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On λ-Power Distributional n-Chaos*

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Abstract For each real number $\lambda \in [0, 1]$, λ-power distributional chaos has been introduced and studied via Furstenberg families recently. The chaoticity gets stronger and stronger as λ varies from 1 to 0, where 1-power distributional chaos is exactly the usual distributional chaos. As a generalization of distributional n-chaos, λ-power distributional n-chaos is defined similarly. Lots of classic results on distributional chaos can be improved to be the versions of λ-power distributional n-chaos accordingly. A practical method for distinguishing 0-power distributional n-chaos is given. A transitive system is constructed to be 0-power distributionally n-chaotic but without any distributionally $(n+1)$-scrambled tuples. For each $\lambda \in [0, 1]$, λ-power distributional n-chaos can still appear in minimal systems with zero topological entropy.

Keywords Furstenberg family, λ-power distributional n-chaos, Minimal system, Topological entropy

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1 Introduction

By a dynamical system (in short, a system) we mean a pair (X, f), where X is a compact metric space with metric d and $f : X \to X$ is a continuous map.

As far as we know, the first topological definition of chaos was introduced by Li and Yorke [6] to describe the complexity of the orbits of points in a system. The Li-Yorke chaos became one of the most discussed topics for the last several decades. Various extensions of Li-Yorke chaos were developed. In 1994, Schweizer B. and Smítal J. [9] introduced distributional chaos and showed that positive topological entropy is equivalent to distributional chaos for interval self-maps. But the equivalence is no longer valid when a general compact metric space is considered (see [4, 7]).

In 2007, Li-Yorke chaos and distributional chaos have been unified by Xiong et al [14] into the frame of $\mathcal{F}$-chaos, where $\mathcal{F}$ is a Furstenberg family. Recently, via $\mathcal{F}$-chaos, Xiong et al [13] have introduced the notion of λ-power distributional chaos to strengthen distributional chaos, where $\lambda \in [0, 1]$. The hierarchical relation of these chaos is intensively discussed in [2].

The above definitions of chaos are expressed in terms of dynamics of pairs. Some authors have realized that notions of chaos can also be stated by means of dynamics of tuples, e.g.,

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n-scrambled tuples (see [5, 12]). Following this idea, we extended distributional chaos to distributional n-chaos for $n \geq 2$ (see [10]). There we discussed several properties of distributional n-chaos and constructed a transitive system which is distributionally n-chaotic without any distributionally $(n+1)$-scrambled tuples.

In this paper, we introduce the notion of $\mathcal{F}$-n-chaos generally for a given Furstenberg family $\mathcal{F}$ and $n \geq 2$. Then we apply $\mathcal{F}$-n-chaos to the study of λ-power distributional n-chaos. Our main aim is to extend some classic results on distributional chaos to be the versions of λ-power distributional n-chaos. This paper is organized as follows. Section 2 is devoted to preliminaries on Furstenberg families and on topological dynamics. In Section 3, for $\lambda \in [0, 1]$, λ-power distributional n-chaos is introduced as a generalization of distributional n-chaos via Furstenberg families, where 0-power distributional n-chaos is the strongest. Then we provide a simple criterion for a system to be 0-power distributionally n-chaotic. In Section 4, we present a transitive system which is 0-power distributionally n-chaotic without any distributionally $(n+1)$-scrambled tuples. Finally in Section 5, for each $\lambda \in [0, 1]$, we show that λ-power distributional n-chaos may exist in minimal systems with zero topological entropy.

2 Preliminaries

2.1 Furstenberg families

We review some notations related to Furstenberg families (see [1]). Denote by $\mathbb{Z}_+, \mathbb{N}$ the set of positive integers and the set of non-negative integers respectively. Denote by $\mathcal{P}$ the collection of all subsets of $\mathbb{Z}_+$.

$\mathcal{F} \subset \mathcal{P}$ is called a Furstenberg family, if it is hereditary upwards, that is, $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. Obviously, the family of all infinite subsets of $\mathbb{Z}_+$ is a Furstenberg family, denoted by $\mathcal{B}$.

For a family $\mathcal{F}$, denote

$$\kappa\mathcal{F} = \{F \in \mathcal{P} : \mathbb{Z}_+ - F \notin \mathcal{F}\} = \{F \in \mathcal{P} : F \cap F' \neq \varnothing \text{ for all } F' \in \mathcal{F}\}.$$

$\kappa\mathcal{F}$ is a Furstenber family, called the dual family of $\mathcal{F}$. It is easy to see that $\kappa\mathcal{B}$ is the family of cofinite subsets.

A subset F of $\mathbb{Z}_+$ is thick if it contains arbitrarily long runs of positive integers. The family of all thick subset of $\mathbb{Z}_+$ is denoted by $\tau\mathcal{B}$. The set in $\kappa\tau\mathcal{B}$ is said to be syndetic. So $F \subset \mathbb{Z}_+$ is syndetic if and only if it is of bounded gaps, i.e., there is N such that $\{i, i+1, \cdots, i+N\} \cap F \neq \varnothing$ for every $i \in \mathbb{Z}_+$.

Let $J \subset \mathbb{Z}_+$. Define

$$\overline{\mu}(J) = \limsup_{n\to\infty} \frac{\#(J \cap \{1, \cdots, n\})}{n}$$

and

$$BD^*(J) = \limsup_{\#I\to\infty} \frac{\#(J \cap I)}{\#I},$$

where I ranges over intervals of $\mathbb{Z}_+$, $\overline{\mu}(J)$ and $BD^*(J)$ are said to be the upper density of J and the upper Banach density of J, respectively. The lower density $\underline{\mu}(J)$ and the lower Banach density $BD_*(J)$ are defined similarly.

For every $t \in [0,1]$, let $\overline{\mathcal{M}}(t) = \{F \in \mathcal{B} : \overline{\mu}(F) \geq t\}$. Obviously, all $\overline{\mathcal{M}}(t)$ are Furstenberg families and $\overline{\mathcal{M}}(0) = \mathcal{B}$.

2.2 Topological dynamics

$\overline{A}$ denotes the closure of the set A in X. For given $\delta > 0$, let $[A]_\delta = \{x \in X : d(x, A) < \delta\}$, where $d(x, A) = \inf\{d(x, y) : y \in A\}$.

Suppose that (X, f) is a system. $A \subset X$ is invariant if $f(A) \subset A$. (Y, f) is a subsystem of (X, f) if $Y \subset X$ is nonempty, closed and invariant. For $U, V \subset X$, $N(U, V) = \{n \in \mathbb{Z}_+ : U \cap f^{-n}V \neq \varnothing\}$ is called the meeting time set of U and V. Specially, if U is a singleton $\{x\}$, $N(\{x\}, V)$ is simply written as $N(x, V)$, called the return time set from x to V. (X, f) or f is said to be transitive if $N(U, V) \neq \varnothing$ for any two nonempty open sets $U, V \subset X$. We write $\mathrm{Orb}(x, f) = \{x, f(x), f^2(x), \cdots\}$ and call it the orbit of x. $x \in X$ is said to be a recurrent point if x is a limit point of the set $\mathrm{Orb}(x, f)$. Clearly, if x is a recurrent point, then $(\overline{\mathrm{Orb}(x, f)}, f)$ is a transitive system.

(X, f) or f is said to be minimal if there is no proper subsystem of (X, f). If a subsystem (Y, f) of (X, f) is minimal, then we say that Y is a minimal set of X. Each point in a minimal set is called a minimal point. It is well known that $x \in X$ is a minimal point if and only if $N(x, U)$ is syndetic for any neighborhood U of x.

For a finite open cover $\mathcal{U}$ of X define

$$h(f, \mathcal{U}) = \limsup_{k \to \infty} \frac{1}{k} \log N\Big(\bigvee_{i=0}^{k-1} f^k(\mathcal{U})\Big),$$

where $N(\mathcal{C})$ is the minimal cardinality among all cardinalities of subcovers of $\mathcal{C}$. The topological entropy of (X, f) is $h(f) = \sup\limits_{\mathcal{U}} h(f, \mathcal{U})$, where the supremum is taken over all finite open covers of X.

Consider the set $E = \{0, 1, \cdots, n-1\}$, $n \geq 2$, endowed with the discrete topology. Let $\Sigma_n = \prod\limits_{i=1}^{\infty} E = \{x \mid x = x_1 x_2 \cdots, x_i \in E,\ i = 1, 2, \cdots\}$ with the product topology. Then Σ_n is a compact metric space, called a symbolic space (on n symbols). A compatible metric on Σ_n is given by $d(x, y) = \frac{1}{k}$, where $k = \min\{i \mid x_i \neq y_i,\ i = 1, 2, \cdots\}$ for any $x = x_1 x_2 \cdots$, $y = y_1 y_2 \cdots \in \Sigma_n$ with $x \neq y$, otherwise 0 when $x = y$. Define $\sigma : \Sigma_n \to \Sigma_n$ by $\sigma(x) = x_2 x_3 \cdots$ for any $x = x_1 x_2 \cdots \in \Sigma_n$. It is obvious that σ is continuous. (Σ_n, σ) is called the full shift (on n symbols). Any subsystem of (Σ_n, σ) is called a subshift.

Each $A \in \bigcup\limits_{k=1}^{\infty} E^k$ is called a word over E, where $E^k = \{x_1 x_2 \cdots x_k \mid x_i \in E,\ 1 \leq i \leq k\}$ is the set of all k-words. If $A = a_1 a_2 \cdots a_l$ is an l-word, then we call that the length of A is l, denoted $|A| = l$. If $B = b_1 b_2 \cdots b_m$ is an m-word, the catenation of A and B is denoted by $AB = a_1 \cdots a_l b_1 \cdots b_m$. Then AB is an $(l+m)$-word. If $A_1, A_2, \cdots$ is a sequence of words, then $A_1 A_2 \cdots$ is regarded as a point of Σ_n. For simplicity, denote $A \cdots A$ (k times), $AA \cdots$ by $A^{(k)}$ and A^∞ respectively. We say that A occurs in B, denoted $A \prec B$, if there is $i \geq 0$ such that $a_j = b_{i+j}$ holds for each $j = 1, 2, \cdots, l$. For a point $x \in \Sigma_n$, we get the definition of $A \prec x$ similarly and say that A occurs in x.

Lemma 2.1 comes from [4].

Lemma 2.1 *Let $x = x_1x_2\cdots \in \Sigma_n$. If for any $k \geq 1$, there exists $K > 0$ such that $x_1x_2\cdots x_k \prec x_ix_{i+1}\cdots x_{i+K}$ holds for each $i \geq 1$, then x is a minimal point of σ.*

Let $Y \subset \Sigma_n$. For $k \geq 1$, denote

$$Q_k(Y) = \{A \in E^k : \text{ there is } x \in Y \text{ such that } A \prec x\}.$$

The following Lemma 2.2 is well known, for example see [11].

Lemma 2.2 *Suppose that $\sigma|_Y : Y \to Y$ is a subshift. Then*

$$h(\sigma|_Y) = \lim_{k\to\infty} \frac{\log \#Q_k(Y)}{k}.$$

3 λ-Power Distributional n-Chaos

Suppose that (X, f) is a system and $\mathcal{F}$ is a Furstenberg family.

Let $A \subset X$ and $\delta > 0$. $x \in X$ is said to be an $\mathcal{F}$-attaching point of A if $N(x, A) \in \mathcal{F}$; an $\mathcal{F}$-adherent point of A if x is an $\mathcal{F}$-attaching point of $[A]_\varepsilon$ for any $\varepsilon > 0$; an $\mathcal{F}$-δ-escape point of A if x is an $\mathcal{F}$-attaching point of the set $X - \overline{[A]_\delta}$; an $\mathcal{F}$-escape point of A if x is an $\mathcal{F}$-δ'-escape point of A for some $\delta' > 0$.

3.1 $\mathcal{F}$-n-chaos

Let $n \geq 2$. Similar to the definition of $\mathcal{F}$-chaos in [14], we introduce $\mathcal{F}$-n-chaos.

Denote by $(X^n, f^{(n)})$ the n-fold product system $(X \times X \times \cdots \times X, f \times f \times \cdots \times f)$. Put $\Delta_n = \{(x, x, \cdots, x) \in X^n : x \in X\}$ and $\Delta^{(n)} = \{(x_1, x_2, \cdots, x_n) \in X^n : x_i = x_j \text{ for some } i \neq j\}$.

Let $\delta > 0$. A tuple $\widetilde{x} \in X^n \setminus \Delta^{(n)}$ is said to be $\mathcal{F}$-δ-n-scrambled if $\widetilde{x}$ is an $\mathcal{F}$-adherent point of Δ_n and an $\mathcal{F}$-δ-escape point of $\Delta^{(n)}$ in the product system $(X^n, f^{(n)})$. A subset C of X is said to be $\mathcal{F}$-δ-n-scrambled if each tuple $\widetilde{x} \in C^n \setminus \Delta^{(n)}$ is $\mathcal{F}$-δ-n-scrambled. A system (X, f) is said to be uniformly $\mathcal{F}$-n-chaotic if there exists an uncountable $\mathcal{F}$-δ'-n-scrambled set for some $\delta' > 0$.

In the same manner, we get the definitions of $\mathcal{F}$-n-scrambled tuples, $\mathcal{F}$-n-scrambled sets and $\mathcal{F}$-n-chaos.

3.2 λ-power distributional n-chaos

For $F \subset \mathbb{Z}_+$, denote $F^c = \mathbb{Z}_+ - F$. Note that

$$\begin{aligned}\overline{\mathcal{M}}(1) &= \Big\{F \in \mathcal{B} : \limsup_{n\to\infty} \frac{\#(F \cap \{1, \cdots, n\})}{n} = 1\Big\}\\ &= \Big\{F \in \mathcal{B} : \liminf_{n\to\infty} \frac{\#(F^c \cap \{1, \cdots, n\})}{n} = 0\Big\}.\end{aligned}$$

In [13], Xiong et al introduced a new class of Furstenberg families as follows: For each $\lambda \in (0, 1]$, put

$$\mathcal{D}_\lambda = \Big\{F \in \mathcal{B} : \liminf_{n\to\infty} \frac{\#(F^c \cap \{1, \cdots, n\})}{n^\lambda} = 0\Big\}, \quad \mathcal{D}_0 = \bigcap_{\lambda\in(0,1]} \mathcal{D}_\lambda.$$

Then $\mathcal{D}_\lambda$ is a Furstenberg family for each $\lambda \in [0,1]$. It is easy to see that $\mathcal{D}_1 = \overline{\mathcal{M}}(1) \subset \tau\mathcal{B}$ and that $\mathcal{D}_{\lambda_1} \subset \mathcal{D}_{\lambda_2}$ for any $0 \le \lambda_1 \le \lambda_2 \le 1$.

Recall that a system is distributionally chaotic if and only if it is $\mathcal{D}_1$-chaotic (see [14]), which inspires us to introduce the following intuitive synonyms.

Let $\lambda \in [0,1]$, $n \ge 2$ and $\delta > 0$. By a λ-power distributionally δ-n-scrambled tuple (or set), we mean a $\mathcal{D}_\lambda$-δ-n-scrambled tuple (or set respectively). Likewise, (uniformly) λ-power distributionally n-chaotic systems means (uniformly) $\mathcal{D}_\lambda$-n-chaotic systems.

Surely, 1-power distributional n-chaos is just distributional n-chaos defined in [10]. λ-power distributional n-chaos gets stronger and stronger as λ varies from 1 to 0. In [2], Fu et al constructed examples to demonstrate that λ_1-power distributional chaos and λ_2-power distributional chaos are not equivalent for any different $\lambda_1, \lambda_2 \in [0,1]$. The examples there, in fact, also show that λ_1-power distributional n-chaos and λ_2-power distributional n-chaos are not equivalent for any different $\lambda_1, \lambda_2 \in [0,1]$. So all of λ-power distributional n-chaos, where $\lambda \in [0,1]$, form a hierarchy of distributional n-chaos.

Corollary 4.4 in [13] has offered a criterion for a system to be uniformly 0-power distributionally chaotic. It can be modified slightly into a version for uniformly 0-power distributionally n-chaotic systems. For proof of Theorem 4.1 here, we merely present a simplified and practical criterion as follows.

Proposition 3.1 *Suppose that (X, f) is a system and $n \ge 2$. If f has n distrinct fixed points p_j, $j = 0, 1, \cdots, n-1$, such that $\bigcup\limits_{i=1}^{\infty} f^{-i}(p_j)$ is dense in X for each j, then (X, f) is uniformly 0-power distributionally n-chaotic.*

In fact, Proposition 3.1 implies that (X, f) is generically uniformly 0-power distributionally n-chaotic, that is, the set of 0-power distributionally δ-n-scrambled tuples is residual in X^n for some fixed $\delta > 0$.

We take the most known system (Σ_n, σ) as an example. Note that all of $\overline{j} = jj\cdots \in \Sigma_n$, $0 \le j \le n-1$ are fixed points of σ and they satisfy the condition stated in Proposition 3.1. Consequently, we give the following example.

Example 3.1 The full shift (Σ_n, σ) is uniformly 0-power distributionally n-chaotic.

4 0-Power Distributionally n-Chaotic Systems Without Distributionally $(n+1)$-Scrambled Tuples

For $n \ge 2$, we have constructed a transitive system which is distributionally n-chaotic without any distributionally $(n+1)$-scrambled tuples (see [10]). Analogously, a transitive system is constructed, which is 0-power distributionally n-chaotic without any distributionally $(n+1)$-scrambled tuples. Before doing it, we need some preparations.

4.1 Technical preparations

The following Lemma 4.1 comes from [1, 8].

Lemma 4.1 *Suppose that (X, f) is a system, and $A \subset X$ is nonempty and closed. Then $x \in X$ is a $\tau\mathcal{B}$-adherent point of A if and only if A contains a minimal set of $\overline{\mathrm{Orb}(x, f)}$.*

Proposition 4.1 *Suppose that (X, f) is a system and $n \geq 2$. If there is a $\tau\mathcal{B}$-escape point of $\Delta^{(n)}$ in the product system $(X^n, f^{(n)})$, then (X, f) has at least n distrinct minimal points.*

Proof Let $\widetilde{x} \in X^n$ be a $\tau\mathcal{B}$-escape point of $\Delta^{(n)}$. Then $\widetilde{x}$ is a $\tau\mathcal{B}$-attaching point of the set $X^n - \overline{[\Delta^{(n)}]_\delta}$ for some $\delta > 0$. By Lemma 4.1, $X^n - [\Delta^{(n)}]_\delta$ contains a minimal point of $(X^n, f^{(n)})$, say $\widetilde{y} = (y_1, \cdots, y_n)$. Therefore, $y_1, \cdots, y_n$ are n distrinct minimal points of (X, f).

Corollary 4.1 *Suppose that (X, f) is a system and $n \geq 2$. If (X, f) has a distributionally n-scrambled tuple, then (X, f) has at least n distrinct minimal points.*

Proof Let $\widetilde{x}$ be a distributionally n-scrambled tuple. It follows that $\widetilde{x}$ is a $\mathcal{D}_1$-escape point of $\Delta^{(n)}$ in the product system $(X^n, f^{(n)})$. Since $\mathcal{D}_1 \subset \tau\mathcal{B}$, the corollary holds by Proposition 4.1.

Corollary 4.1 tells us, if a system does not admit n distrinct minimal points, then it has no distributionally n-scrambled tuple. Needless to say, it is not distributionally n-chaotic.

4.2 Construction of examples

Suppose $n \geq 2$. Below, we define a sequence of words $\{A_k\}_{k \in \mathbb{Z}_+}$ inductively.

Let $A_1 = 1$. For $k \geq 2$, suppose that A_{k-1} is defined, and denote $l_{k-1} = |A_{k-1}|$, the length of A_{k-1}. If $k = ni + j$, where $i \in \mathbb{N},\ 0 \leq j \leq n-1$, define

$$A_k = A_{k-1} j^{(l_{k-1}^2)} A_{k-1}.$$

Let $x = \lim\limits_{k \to \infty} A_k 000 \cdots$, and $X = \overline{\mathrm{Orb}(x, \sigma)}$.

Proposition 4.2 *The point x defined above is a recurrent point of σ. Therefore, the subshift (X, σ) is transitive.*

By the definition of A_k, it is easy to see that $l_1 = 1,\ l_k = 2l_{k-1} + l_{k-1}^2,\ k \geq 2$.

Proposition 4.3 (X, σ) *processes the following properties:*

(1) *For each $k \in \mathbb{Z}_+$, x can be expressed as*

$$x = A_k j_0^{(l_k^2)} A_k j_1^{(l_{k+1}^2)} A_k j_0^{(l_k^2)} A_k j_2^{(l_{k+2}^2)} A_k j_0^{(l_k^2)} A_k j_1^{(l_{k+1}^2)} A_k j_0^{(l_k^2)} A_k \cdots,$$

where $0 \leq j_0 \leq n-1$, $j_p = (j_0 + p)_{\bmod n},\ p \in \mathbb{Z}_+$.

(2) $\lim\limits_{k \to \infty} \frac{l_k^2}{l_{k+1}} = 1$, *or equivalently,* $\lim\limits_{k \to \infty} \frac{l_k}{l_{k+1}} = 0$.

Suppose that $A = a_1 \cdots a_l$ is a word with length l. For $k \leq l$, put

$$r_k(A) = \#\{i \mid a_i \cdots a_{i+k-1} = j^{(k)} \text{ for some } 0 \leq j \leq n-1\}.$$

Obviously, $\overline{j} = jj \cdots \in X,\ 0 \leq j \leq n-1$, are all fixed points of σ.

Put $\underline{\mathcal{M}}_{BD}(1) = \{F \in \mathcal{P} : BD_*(F) = 1\}$. It is verified that $\underline{\mathcal{M}}_{BD}(1) \subset \kappa\tau\mathcal{B}$.

Proposition 4.4 *In the system (X, σ), x is an $\underline{\mathcal{M}}_{BD}(1)$-adherent point of $\{\overline{j},\ 0 \leq j \leq n-1\}$.*

Proof Suppose that U is an arbitrary neighborhood of $\{\overline{j},\ 0 \leq j \leq n-1\}$ in X. Without loss of generality, assume that $U = \bigcup\limits_{j=0}^{n-1} U_j$, where $U_j = B(\overline{j}, \frac{1}{p})$ for some $p \in \mathbb{Z}_+$.

For any given $M \in \mathbb{Z}_+$, there exists m such that

$$l_m + l_m^2 \le M < l_{m+1} \text{ or } l_{m+1} \le M < l_{m+1} + l_{m+1}^2.$$

By Proposition 4.3(1), when $l_m + l_m^2 \le M < l_{m+1}$, we have

$$\frac{r_k(x_{i+1} \cdots x_{i+M})}{M} \ge \frac{l_m^2 - p}{l_{m+1}} = 1 - \frac{2l_m + p}{l_{m+1}}.$$

Otherwise,

$$\begin{aligned}\frac{r_k(x_{i+1} \cdots x_{i+M})}{M} &\ge \frac{l_m^2 + M - l_{m+1} - 2p}{M} \\ &= 1 - \frac{l_{m+1} - l_m^2 + 2p}{M} \\ &= 1 - \frac{2l_m + 2p}{M} \\ &\ge 1 - \frac{2l_m + 2p}{l_{m+1}}.\end{aligned}$$

Note that $M \to \infty$ implies $m \to \infty$. Thus, according to Proposition 4.3(2),

$$\lim_{M\to\infty} \frac{r_k(x_{i+1} \cdots x_{i+M})}{M} = 1.$$

Since M and i are arbitrary, x is an $\underline{\mathcal{M}}_{BD}(1)$-adherent point of $\{\overline{j},\ 0 \le j \le n-1\}$.

Proposition 4.5 *The subshift (X, σ) has exactly n distrinct minimal points, which are the fixed points $\overline{j},\ 0 \le j \le n-1$.*

Proof Firstly, these fixed points $\overline{j},\ 0 \le j \le n-1$, are all minimal points of (X, σ).

Next, we want to show that any minimal point of (X, σ) is just one of $\Lambda = \{\overline{j},\ 0 \le j \le n-1\}$. If not, assume that there is another distrinct minimal point ω. Then $\overline{\mathrm{Orb}(\omega, \sigma)} \cap \Lambda = \varnothing$. Let U, V be two disjoint neighborhoods of Λ and of $\overline{\mathrm{Orb}(\omega, \sigma)}$ respectively. By Lemma 4.1 and Proposition 4.4, we have

$$N(x, U) \in \underline{\mathcal{M}}_{BD}(1), \quad N(x, V) \in \tau\mathcal{B}.$$

Since $\underline{\mathcal{M}}_{BD}(1) \subset \kappa\tau\mathcal{B}$, it follows that $N(x, U) \cap N(x, V) \ne \varnothing$, which contradicts $U \cap V \ne \varnothing$.

Theorem 4.1 *The subshift (X, σ) is uniformly 0-power distributionally n-chaotic, without any distributionally $(n+1)$-scrambled tuple. Of course, (X, σ) is not 0-power distributionally $(n+1)$-chaotic.*

Proof For each word $A = a_1 a_2 \cdots a_l \prec x$, there is k such that $A \prec A_k$. For any $0 \le j \le n-1$ and any $m \in \mathbb{Z}_+$, it follows from Proposition 4.3(1) that $A_k j^{(m)} \prec x$. Thus, there is a word B such that $ABj^{(m)} \prec x$. It follows that $\bigcup\limits_{k=1}^{\infty} \sigma^{-k}(\overline{j})$ is dense in X for each fixed point $\overline{j},\ 0 \le j \le n-1$. By Proposition 3.1, (X, σ) is uniformly 0-power distributionally n-chaotic.

However, according to Proposition 4.5 and Corollary 4.1, (X, σ) has no distributionally $(n+1)$-scrambled tuple.

5 Minimal Systems

Liao et al [4] constructed in a symbolic space a minimal and distributionally chaotic system with topological entropy 0. Oprocha [7] obtained an uncountable family of such systems in a symbolic space. Following the ideas in [4, 7], for each $\lambda \in [0,1]$ and $n \geq 2$, we construct in a symbolic space a uniformly λ-power distributionally n-chaotic and minimal system with zero topological entropy.

Let $n \geq 2$ and $t = (t_1, t_2, \cdots) \in \mathbb{Z}_+^{\mathbb{Z}_+}$ be any given sequence of positive integer. Recall that $E = \{0, 1, \cdots, n-1\}$. Below, we will define a sequence of words over E. Firstly, define a map $\eta : E \to E$ by $\eta(i) = (i+1)_{\mathrm{mod}\, n}$. We can extend η naturally to be defined on any word $A \in \bigcup\limits_{m=1}^{\infty} E^m$ or on any point $x \in \Sigma_n$.

Then, let A_{10} be a given word over E. For each $1 \leq j \leq i-1$, put $A_{1j} = \eta^j(A_{10})$. Assume that for some $k \in \mathbb{Z}_+$, words $A_{ij}, 1 \leq i \leq k,\ 0 \leq j \leq n-1$ are all already defined. Let

$$J_{ij} = A_{ij}^{(t_i)}.$$

We define a set $P_k \subset \bigcup\limits_{m=1}^{\infty} E^m$ as follows:

$$P_k = \{J_{1a_1} J_{2a_2} \cdots J_{ka_k} : a_1 a_2 \cdots a_k \in E^k\}.$$

We enumerate elements of P_k, say $P_k = \{w_1, w_2, \cdots, w_{n^k}\}$. Put

$$A_{(k+1)0} = w_1 w_2 \cdots w_{n^k},$$

i.e., $A_{(k+1)0}$ is a permutation of elements of P_k. For each $1 \leq j \leq n-1$, put

$$A_{(k+1)j} = \eta^j(A_{(k+1)0}).$$

Define $x^{(t)} = J_{10} J_{20} \cdots$ and $M^{(t)} = \overline{\mathrm{Orb}(x^{(t)}, \sigma)}$. Denote $s_k^{(t)} = |A_{k0}|,\ r_k^{(t)} = |J_{10} J_{20} \cdots J_{k0}| = \sum\limits_{i=1}^{k} t_i s_i^{(t)}$.

Proposition 5.1 *For any $t = (t_1, t_2, \cdots) \in \mathbb{Z}_+^{\mathbb{Z}_+}$ and any $k \in \mathbb{Z}_+$, $x^{(t)}$, $s_k^{(t)}$ and $r_k^{(t)}$ are define as above. Then*

(1) $s_{k+1}^{(t)} = n^k r_k^{(t)}$.

(2) *$x^{(t)}$ is an infinite catenation of elements of P_k.*

Proposition 5.2 *For any $t \in \mathbb{Z}_+^{\mathbb{Z}_+}$, $x^{(t)}$ is a minimal point of σ. Namely, $(M^{(t)}, \sigma)$ is a minimal subshift.*

Proof Suppose $x^{(t)} = x_1 x_2 \cdots$. For each $k \geq 1$, $x_1 x_2 \cdots x_k \prec J_{10} J_{20} \cdots J_{k0}$. By the definition of $A_{(k+1)0}$, for each $0 \leq j \leq n-1$, one has

$$J_{1j} J_{2j} \cdots J_{kj} \prec A_{(k+1)0}.$$

Under the action of η^{n-j}, it follows that

$$J_{10} J_{20} \cdots J_{k0} \prec A_{(k+1)(n-j)}.$$

Consequently, whenever $J_{1a_1}J_{2a_2}\cdots J_{(k+1)a_{k+1}} \in P_{k+1}$, where $a_1a_2\cdots a_{k+1} \in E^{k+1}$, one has

$$J_{10}J_{20}\cdots J_{k0} \prec J_{(k+1)a_{k+1}} \prec J_{1a_1}J_{2a_2}\cdots J_{(k+1)a_{k+1}}.$$

Then, for given $k \geq 1$, we choose $K = 2r^{(t)}_{k+1}$. By Proposition 5.1, $x^{(t)}$ is an infinite catenation of elements of P_{k+1}. Therefore, for any $i \in \mathbb{Z}_+$, there is some $J_{1a_1}J_{2a_2}\cdots J_{(k+1)a_{k+1}} \in P_{k+1}$ occurring in $x_ix_{i+1}\cdots x_{i+K}$, that is,

$$J_{1a_1}J_{2a_2}\cdots J_{(k+1)a_{k+1}} \prec x_ix_{i+1}\cdots x_{i+K}.$$

This is because the length of any word in P_{k+1} is $r^{(t)}_{k+1}$. So

$$x_1x_2\cdots x_k \prec x_ix_{i+1}\cdots x_{i+K}.$$

By Lemma 2.1, $x^{(t)}$ is a minimal point of σ.

Proposition 5.3 *For any $t \in \mathbb{Z}_+^{\mathbb{Z}_+}$, the topological entropy of $(M^{(t)}, \sigma)$ is* 0.

Proof Every point $x \in M^{(t)}$ is an infinite catenation of words in P_k, and all words in P_k are of the same length $l_k = r^{(t)}_k$. So each word with length l_k, occurring in x, must be a subword of some uw, where $u, w \in P_k$. This implies that

$$\#Q_{l_k}(M^{(t)}) \leq l_k(\#P_k)^2 = l_k(n^k)^2 = n^{2k}l_k.$$

By Lemma 2.2, one has

$$h(\sigma|_{M^{(t)}}) = \lim_{k\to\infty} \frac{\log \#Q_{l_k}(M^{(t)})}{l_k} \leq \lim_{k\to\infty}\Big(\frac{\log l_k}{l_k} + \frac{2k\log n}{l_k}\Big) = 0.$$

Theorem 5.1 *For $t \in \mathbb{Z}_+^{\mathbb{Z}_+}$ and $\lambda \in (0, 1]$, if*

$$\lim_{k\to\infty} \frac{r^{(t)}_{k-1}}{(t_ks^{(t)}_k)^\lambda} = 0,$$

then $(M^{(t)}, \sigma)$ is uniformly λ-power distributionally n-chaotic.

Proof Choose an uncountable subset C of Σ_n such that if $c_j = c^j_1c^j_2\cdots$ is n distrinct points of C, where $0 \leq j \leq n-1$, then $c^0_p = c^1_p = \cdots = c^{n-1}_p$ for infinitely many $p \in \mathbb{Z}_+$ and c^j_q, $0 \leq j \leq n-1$, are pairwise different for infinitely many $q \in \mathbb{Z}_+$. By Example 3.1, such a subset does exist in Σ_n.

Define $\xi : C \to \Sigma_n$ such that for any $c = c_1c_2\cdots \in C$, $\xi(x) = J_{1c_1}J_{2c_2}\cdots$. For each fixed $k \in \mathbb{Z}_+$, we always have $J_{1c_1}\cdots J_{kc_k} \prec A_{(k+1)0} \prec x^{(t)}$. Therefore, there is some $h \geq 0$ such that the beginning $r^{(t)}_k$-word of $\sigma^h(x^{(t)})$ is $J_{1c_1}\cdots J_{kc_k}$. This implies that $\xi(c) \in M^{(t)}$ for any $c \in C$. Put $D = \xi(C)$. Then $D \subset M^{(t)}$. Since C is uncountable and ξ is an injective, D is also uncountable.

Let

$$w_j = J^j_{1c_1}J^j_{2c_2}\cdots$$

be arbitrary n distrinct points of D, where $0 \leq j \leq n-1$. Then there are two strictly increasing sequences $\{p_k\}$ and $\{q_k\}$ of positive integers, such that for each k, $J^0_{p_kc_{p_k}} = J^1_{p_kc_{p_k}} = \cdots = J^{n-1}_{p_kc_{p_k}}$, and the corresponding components of $J^j_{q_kc_{q_k}}$, $0 \leq j \leq n-1$, are pairwise different.

On the one hand, for given $p_k > 1$, when

$$t_{p_k-1}s^{(t)}_{p_k-1} \le i \le t_{p_k}s^{(t)}_{p_k} - t_{p_k-1}s^{(t)}_{p_k-1},$$

the beginning $t_{p_k-1}s^{(t)}_{p_k-1}$ components of $\sigma^i(w_j)$, $0 \le j \le n-1$, are correspondingly identical. Hence, for any $0 \le j \ne m \le n-1$, one has

$$d(\sigma^i(w_j), \sigma^i(w_m)) \le \frac{1}{t_{p_k-1}s^{(t)}_{p_k-1}}.$$

So for any given $\varepsilon > 0$, whenever p_k is large enough, one has

$$\max_{0\le j\ne m\le n-1} d(\sigma^i(w_j), \sigma^i(w_m)) < \varepsilon.$$

Then

$$\begin{aligned}
&\frac{1}{[r^{(t)}_{p_k}]^\lambda}\Big(r^{(t)}_{p_k} - \#\Big\{i \in \mathbb{Z}_+ : \max_{0\le j\ne m\le n-1} d(\sigma^i(w_j), \sigma^i(w_m)) < \varepsilon\Big\}\Big)\\
&\le \frac{r^{(t)}_{p_k} - t_{p_k}s^{(t)}_{p_k} + 2t_{p_k-1}s^{(t)}_{p_k-1}}{[r^{(t)}_{p_k}]^\lambda} = \frac{r^{(t)}_{p_k-1} + 2t_{p_k-1}s^{(t)}_{p_k-1}}{[r^{(t)}_{p_k}]^\lambda}\\
&\le \frac{3r^{(t)}_{p_k-1}}{[r^{(t)}_{p_k-1} + t_{p_k}s_{p_k}]^\lambda} = \frac{3r^{(t)}_{p_k-1}}{[t_{p_k}s^{(t)}_{p_k}]^\lambda}\cdot\Big(\frac{r^{(t)}_{p_k-1}}{t_{p_k}s^{(t)}_{p_k}} + 1\Big)^{-1}\\
&= \frac{3r^{(t)}_{p_k-1}}{[t_{p_k}s^{(t)}_{p_k}]^\lambda}\cdot\Big(\frac{1}{n^{p_k-1}t_{p_k}} + 1\Big)^{-1} \to 0, \quad \text{as } p_k \to \infty.
\end{aligned}$$

Namely, for any given $\varepsilon > 0$,

$$\Big\{i \in \mathbb{Z}_+ : \max_{0\le j\ne m\le n-1} d(\sigma^i(w_j), \sigma^i(w_m)) < \varepsilon\Big\} \in \mathcal{D}_\lambda.$$

On the other hand, for given $q_k > 1$, when

$$t_{q_k-1}s^{(t)}_{q_k-1} \le i \le t_{q_k}s^{(t)}_{q_k} - t_{q_k-1}s^{(t)}_{q_k-1},$$

the beginning $t_{q_k-1}s^{(t)}_{q_k-1}$ components of $\sigma^i(w_j)$, $0 \le j \le n-1$ are correspondingly pairwise distrinct. Hence, for any $0 \le j \ne m \le n-1$, one has

$$d(\sigma^i(w_j), \sigma^i(w_m)) = 1.$$

Choose $\varepsilon_0 = 1$. Then

$$\begin{aligned}
&\frac{1}{[r^{(t)}_{q_k}]^\lambda}\Big(r^{(t)}_{q_k} - \#\Big\{i \in \mathbb{Z}_+ : \min_{0\le j\ne m\le n-1} d(\sigma^i(w_j), \sigma^i(w_m)) = 1\Big\}\Big)\\
&\le \frac{r^{(t)}_{q_k} - t_{q_k}s^{(t)}_{q_k} + 2t_{q_k-1}s^{(t)}_{q_k-1}}{[r^{(t)}_{q_k}]^\lambda} = \frac{r^{(t)}_{q_k-1} + 2t_{q_k-1}s^{(t)}_{q_k-1}}{[r^{(t)}_{q_k}]^\lambda}\\
&\le \frac{3r^{(t)}_{q_k-1}}{[r^{(t)}_{q_k-1} + t_{q_k}s_{q_k}]^\lambda} = \frac{3r^{(t)}_{q_k-1}}{[t_{q_k}s^{(t)}_{q_k}]^\lambda}\cdot\Big(\frac{r^{(t)}_{q_k-1}}{t_{q_k}s^{(t)}_{q_k}} + 1\Big)^{-1}\\
&= \frac{3r^{(t)}_{q_k-1}}{[t_{q_k}s^{(t)}_{q_k}]^\lambda}\cdot\Big(\frac{1}{n^{q_k-1}t_{q_k}} + 1\Big)^{-1} \to 0, \quad \text{as } q_k \to \infty.
\end{aligned}$$

Namely,

$$\Big\{i \in \mathbb{Z}_+ : \min_{0\le j\ne m\le n-1} d(\sigma^i(w_j), \sigma^i(w_m)) = 1\Big\} \in \mathcal{D}_\lambda.$$

So D is a $\mathcal{D}_\lambda$-1-n-scrambled set. Thus, $(M^{(t)}, \sigma)$ is uniformly λ-power distributionally n-chaotic.

By Theorem 5.1 and the definition of $\mathcal{D}_0$, we have the following theorem.

Theorem 5.2 *For $t \in \mathbb{Z}_+^{\mathbb{Z}_+}$, if for each $\lambda \in (0, 1]$,*

$$\lim_{k\to\infty} \frac{r_{k-1}^{(t)}}{(t_k s_k^{(t)})^\lambda} = 0,$$

then $(M^{(t)}, \sigma)$ is uniformly 0-power distributionally n-chaotic.

In the final section, we present some examples. For this purpose, it is a key to find the sequence $t \in \mathbb{Z}_+^{\mathbb{Z}_+}$ which satisfies the condition of Theorem 5.1 or Theorem 5.2.

Example 5.1 Let $\lambda \in (0, 1]$ and $n \ge 2$. Define three sequences $\{r_i\}_{i=1}^\infty, \{s_i\}_{i=1}^\infty$ and $\{t_i\}_{i=1}^\infty$ inductively as follows.

Let $t_1 = 1,\ r_1 = s_1 = 2$. Assume that for $k \ge 2$, $r_{k-1}, s_{k-1}, t_{k-1}$ are defined. Then put

$$s_k = n^{k-1} r_{k-1}, \quad t_k = \lfloor s_k^{\frac{1}{\lambda}-1} \rfloor, \quad r_k = \sum_{i=1}^k t_i s_i,$$

where $\lfloor a \rfloor$ denotes the largest integer not greater than the real number a. It follows that

$$\lim_{k\to\infty} \frac{r_{k-1}}{(t_k s_k)^\lambda} = \lim_{k\to\infty} \frac{r_{k-1}}{(s_k \lfloor s_k^{\frac{1}{\lambda}-1} \rfloor)^\lambda} = \lim_{k\to\infty} \frac{r_{k-1}}{s_k} = \lim_{k\to\infty} \frac{1}{n^{k-1}} = 0.$$

Let $t = \{t_i\}_{i=1}^\infty$. Then $t \in \mathbb{Z}_+^{\mathbb{Z}_+}$ satisfies the condition of Theorem 5.1. So the minimal system $(M^{(t)}, \sigma)$ with zero topological entropy is uniformly λ-power distributionally n-chaotic.

Example 5.2 Let $n \ge 2$. Define three sequences $\{r_i\}_{i=1}^\infty, \{s_i\}_{i=1}^\infty$ and $\{t_i\}_{i=1}^\infty$ inductively as follows.

Let $t_1 = 1,\ r_1 = s_1 = 2$. Assume that for $k \ge 2$, $r_{k-1}, s_{k-1}, t_{k-1}$ are defined. Then put

$$s_k = n^{k-1} r_{k-1}, \quad t_k = s_k^{k-1}, \quad r_k = \sum_{i=1}^k t_i s_i.$$

For any $\lambda \in (0, 1]$, we have

$$\lim_{k\to\infty} \frac{r_{k-1}}{(t_k s_k)^\lambda} = \lim_{k\to\infty} \frac{r_{k-1}}{s_k^{k\lambda}} = 0.$$

The last equality holds because $k\lambda \ge 1$ when k is large enough.

Let $t = \{t_i\}_{i=1}^\infty$. Then $t \in \mathbb{Z}_+^{\mathbb{Z}_+}$ satisfies the condition of Theorem 5.2. So the minimal system $(M^{(t)}, \sigma)$ with zero topological entropy is uniformly 0-power distributionally n-chaotic.

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