, this result shows that QWSP and SWSP are strictly weaker than WSP.

2 Preliminaries and Basic Concepts

Let (X, f) be a dynamical system. In this paper, we use $\mathbb{Z}_+$ to denote the set of all nonnegative integers and use $\mathbb{N}$ to denote the set of all positive integers and denote the sets of periodic points, almost periodic points, recurrent points, and non-wandering points of f by $P(f)$, $A(f)$, $R(f)$ and $\Omega(f)$, respectively. Let $x \in X$, denote by $\mathrm{orb}(x, f)$ and $\omega(x, f)$ the orbit of x and the ω-limit set of x under f, respectively.

Denote $B(x, \varepsilon)$ by the ε-neighborhood of x, that is $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$.

We set, for nonempty open subsets U, V of X,

$$N(U, V) = \{n \in \mathbb{Z}_+ \mid f^n(U) \cap V \neq \varnothing\}.$$

We say that:

f is (topologically) transitive, if for any two nonempty open sets $U, V \subset X$, $N(U,V) \neq \varnothing$;

f is strongly mixing, if $N(U,V)$ is cofinite, namely, there exists $N \in \mathbb{N}$ such that for any $n > N$, $f^n(U) \cap V \neq \varnothing$.

Let (X,f), (Y,g) be two dynamical systems with metric d, d_1, respectively. The product system of (X,f) and (Y,g) is denoted by $(X \times Y, f \times g)$. The metric $\widetilde{d}$ on $X \times Y$ is defined as

$$\widetilde{d}((x_1,y_1),(x_2,y_2)) = \max\{d(x_1,x_2), d_1(y_1,y_2)\},$$

whenever $(x_1,y_1),(x_2,y_2) \in X \times Y$.

If there is a continuous surjective map $\phi : X \to Y$ with $\phi \circ f = g \circ \phi$, we will say that f and g are semi-conjugate (by ϕ). The map ϕ is called a semi-conjugacy or a factor map (from f to g). The map g is called a factor of f and the map f is called an extension of g. If ϕ is a homeomorphism, then we call it a conjugacy (from f to g).

Next, we introduce some basic notations of symbolic dynamical systems.

Suppose that $S = \{0,1\}$ and $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ is the one-sided symbolic space on S. A distance on Σ_2 is defined as follows: For $x = (x_1x_2x_3\cdots)$, $y = (y_1y_2y_3\cdots) \in \Sigma_2$,

$$\rho(x,y) = \begin{cases} 0, & \text{if } \ x = y, \\ 2^{-\min\{n \in \mathbb{N} | x_n \neq y_n\}}, & \text{if } \ x \neq y. \end{cases}$$

Then (Σ_2, ρ) is a compact metric space. A shift map $\sigma : \Sigma_2 \to \Sigma_2$ is defined as follows: $\sigma(x) = (x_2x_3\cdots)$ for any $x = (x_1x_2x_3\cdots) \in \Sigma_2$. Then (Σ_2, σ) is called the one-sided symbolic dynamical system.

Call V a tuple of S, if V is a finite arrangement of some elements of S. If $V = v_1v_2\cdots v_r$, where $v_i \in S$ for $i = 1, \cdots, r$, then we call r the length of V, denoted by $|V|$. Denote by S^* the set of all the tuples of S. Let $W = w_1w_2\cdots w_s$ be another tuple of S, denote

$$VW = v_1v_2\cdots v_rw_1w_2\cdots w_s.$$

Then VW is also a tuple of S. V is said to occur in W, denoted by $V \prec W$, if there is $p \geq 0$ such that $v_q = w_{p+q}$, $q = 1, 2, \cdots, r$. Otherwise, denoted by $V \not\prec W$. Let $x = (x_1x_2x_3\cdots) \in \Sigma_2$ and $V = v_1v_2\cdots v_r$ be a tuple of S. We say that V is a tuple of x, if there exists $i \geq 1$ such that $V = x_ix_{i+1}\cdots x_{i+r-1}$; we say that V occurs in x infinite times, if there exists a positive integer sequence $\{n_i\}_{i=1}^{\infty}$ such that $v_j = x_{n_i+j}$, $j = 1, 2, \cdots, r$ for any $i \geq 1$.

3 Main Results and Proofs

Firstly, we present some properties of WSP.

Proposition 3.1 *Let (X,f) and (Y,g) be two dynamical systems and f, g be semi-conjugate. If (X,f) has WSP, so does (Y,g).*

Proof Let $\phi : X \to Y$ be the semi-conjugate map from f to g. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_1(\phi(x), \phi(y)) < \varepsilon$ for all $x, y \in X$ with $d(x,y) < \delta$. Let N_δ be such a positive integer corresponding to δ as in the definition of WSP. For the above ε, take $N_\varepsilon = N_\delta > 0$. Then for any $y_1, y_2 \in Y$, there exist $x_1, x_2 \in X$ such that $\phi(x_1) = y_1$, $\phi(x_2) = y_2$. For any sequence $0 = j_1 \leq k_1 < j_2 \leq k_2$ with $j_2 - k_1 \geq N_\varepsilon = N_\delta$, there is a point $z \in X$ such that for $m = 1, 2$ and all integers i with $j_m \leq i \leq k_m$,

$$d(f^i(z), f^i(x_m)) < \delta.$$

Let $w = \phi(z)$, then

$$d_1(g^i(w), g^i(y_m)) = d_1(g^i(\phi(z)), g^i(\phi(x_m))) = d_1(\phi(f^i(z)), \phi(f^i(x_m))) < \varepsilon.$$

Thus (Y, g) has WSP.

Proposition 3.2 *Let (X, f) and (Y, g) be two dynamical systems. If (X, f) and (Y, g) have WSP, then $(X \times Y, f \times g)$ has WSP.*

Proof For any $\varepsilon > 0$, let N_1 and N_2 be such positive integers given by WSP of (X, f) and (Y, g), respectively. Take $N_\varepsilon = \max\{N_1, N_2\}$. For any $(x_1, y_1), (x_2, y_2) \in X \times Y$ and any sequence $0 = j_1 \le k_1 < j_2 \le k_2$ with $j_2 - k_1 \ge N_\varepsilon$, there exist $x \in X$, $y \in Y$ such that for each positive integer $m = 1, 2$ and all integers i with $j_m \le i \le k_m$, the following conditions hold:

$$d(f^i(x), f^i(x_m)) < \varepsilon,$$
$$d_1(g^i(y), g^i(y_m)) < \varepsilon.$$

Then,

$$\widetilde{d}((f \times g)^i(x, y), (f \times g)^i(x_m, y_m)) = \max\{d(f^i(x), f^i(x_m)), d_1(g^i(y), g^i(y_m))\} < \varepsilon.$$

Hence $(X \times Y, f \times g)$ has WSP.

Proposition 3.3 *Let (X, f) be a dynamical system. Then (X, f) has WSP if and only if (X, f^n) has WSP for any $n \ge 1$.*

Proof The sufficiency is obvious, we prove the necessity.

For any $\varepsilon > 0$, let N_1 be such a positive integer corresponding to ε as appears in the definition of the WSP of f. Take $N_\varepsilon = \left[\frac{N_1}{n}\right] + 1$, where $\left[\frac{N_1}{n}\right]$ denotes the maximum integer not more than $\frac{N_1}{n}$. For any $x_1, x_2 \in X$ and any sequence $0 = j_1 \le k_1 < j_2 \le k_2$ with $j_2 - k_1 \ge N_\varepsilon$, obviously, $0 = nj_1 \le nk_1 < nj_2 \le nk_2$ and $nj_2 - nk_1 \ge nN_\varepsilon = n\left[\frac{N_1}{n}\right] + n > N_1$; thus, there exists $x \in X$ such that for each positive integer $m = 1, 2$ and all integers i with $nj_m \le i \le nk_m$, the following result holds:

$$d(f^i(x), f^i(x_m)) < \varepsilon.$$

Then

$$d(f^{nj}(x), f^{nj}(x_m)) < \varepsilon$$

for all integers j with $j_m \le j \le k_m$. Therefore (X, f^n) has WSP.

Remark 3.1 Propositions 3.1–3.3 are also true for QWSP and SWSP.

Next we list two lemmas, which are helpful for the proofs of our main results.

Lemma 3.1 (see [8]) *Suppose that there exists $x \in X$ such that $\omega(x, f) = X$, then f is strongly mixing if and only if for any $\varepsilon > 0$ there is $N > 0$ such that*

$$f^n(B(x, \varepsilon)) \cap B(x, \varepsilon) \ne \varnothing$$

for all $n \ge N$.

Lemma 3.2 *Let (X, f) be a dynamical system. If $\Omega(f) = X$, then f is surjective.*

Proof The proof is simple, so we omit it.

As is well known that a dynamical system with WSP is strongly mixing, how about the converse? The following Theorem 3.1 shows that the converse may not be true.

Theorem 3.1 *There exists a sub-shift which is strongly mixing but has no WSP.*

Proof For the sake of convenience, we denote $\overbrace{A\cdots A}^{k \text{ many times}}$ by $\overbrace{A\cdots A}^{k}$, where $A \in S^*$.

Let $D_1 = 1,\ B_1 = 1001$ be two tuples of S, $D_2 = 10$ be the tuple consisting of the first two terms of B_1 and $B_2 = D_1 00 D_1 0000 D_1 D_1 D_1 D_1 0000 D_2 D_2 D_2 D_2$. Let D_3 be the tuple consisting of the first three terms of B_2. For $k \geq 2$, we define B_k and D_k by induction as follows:

$$B_k = B_{k-1}\overbrace{00\cdots 0}^{2k}\overbrace{D_1D_1\cdots D_1}^{2k}\overbrace{00\cdots 0}^{2k}\overbrace{D_2D_2\cdots D_2}^{2k}\cdots\overbrace{00\cdots 0}^{2k}\overbrace{D_kD_k\cdots D_k}^{2k},$$

and D_k is the tuple of S consisting of the first k terms of B_{k-1}.

Let $x = \lim\limits_{k\to\infty}(B_k 00\cdots) \in \Sigma_2$ and $X = \omega(x,\sigma)$. The restriction of σ to X is denoted by σ_1, then $\sigma_1 : X \to X$ and (X,σ_1) is a sub-shift of (Σ_2,σ). To complete the proof of the result, we first prove two claims.

Claim 1 Let $\sigma_1 : X \to X$ be as above, then σ_1 is strongly mixing.

Proof of Claim 1 For any $\varepsilon > 0$, there exists $N > 0$ such that

$$(D_n D_n \cdots) \in B(x,\varepsilon)$$

for any $n > N$. Note that for any $n > N$, $(D_n D_n\cdots) \in X$ and $\sigma_1^n(D_n D_n\cdots) = (D_n D_n \cdots) \in B(x,\varepsilon)$, thus

$$\sigma_1^n(B(x,\varepsilon)) \cap B(x,\varepsilon) \neq \varnothing.$$

By Lemma 3.1, σ_1 is strongly mixing.

Claim 2 Let $\sigma_1 : X \to X$ be as above, then (X,σ_1) has no WSP.

Proof of Claim 2 Let $\varepsilon_0 = \frac{1}{3}$ and choose

$$x_1 = (D_2D_2D_2D_2\cdots),\ x_2 = (D_1D_1D_1D_1\cdots) \in X.$$

For any $N > 0$, take $j_1 = 0,\ k_1 = 4N+3,\ j_2 = 5N+5,\ k_2 = 11N+4$. For any $z \in X$, one of the following conclusions holds:

$$\rho(\sigma^i(z),\sigma^i(x_1)) \geq \frac{1}{3}, \quad \exists i \in [0, 4N+3], \tag{3.1}$$

$$\rho(\sigma^j(z),\sigma^j(x_2)) \geq \frac{1}{3}, \quad \exists j \in [5N+5, 11N+4]. \tag{3.2}$$

Otherwise, we have

$$\rho(\sigma^i(z),\sigma^i(x_1)) < \frac{1}{3}, \quad \forall i \in [0, 4N+3] \tag{3.3}$$

and

$$\rho(\sigma^j(z),\sigma^j(x_2)) < \frac{1}{3}, \quad \forall j \in [5N+5, 11N+4]. \tag{3.4}$$

Then such a z satisfying formulas (3.3)–(3.4) must have the following characteristics:

$$z = (\overbrace{D_2D_2\cdots D_2}^{2N+2}\overbrace{\cdots}^{N+1}\overbrace{D_1D_1\cdots D_1}^{6N}\overbrace{\cdots}^{\cdots}),$$

where $\overbrace{\cdots}^{N+1}$ denotes a tuple of S with length $N+1$, similarly hereinafter. Since $z \in \omega(x,\sigma)$, that is, all the tuples of z must occur in x infinite times. Obviously, $\overbrace{D_2D_2\cdots D_2}^{2N+2}\overbrace{\cdots}^{N+1}\overbrace{D_1D_1\cdots D_1}^{6N}$ is a tuple of z. Thus, $\overbrace{D_2D_2\cdots D_2}^{2N+2}\overbrace{\cdots}^{N+1}\overbrace{D_1D_1\cdots D_1}^{6N}$ must occur in x infinite times. By the construction of x, we know that the tuple $\overbrace{D_2D_2\cdots D_2}^{2N+2}\overbrace{\cdots}^{N+1}\overbrace{D_1D_1\cdots D_1}^{6N}$ can only occur in such tuples as $\overbrace{D_2D_2\cdots D_2}^{2m}\overbrace{00\cdots 0}^{2m}\overbrace{\cdots}^{5m}$ $(m \geq N+1)$ or D_kD_k of x (here k is an enough large positive integer), that is,

$$\overbrace{D_2D_2\cdots D_2}^{2N+2}\overbrace{\cdots}^{N+1}\overbrace{D_1D_1\cdots D_1}^{6N} \prec \overbrace{D_2D_2\cdots D_2}^{2m}\overbrace{00\cdots 0}^{2m}\overbrace{\cdots}^{5m} \tag{3.5}$$

or

$$\overbrace{D_2D_2\cdots D_2}^{2N+2}\overbrace{\cdots}^{N+1}\overbrace{D_1D_1\cdots D_1}^{6N} \prec D_kD_k. \tag{3.6}$$

Next, we discuss the following two cases:

Case 1 We show that it is impossible for the case of (3.5).

Since $2m \geq 2N+2 \geq N+3$, then $\overbrace{D_2D_2\cdots D_2}^{2N+2}\overbrace{\cdots}^{N+1}D_1D_1 \not\prec \overbrace{D_2D_2\cdots D_2}^{2m}\overbrace{00\cdots 0}^{2m}$. That is,

$$\overbrace{D_2D_2\cdots D_2}^{2N+2}\overbrace{\cdots}^{N+1}\overbrace{D_1D_1\cdots D_1}^{6N} \not\prec \overbrace{D_2D_2\cdots D_2}^{2m}\overbrace{00\cdots 0}^{2m}\overbrace{\cdots}^{5m}.$$

Case 2 We show that it is also impossible for the case of (3.6).

If $\overbrace{D_2D_2\cdots D_2}^{2N+2}\overbrace{\cdots}^{N+1}\overbrace{D_1D_1\cdots D_1}^{6N} \prec D_kD_k$, then $\overbrace{D_1D_1\cdots D_1}^{6N-1} \prec D_k$. By the construction of x, one can see that $\overbrace{00\cdots 0}^{6N-1}\overbrace{D_1D_1\cdots D_1}^{6N-1} \prec D_k$. Since $6N-1 > N+3$, then $D_2\overbrace{\cdots}^{N+1}\overbrace{D_1D_1\cdots D_1}^{6N} \not\prec D_kD_k$. Clearly,

$$\overbrace{D_2D_2\cdots D_2}^{2N+2}\overbrace{\cdots}^{N+1}\overbrace{D_1D_1\cdots D_1}^{6N} \not\prec D_kD_k.$$

Summarizing Case 1 and Case 2, we obtain that the tuple $\overbrace{D_2D_2\cdots D_2}^{2N+2}\overbrace{\cdots}^{N+1}\overbrace{D_1D_1\cdots D_1}^{6N}$ can not occur in x. Thus $z \notin \omega(x,\sigma)$, which is contrary to $z \in X = \omega(x,\sigma)$. This contradiction gives that (3.1) or (3.2) is true. Thus, we obtain immediately that (X,σ_1) has no WSP. The proof is ended.

Lemma 3.3 *Let (X,f) be a dynamical system, then (X,f) has QWSP if and only if f is strongly mixing.*

Proof First, we prove the necessity.

Let $U,V \subset X$ be any nonempty open sets, then there exist $x \in U$, $y \in V$ and $\delta_0 > 0$, such that $B(x,\delta_0) \subset U$ and $B(y,\delta_0) \subset V$. Let N_{δ_0} be such a positive integer corresponding to δ_0 as appears in the definition of QWSP. Since f is surjective, for any $n \geq N_{\delta_0}$, there exists $z \in X$ such that $y = f^n(z)$. By the definition of QWSP, there is $r \in X$ such that $d(r,x) < \delta_0$ and

$d(f^n(r), f^n(z)) < \delta_0$. Thus $r \in B(x, \delta_0) \subset U$ and $f^n(r) \in B(y, \delta_0) \subset V$. So $f^n(U) \cap V \neq \varnothing$, which implies that f is strongly mixing.

Next, we prove the sufficiency.

Since f is strongly mixing, $\Omega(f) = X$. By Lemma 3.2, f is surjective.

For any $\delta > 0$, there exists $A = \{x_1, x_2, \cdots, x_m\} \subset X$, such that $X \subset \bigcup\limits_{i=1}^{m} B(x_i, \frac{\delta}{2})$. Since f is strongly mixing, there exists $N_{\frac{\delta}{2}} > 0$ such that $f^n(B(x_i, \frac{\delta}{2})) \cap B(x_j, \frac{\delta}{2}) \neq \varnothing$ for any $i, j \in \{1, 2, \cdots, m\}$ and $n \geq N_{\frac{\delta}{2}}$. For any two points $y_1, y_2 \in X$ and $n \geq N_{\frac{\delta}{2}}$, let $y = f^n(y_2)$, there exist $x_p, x_q \in A$ such that $y_1 \in B(x_p, \frac{\delta}{2}),\ y \in B(x_q, \frac{\delta}{2})$. Obviously, there is $x_0 \in B(x_p, \frac{\delta}{2})$ and $f^n(x_0) \in B(x_q, \frac{\delta}{2})$. Thus,

$$
\begin{aligned}
&d(x_0, y_1) < d(x_0, x_p) + d(x_p, y_1) < \delta,\\
&d(f^n(x_0), f^n(y_2)) = d(f^n(x_0), y) < d(f^n(x_0), x_q) + d(x_q, y) < \delta.
\end{aligned}
$$

Therefore (X, f) has QWSP.

Remark 3.2 By Lemma 3.3 and noting that WSP implies the strongly mixing property, we know that WSP implies QWSP. But Theorem 3.1 gives that the converse is not true.

Lemma 3.4 *Let (X, f) be a dynamical system. If (X, f) has SWSP, then f is strongly mixing.*

Proof Let $U, V \subset X$ be any nonempty open sets. There exist $x_1 \in U,\ x_2 \in V$ and $\varepsilon > 0$, such that $B(x_1, \varepsilon) \subset U$ and $B(x_2, \varepsilon) \subset V$. Note the continuity of f, for the above ε, there exists $0 < \delta < \varepsilon$ such that $d(f(x), f(y)) < \varepsilon$ for all $x, y \in X$ with $d(x, y) < \delta$. Let N_δ be such a positive integer corresponding to δ as appears in the definition of SWSP. Take $j_1 = 0,\ k_1 = 1,\ j_2 = N_\delta + 2,\ k_2 = N_\delta + 3$. Since f is surjective, there are $y_1, y_2 \in X$ such that $x_1 = f(y_1),\ x_2 = f^{N_\delta+3}(y_2)$. By the definition of SWSP, we consider the following four cases:

Case 1 If there exists x such that $d(x, y_1) < \delta$ and $d(f^{N_\delta+2}(x), f^{N_\delta+2}(y_2)) < \delta$, then by the continuity of f,

$$
d(f(x), f(y_1)) < \varepsilon, \quad d(f^{N_\delta+3}(x), f^{N_\delta+3}(y_2)) < \varepsilon.
$$

Obviously, $f(x) \in B(x_1, \varepsilon) \subset U$ and $f^{N_\delta+3}(x) \subset B(x_2, \varepsilon) \subset V$. Thus $f^{N_\delta+2}(U) \cap V \neq \varnothing$.

Case 2 If there exists x such that $d(x, y_1) < \delta$ and $d(f^{N_\delta+3}(x), f^{N_\delta+3}(y_2)) < \delta < \varepsilon$, then

$$
d(f(x), f(y_1)) < \varepsilon, \quad d(f^{N_\delta+3}(x), f^{N_\delta+3}(y_2)) < \delta < \varepsilon.
$$

Clearly, $f(x) \in B(x_1, \varepsilon) \subset U$ and $f^{N_\delta+3}(x) \subset B(x_2, \varepsilon) \subset V$. Hence $f^{N_\delta+2}(U) \cap V \neq \varnothing$.

Case 3 If there exists x such that $d(f(x), f(y_1)) < \delta < \varepsilon$ and $d(f^{N_\delta+2}(x), f^{N_\delta+2}(y_2)) < \delta$, then

$$
d(f(x), f(y_1)) < \delta < \varepsilon, \quad d(f^{N_\delta+3}(x), f^{N_\delta+3}(y_2)) < \varepsilon.
$$

Evidently, $f(x) \in B(x_1, \varepsilon) \subset U$ and $f^{N_\delta+3}(x) \subset B(x_2, \varepsilon) \subset V$. So $f^{N_\delta+2}(U) \cap V \neq \varnothing$.

Case 4 If there exists x such that

$$
d(f(x), f(y_1)) < \delta < \varepsilon, \quad d(f^{N_\delta+3}(x), f^{N_\delta+3}(y_2)) < \delta < \varepsilon.
$$

Distinctly, $f(x) \in B(x_1, \varepsilon) \subset U$ and $f^{N_\delta+3}(x) \subset B(x_2, \varepsilon) \subset V$. Therefore $f^{N_\delta+2}(U) \cap V \neq \varnothing$.

Summarizing all the cases, we get immediately that $f^{N_\delta+2}(U) \cap V \neq \varnothing$ for $j_1 = 0,\ k_1 = 1,\ j_2 = N_\delta + 2,\ k_2 = N_\delta + 3$. Then by a similar method, we can prove that $f^{N_\delta+3}(U) \cap V \neq \varnothing$ for $j_1 = 0,\ k_1 = 1,\ j_2 = N_\delta + 3,\ k_2 = N_\delta + 4$, and the like $f^n(U) \cap V \neq \varnothing$ for any $n \geq N_\delta + 2$. Therefore f is strongly mixing.

Theorem 3.2 *Let (X, f) be a dynamical system, then the following statements are equivalent:*

(1) *f is strongly mixing;*

(2) *(X, f) has QWSP;*

(3) *(X, f) has SWSP.*

Proof By Lemma 3.3, (X, f) has QWSP if and only if f is strongly mixing, and by Lemma 3.4, we have that if (X, f) has SWSP, then f is strongly mixing. Thus, we only need to prove that (X, f) has SWSP if (X, f) has QWSP.

Suppose that (X, f) has QWSP. For any $\delta > 0$, let N_δ be such a positive integer corresponding to δ as appears in the definition of QWSP. For any two points y_1, y_2 and any sequence $0 \leq j_1 < k_1 < j_2 < k_2$ with $j_2 - k_1 \geq N_\delta$, by the surjective property of f, there exist y_3, y_4 such that $y_3 = f^{j_1}(y_1),\ y_4 = f^{j_1}(y_2)$. Let $n = j_2 - j_1 > j_2 - k_1 > N_\delta$. Since (X, f) has QWSP, there exists $z \in X$ such that $d(z, y_3) < \delta$ and $d(f^n(z), f^n(y_4)) < \delta$. Note that f is surjective, then there is $z_1 \in X$ such that $z = f^{j_1}(z_1)$. Thus

$$\begin{aligned} &d(f^{j_1}(z_1), f^{j_1}(y_1)) < \delta, \\ &d(f^{n+j_1}(z_1), f^{n+j_1}(y_2)) = d(f^{j_2}(z_1), f^{j_2}(y_2)) < \delta. \end{aligned}$$

Hence (X, f) has SWSP.

Remark 3.3 By the main results of this paper, one can deduce that both QWSP and SWSP are strictly weaker than WSP.

Remark 3.4 In particular, WSP, QWSP, SWSP and strongly mixing are equivalent for the case of interval maps. And we believe that WSP, QWSP, SWSP and strongly mixing are equivalent for the case of sub-shifts of finite type.

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