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Möbius Homogeneous Hypersurfaces with Three Distinct Principal Curvatures in $\mathbb{S}^{n+1}$ *

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Abstract Let $x : M^n \to \mathbb{S}^{n+1}$ be an immersed hypersurface in the $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$. If, for any points $p, q \in M^n$, there exists a Möbius transformation $\phi : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$ such that $\phi \circ x(M^n) = x(M^n)$ and $\phi \circ x(p) = x(q)$, then the hypersurface is called a Möbius homogeneous hypersurface. In this paper, the Möbius homogeneous hypersurfaces with three distinct principal curvatures are classified completely up to a Möbius transformation.

Keywords Möbius transformation group, Conformal transformation group, Möbius homogeneous hypersurfaces, Möbius isoparametric hypersurfaces
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1 Introduction

A diffeomorphism $\phi : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$ is said to be a Möbius transformation if ϕ takes the set of round n-spheres into the set of round n-spheres. All Möbius transformations form a transformation group, which is called the Möbius transformation group of $\mathbb{S}^{n+1}$ and denoted by $M(\mathbb{S}^{n+1})$. It is well-known that, for $n \geq 2$, the Möbius transformation group $M(\mathbb{S}^{n+1})$ of $\mathbb{S}^{n+1}$ coincides with the conformal transformation group $C(\mathbb{S}^{n+1})$ of $\mathbb{S}^{n+1}$. In [11], Wang introduced complete Möbius invariants for a submanifold $x : M^m \to \mathbb{S}^{n+1}$, and obtained a congruent theorem of hypersurfaces in $\mathbb{S}^{n+1}$ (also see [1]). Recently some special hypersurfaces, including the Möbius isoparametric hypersurfaces, the Blaschke isoparametric hypersurfaces and so on, have been extensively studied in the context of Möbius geometry (see [4–5, 7]).

Another special hypersurface is the Möbius homogeneous hypersurface. A hypersurface $x : M^n \to \mathbb{S}^{n+1}$ is called a Möbius homogeneous hypersurface if for any two points $p, q \in M^n$, there exists a Möbius transformation $\phi \in M(\mathbb{S}^{n+1})$ such that $\phi \circ x(M^n) = x(M^n)$ and $\phi \circ x(p) = x(q)$. Let $x : M^n \to \mathbb{S}^{n+1}$ be a Möbius homogeneous hypersurface, we define

$$\Pi = \{\phi \in M(\mathbb{S}^{n+1}) \mid \phi \circ x(M^n) = x(M^n)\}.$$

Then Π is a subgroup of the Möbius group $M(\mathbb{S}^{n+1})$, and the hypersurface x is the orbit of the subgroup Π. Thus the Möbius scalar invariants on the hypersurface are constant.

Standard examples of Möbius homogeneous hypersurfaces in $\mathbb{S}^{n+1}$ are the image of homogeneous hypersurfaces in $\mathbb{S}^{n+1}$ under the Möbius transformations. The homogeneous hypersurface

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in $\mathbb{S}^{n+1}$ is the isoparametric hypersurface, which was systematically studied (see [2–3]). Another standard examples come from homogeneous hypersurfaces in $\mathbb{R}^{n+1}$. The inverse of the stereographic projection $\sigma : \mathbb{R}^{n+1} \mapsto \mathbb{S}^{n+1}$ is defined by

$$\sigma(u) = \Big(\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2}\Big).$$

The conformal map σ assigns any hypersurface in $\mathbb{R}^{n+1}$ to a hypersurface in $\mathbb{S}^{n+1}$. In [8], authors proved that the Möbius invariants on $f : M^n \mapsto \mathbb{R}^{n+1}$ are the same as the Möbius invariants on $\sigma \circ f : M^n \mapsto \mathbb{S}^{n+1}$. If $f : M^n \to \mathbb{R}^{n+1}$ is a homogeneous hypersurface, then the hypersurface $x = \sigma \circ f$ is a Möbius homogeneous hypersurface.

Next we give a method to construct the Möbius homogeneous hypersurface in $\mathbb{S}^{n+1}$.

Proposition 1.1 *Let $u : M^m \to \mathbb{S}^{m+1}$ be an immersed hypersurface. We define the cone over u as*

$$\begin{aligned} &f : M^m \times R^+ \times \mathbb{R}^{n-m-1} \to \mathbb{R}^{n+1}, \quad 1 \le m \le n-1, \\ &f(p,t,y) = (tu(p), y). \end{aligned}$$

If $u : M^m \to \mathbb{S}^{m+1}$ be a homogeneous hypersurface, then the image of σ of the cone hypersurface f over u is a Möbius homogeneous hypersurface in $\mathbb{S}^{n+1}$.

These examples above come from homogeneous hypersurfaces in $\mathbb{S}^{n+1}$ or $\mathbb{R}^{n+1}$. But there are some examples of Möbius homogeneous hypersurfaces which can not be obtained in this way. In [10], Sulanke constructed a Möbius homogeneous surface, which is a cylinder over a logarithmic spiral in $\mathbb{R}^2$, and classified the Möbius homogeneous surfaces in $\mathbb{R}^3$. In [6], authors constructed a Möbius homogeneous hypersurface, a logarithmic spiral cylinder, which is a high dimensional version of Sulanke's example and classified the Möbius homogeneous hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures. In addition, in [6], authors also classified the Möbius homogeneous hypersurfaces in $\mathbb{S}^4$.

In this paper, the Möbius homogeneous hypersurfaces with three distinct principal curvatures are classified, and the main results are as follows.

Theorem 1.1 *Let $x : M^n \to \mathbb{S}^{n+1}$ be a Möbius homogeneous hypersurface with three distinct principal curvatures. Then x is Möbius equivalent to one of the following hypersurfaces:*

(1) *The isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ with three distinct principal curvatures;*

(2) *the image of σ of the cone over a standard torus*

$$u : \mathbb{S}^k(r) \times \mathbb{S}^{m-k}(\sqrt{1-r^2}) \to \mathbb{S}^{m+1}, \quad 1 \le k \le m-1, \ k < m \le n-1;$$

(3) *the image of σ of the cone over the cartan's minimal isoparametric hypersurface $u : M^{n-1} \to \mathbb{S}^n$ with three distinct principal curvatures.*

Remark 1.1 Two hypersurfaces $x, \widetilde{x} : M^n \to \mathbb{S}^{n+1}$ are Möbius equivalent, if there exists a Möbius transformation $\phi \in M(\mathbb{S}^{n+1})$ such that $\phi \circ x(M^n) = \widetilde{x}(M^n)$.

According to the classification results in [5], combining Proposition 1.1 and our main Theorem 1.1, we can derive the following corollary.

Corollary 1.1 *A Möbius homogeneous hypersurface with three distinct principal curvatures is a Möbius isoparametric hypersurface. Conversely, a Möbius isoparametric hypersurface with three distinct principal curvatures is a Möbius homogeneous hypersurface.*

We organize the paper as follows. In Section 2, we give the elementary facts about Möbius geometry for hypersurfaces in $\mathbb{S}^{n+1}$. In Section 3, we prove Proposition 1.1 and give a characterization of the cone hypersurfaces. In Section 4, we prove that the Möbius form of the Möbius homogeneous hypersurfaces with three distinct principal curvatures vanishes. In Section 5, we give the proof of Theorem 1.1.

2 Möbius Invariants for Hypersurfaces in $\mathbb{S}^{n+1}$

In this section, we recall some facts about the Möbius transformation group and define Möbius invariants of hypersurfaces in $\mathbb{S}^{n+1}$. For details we refer to [11].

Let $\mathbb{R}^{n+3}_1$ be the Lorentz space, i.e., $\mathbb{R}^{n+3}$ with the inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_{n+2}y_{n+2}$$

for $x = (x_0, x_1, \cdots, x_{n+2}),\ y = (y_0, y_1, \cdots, y_{n+2}) \in \mathbb{R}^{n+3}$.

Let $O(n+2,1)$ be the Lorentz group of $\mathbb{R}^{n+3}_1$ defined by

$$O(n+2,1) = \{T \in GL(\mathbb{R}^{n+3}) \mid {}^tTI_1T = I_1\},$$

where ${}^tT$ denotes the transpose of T and $I_1 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & I \end{smallmatrix}\right)$.

Let $C^{n+2}_+ = \{y = (y_0, y_1) \in R \times \mathbb{R}^{n+2} \mid \langle y, y\rangle = 0, y_0 > 0\} \subset \mathbb{R}^{n+3}_1$, and $O^+(n+2,1)$ denote the subgroup of $O(n+2,1)$ defined by

$$O^+(n+2,1) = \{T \in O(n+2,1) \mid T(C^{n+2}_+) = C^{n+2}_+\}.$$

Lemma 2.1 (see [9]) *Let* $T = \left(\begin{smallmatrix} w & u \\ v & B \end{smallmatrix}\right) \in O(n+2,1)$*. then* $T \in O^+(n+2,1)$ *if and only if* $w > 0$.

It is well-known that the subgroup $O^+(n+2,1)$ is isomorphic to the Möbius transformation group $M(\mathbb{S}^{n+1})$. In fact, for any

$$T = \begin{pmatrix} w & u \\ v & B \end{pmatrix} \in O^+(n+2,1),$$

we can define the Möbius transformation $L(T) : \mathbb{S}^{n+1} \mapsto \mathbb{S}^{n+1}$ by

$$L(T)(x) = \frac{Bx + v}{ux + w}, \quad x = {}^t(x_1, \cdots, x_{n+2}) \in S^{n+1}.$$

Then the map $L : O^+(n+2,1) \mapsto M(\mathbb{S}^{n+1})$ is a group isomorphism.

Let $x : M^n \mapsto \mathbb{S}^{n+1}$ be a hypersurface without umbilical points, and e_{n+1} the unit normal vector field. Let $\mathrm{II} = \sum\limits_{ij} h_{ij}\theta_i\theta_j$ and $H = \frac{1}{n}\sum\limits_i h_{ii}$ be the second fundamental form and the mean curvature of x, respectively. The Möbius position vector $Y : M^n \mapsto \mathbb{R}^{n+3}_1$ of x is defined by

$$Y = \rho(1, x), \quad \rho^2 = \frac{n}{n-1}(\|\mathrm{II}\|^2 - nH^2).$$

Theorem 2.1 (see [11]) *Two hypersurfaces $x, \widetilde{x} : M^n \mapsto \mathbb{S}^{n+1}$ are Möbius equivalent if and only if there exists $T \in O^+(n+2,1)$ such that $\widetilde{Y} = YT$.*

It follows immediately from Theorem 2.1 that

$$g = \langle dY, dY \rangle = \rho^2 \mathrm{d}x \cdot \mathrm{d}x$$

is a Möbius invariant, which is called the Möbius metric of x (see [11]).

Let Δ be the Laplacian operator with respect to g, we define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y \rangle Y.$$

Then we have

$$\langle Y, Y \rangle = 0, \quad \langle N, Y \rangle = 1, \quad \langle N, N \rangle = 0.$$

Let $\{E_1, \cdots, E_n\}$ be a local orthonormal basis for (M^n, g) with the dual basis $\{\omega_1, \cdots, \omega_n\}$, and write $Y_i = E_i(Y)$, then we have

$$\langle Y_i, Y \rangle = \langle Y_i, N \rangle = 0, \quad \langle Y_i, Y_j \rangle = \delta_{ij}, \quad 1 \le i, j \le n.$$

We define the conformal Gauss map of x

$$G = (H, Hx + e_{n+1}).$$

By direct computations, we have

$$\langle G, Y \rangle = \langle G, N \rangle = \langle G, Y_i \rangle = 0, \quad \langle G, G \rangle = 1.$$

Then $\{Y, N, Y_1, \cdots, Y_n, G\}$ forms a moving frame in $\mathbb{R}^{n+3}_1$ along M^n. We use the following range of indices in this section: $1 \le i, j, k, l \le n$. We can write the structure equations as follows:

$$\begin{aligned}
\mathrm{d}Y &= \sum_i Y_i \omega_i, \\
\mathrm{d}N &= \sum_{ij} A_{ij}\omega_i Y_j + \sum_i C_i \omega_i G, \\
\mathrm{d}Y_i &= -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_j B_{ij}\omega_j G, \\
\mathrm{d}G &= -\sum_i C_i \omega_i Y - \sum_{ij} \omega_j B_{ij} Y_i,
\end{aligned}$$

where ω_{ij} is the connection form of the Möbius metric g and $\omega_{ij} + \omega_{ji} = 0$. The tensors $A = \sum\limits_{ij} A_{ij}\omega_i \otimes \omega_j$, $C = \sum\limits_i C_i \omega_i$ and $B = \sum\limits_{ij} B_{ij}\omega_i \otimes \omega_j$ are called the Blaschke tensor, the Möbius form and the Möbius second fundamental form of x, respectively. The eigenvalues of (B_{ij}) are called the Möbius principal curvatures of x. The covariant derivative of C_i, A_{ij}, B_{ij} are defined by

$$\begin{aligned}
\sum_j C_{i,j}\omega_j &= \mathrm{d}C_i + \sum_j C_j \omega_{ji}, \\
\sum_k A_{ij,k}\omega_k &= \mathrm{d}A_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \\
\sum_k B_{ij,k}\omega_k &= \mathrm{d}B_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki},
\end{aligned}$$

respectively. The integrability conditions for the structure equations are given by

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k, \tag{2.1}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{jk}A_{ki}), \tag{2.2}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \quad \sum_j B_{ij,j} = -(n-1)C_i, \tag{2.3}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}, \tag{2.4}$$

$$\sum_i B_{ii} = 0, \quad \sum_{ij}(B_{ij})^2 = \frac{n-1}{n}, \quad \mathrm{tr}A = \sum_i A_{ii} = \frac{1}{2n}(1+n^2 s), \tag{2.5}$$

where R_{ijkl} denote the curvature tensor of g, $s = \frac{1}{n(n-1)}\sum\limits_{ij} R_{ijij}$ is the normalized Möbius scalar curvature. When $n \geq 3$, we know that all coefficients in the structure equations are determined by $\{g, B\}$, and we have the following theorem.

Theorem 2.2 (see [11]) *Two hypersurfaces $x : M^n \mapsto \mathbb{S}^{n+1}$ and $\widetilde{x} : M^n \mapsto \mathbb{S}^{n+1}$ $(n \geq 3)$ are Möbius equivalent if and only if there exists a diffeomorphism $\varphi : M^n \to M^n$, which preserves the Möbius metric g and the Möbius second fundamental form B.*

The coefficients of the Möbius second fundamental form can be written by

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}).$$

Clearly the number of distinct Möbius principal curvatures is the same as that of its distinct principal curvatures.

3 A Method to Construct the Möbius Homogeneous Hypersurface

In this section, we prove Proposition 1.1 and our Theorem 1.1 for dimension $n = 3$.

Let $u : M^m \to \mathbb{S}^{m+1}$ be an immersed hypersurface in sphere, then the cone over u is defined by

$$f : M^m \times R^+ \times \mathbb{R}^{n-m-1} \to \mathbb{R}^{n+1}, \quad f(p,t,y) = (tu(p), y), \quad 1 \leq m \leq n-1.$$

The Möbius position vector $Y : M^m \times R^+ \times \mathbb{R}^{n-m-1} \longrightarrow \mathbb{R}^{n+3}_1$ of the cone f is

$$Y = \rho_0\Big(\frac{1+t^2+|y|^2}{2t}, \frac{1-t^2-|y|^2}{2t}, \frac{y}{t}, u\Big), \tag{3.1}$$

where $\rho_0^2 = \frac{n}{n-1}(|\mathrm{II}_u|^2 - \frac{m^2}{n}H_u^2) : M^m \longrightarrow \mathbb{R}$, and $y : \mathbb{R}^{n-m-1} \longrightarrow \mathbb{R}^{n-m-1}$ is the identity map. Let

$$\mathbb{H}^{n-m} = \{(y_0, y) \in \mathbb{R}^{n-m+1} \mid -y_0^2 + |y|^2 = -1, y_0 \geq 1\} \cong R^+ \times \mathbb{R}^{n-m-1},$$

then $\big(\frac{1+t^2+|y|^2}{2t}, \frac{1-t^2-|y|^2}{2t}, \frac{y}{t}\big) : R^+ \times \mathbb{R}^{n-m-1} = \mathbb{H}^{n-m} \to \mathbb{H}^{n-m}$ is nothing else but the identity map. And from (3.1), the Möbius position vector Y of the cone f can be written as

$$Y = \rho_0(u, \mathrm{id}) : M^m \times \mathbb{H}^{n-m} \to \mathbb{S}^{m+1} \times \mathbb{H}^{n-m} \subset \mathbb{R}^{n+3}_1, \tag{3.2}$$

where $\rho_0 \in C^\infty(M^m)$ and $\mathrm{id} : \mathbb{H}^{n-m} \to \mathbb{H}^{n-m}$ is an identity map. Thus we have the following result.

Proposition 3.1 *Let $f : M^n \to \mathbb{R}^{n+1}$ be an immersed submanifold without umbilical points. If there exist $\rho_0 \in C^\infty(M^m)$ and a submanifold $u : M^m \to \mathbb{S}^{m+1}$ such that the Möbius position vector of f is*

$$Y = \rho_0(u, \mathrm{id}) : M^m \times \mathbb{H}^{n-m} \to \mathbb{S}^{m+1} \times \mathbb{H}^{n-m} \subset \mathbb{R}^{n+3}_1.$$

Then f is a cone over u.

Proof of Proposition 1.1 If $u : M^m \to \mathbb{S}^{m+1}$ is a homogeneous hypersurface, from (3.2), we know that $Y = \rho_0(u, \mathrm{id}) : M^m \times \mathbb{H}^{n-m} \to \mathbb{S}^{m+1} \times \mathbb{H}^{n-m} \subset \mathbb{R}^{n+3}_1$ is homogeneous. Thus the cone f is Möbius homogeneous, and we finish the proof of Proposition 1.1.

Next, we give the proof of Theorem 1.1 for the case $n = 3$. Let $x : M^3 \to \mathbb{S}^4$ be a Möbius homogeneous hypersurface with three distinct principal curvatures. From [6, 12], we know that x is Möbius equivalent to the two classes of hypersurfaces. One is the 1-parameter family of isoparametric hypersurfaces with three principal curvatures. Another is the images of σ of the cone over the 1-parameter family of isoparametric torus in S^3. Thus, Theorem 1.1 holds for the hypersurfaces in $\mathbb{S}^4$.

4 The Möbius Homogeneous Hypersurfaces with Three Distinct Principal Curvatures in $\mathbb{S}^{n+1}$

Let $x : M^n \mapsto \mathbb{S}^{n+1}$ $(n \geq 4)$ be a Möbius homogeneous hypersurface with three distinct principal curvatures. We can choose a local orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ with respect to the Möbius metric g such that

$$(B_{ij}) = \mathrm{diag}(\underbrace{b_1, \cdots, b_1}_{m_1}, \underbrace{b_2, \cdots, b_2}_{m_2}, \underbrace{b_3, \cdots, b_3}_{m_3}). \tag{4.1}$$

From (2.5), we have

$$m_1 b_1 + m_2 b_2 + m_3 b_3 = 0, \quad m_1 b_1^2 + m_2 b_2^2 + m_3 b_3^2 = \frac{n-1}{n}. \tag{4.2}$$

Since x is a Möbius homogeneous hypersurface, the Möbius principal curvatures $b_1,\ b_2,\ b_3$ are constant. From $\sum\limits_k B_{ij,k}\omega_k = \mathrm{d}B_{ij} + \sum\limits_k B_{ik}\omega_{kj} + \sum\limits_k B_{kj}\omega_{ki}$ and (4.1), we have

$$(b_i - b_j)\omega_{ij} = \sum_m B_{ij,m}\omega_m, \qquad B_{ii,m} = 0. \tag{4.3}$$

Let $[b_i] = \{k \mid b_k = b_i\}$. It follows from (4.3) that

$$\begin{cases} B_{ij,k} = 0, & [b_i] = [b_j],\ 1 \leq k \leq n, \\ \omega_{ij} = \sum\limits_k \dfrac{B_{ij,k}}{b_i - b_j}, & [b_i] \neq [b_j]. \end{cases} \tag{4.4}$$

Proposition 4.1 *Let $x : M^n \mapsto \mathbb{S}^{n+1}$ $(n \geq 4)$ be a Möbius homogeneous hypersurface with three distinct principal curvatures. Then, the Möbius form of x vanishes, i.e., $C = 0$.*

The proof of Proposition 4.1 is divided into the following three lemmas.

Lemma 4.1 *Let $x : M^n \mapsto \mathbb{S}^{n+1}$ $(n \geq 4)$ be a Möbius homogeneous hypersurface with three distinct principal curvatures. If $m_1 \geq 2,\ m_2 \geq 2,\ m_3 \geq 2$, then the Möbius form of x vanishes, i.e., $C = 0$.*

Proof Since $m_i \geq 2$, for each i fixed, we can choose $j, k \in [b_i]$ such that $j \neq k$. From (4.4), we have

$$B_{jj,k} = 0, \quad B_{jk,m} = 0, \quad 1 \leq m \leq n,$$

combining (2.3),

$$0 = B_{jj,k} - B_{jk,j} = C_k, \quad k \in [b_i].$$

Thus $C = 0$, and we finish the proof of Lemma 4.1.

Lemma 4.2 *Let $x : M^n \mapsto \mathbb{S}^{n+1}$ $(n \geq 4)$ be a Möbius homogeneous hypersurface with three distinct principal curvatures. If $m_1 = m_2 = 1,\ m_3 \geq 2$, then the Möbius form of x vanishes, i.e., $C = 0$.*

Proof The distributions $V_1 = \mathrm{span}\{E_1\},\ V_2 = \mathrm{span}\{E_2\},\ V_3 = \mathrm{span}\{E_3, \cdots, E_n\}$, determined by eigenvectors of the Möbius second fundamental form are invariant under the subgroup Π. Since $\dim V_1 = \dim V_2 = 1$, the eigenvectors E_1, E_2 are invariant under the subgroup Π. Therefore, the data $A_{11} = A(E_1, E_1)$, $A_{12} = A(E_1, E_2)$, $C_{1,2} = \nabla C(E_1, E_2)$, R_{1212}, and so on, are constants.

Using (4.4) and (2.3), we have

$$\begin{cases} C_a = 0, \quad B_{1a,1} = B_{2a,2} = 0, & 3 \leq a \leq n, \\ B_{1a,a} = -C_1, \quad B_{2a,a} = -C_2, \quad B_{ab,m} = 0, & 3 \leq a, b \leq n. \end{cases} \tag{4.5}$$

From (4.4)–(4.5), we obtain

$$\begin{cases} \omega_{12} = \sum\limits_m \dfrac{B_{12,m}}{b_1 - b_2}\omega_m, \quad \omega_{1a} = \dfrac{B_{1a,2}}{b_1 - b_3}\omega_2 + \dfrac{C_1}{b_3 - b_1}\omega_a, \\ \omega_{2a} = \dfrac{B_{2a,1}}{b_2 - b_3}\omega_1 + \dfrac{C_2}{b_3 - b_2}\omega_a, \quad 3 \leq a \leq n. \end{cases} \tag{4.6}$$

Let $E = \sum\limits_{a\geq 3} B_{12,a}E_a \in V_3$. It follows that E is invariant under the subgroup Π. We divide the proof into two cases:

Case 1 $E = 0$.

Case 2 $E \neq 0$.

First, we consider Case 1, $E = 0$. We have $B_{12,a} = 0,\ 3 \leq a \leq n$, and from (4.6), we have

$$\begin{cases} \omega_{12} = \dfrac{C_2}{b_2 - b_1}\omega_1 + \dfrac{C_1}{b_2 - b_1}\omega_2, \\ \omega_{1a} = \dfrac{C_1}{b_3 - b_1}\omega_a, \quad \omega_{2a} = \dfrac{C_2}{b_3 - b_2}\omega_a, \quad 3 \leq a \leq n. \end{cases} \tag{4.7}$$

Note that C_1, C_2 are constant, and we use $\mathrm{d}\omega_{ij} - \sum\limits_m \omega_{im} \wedge \omega_{mj} = -\sum\limits_{k<l} R_{ijkl}\omega_k \wedge \omega_l$ and (4.7)

to obtain the following equations:

$$\begin{cases} -\sum\limits_{k<l} R_{12kl}\omega_k\wedge\omega_l = \dfrac{C_1^2+C_2^2}{(b_1-b_2)^2}\omega_1\wedge\omega_2, \\ -\sum\limits_{k<l} R_{1akl}\omega_k\wedge\omega_l = \Big[\dfrac{C_1^2}{(b_1-b_3)^2}+\dfrac{C_2^2}{(b_1-b_2)(b_3-b_2)}\Big]\omega_1\wedge\omega_a \\ \qquad\qquad -\dfrac{C_1C_2}{(b_1-b_3)(b_1-b_2)}\omega_2\wedge\omega_a, \quad a\geq 3, \\ -\sum\limits_{k<l} R_{2akl}\omega_k\wedge\omega_l = \Big[\dfrac{C_1^2}{(b_1-b_3)(b_1-b_2)}+\dfrac{C_2^2}{(b_3-b_2)^2}\Big]\omega_2\wedge\omega_a \\ \qquad\qquad -\dfrac{C_1C_2}{(b_2-b_3)(b_2-b_1)}\omega_1\wedge\omega_a, \quad a\geq 3. \end{cases} \tag{4.8}$$

From (2.4) and (4.8), we can obtain

$$\begin{cases} A_{1a}=A_{2a}=A_{ab}=0, & 3\leq a,b\leq n, \quad a\neq b, \\ R_{1a2a}=A_{12}=\dfrac{C_1C_2}{(b_1-b_3)(b_1-b_2)}, & 3\leq a\leq n, \\ R_{2a1a}=A_{12}=\dfrac{C_1C_2}{(b_2-b_3)(b_2-b_1)}, & 3\leq a\leq n, \\ R_{1a1a}=-\Big[\dfrac{C_1^2}{(b_1-b_3)^2}+\dfrac{C_2^2}{(b_1-b_2)(b_3-b_2)}\Big], & 3\leq a\leq n, \\ R_{2a2a}=-\Big[\dfrac{C_1^2}{(b_1-b_3)(b_1-b_2)}+\dfrac{C_2^2}{(b_3-b_2)^2}\Big], & 3\leq a\leq n. \end{cases} \tag{4.9}$$

From the second and third formula of (4.9), we have

$$(2b_3-b_1-b_2)C_1C_2=0. \tag{4.10}$$

From (2.4) and the fourth and fifth formulas of (4.9), we have

$$A_{11}-A_{22}=\frac{C_1^2(b_2-b_3)}{(b_1-b_3)^2(b_1-b_2)}+\frac{C_2^2(b_1-b_3)}{(b_2-b_3)^2(b_1-b_2)}+(b_2-b_1)b_3. \tag{4.11}$$

If $C_1C_2\neq 0$, from (4.10), $2b_3-b_1-b_2=0$. By (4.2), $b_1+b_2+(n-2)b_3=0$. Thus we have $b_3=0$ and $b_1=-b_2$. From (4.9), we have $A_{12}=\frac{C_1C_2}{2b_1^2}$. On the other hand, from $\mathrm{d}A_{ij}+\sum\limits_m A_{mj}\omega_{mi}+\sum\limits_m A_{im}\omega_{mj}=\sum\limits_m A_{ij,m}\omega_m$, we have

$$\begin{cases} A_{11,2}=\dfrac{C_1^2C_2}{2b_1^3}, & A_{22,1}=\dfrac{-C_1C_2^2}{2b_1^3}, \\ A_{12,1}=(A_{11}-A_{22})\dfrac{C_2}{b_2-b_1}, & A_{12,2}=(A_{11}-A_{22})\dfrac{C_1}{b_2-b_1}. \end{cases} \tag{4.12}$$

Combining (2.1), (4.11)–(4.12), we obtain

$$C_2[C_1^2+C_2^2+4b_1^4]=0, \quad C_1[C_1^2+C_2^2+4b_1^4]=0,$$

which is a contradiction. Thus $C_1C_2=0$.

Since $C_1C_2=0$, from (4.9), we have $A_{12}=0$. On the other hand, using $A_{1a}=A_{2a}=0,\ 3\leq a\leq n$, from $\mathrm{d}A_{ij}+\sum\limits_m A_{mj}\omega_{mi}+\sum\limits_m A_{im}\omega_{mj}=\sum\limits_m A_{ij,m}\omega_m$, we have

$$A_{11,2}=A_{22,1}=0, \quad A_{12,1}=(A_{11}-A_{22})\frac{C_2}{b_2-b_1}, \quad A_{12,2}=(A_{11}-A_{22})\frac{C_1}{b_2-b_1}. \tag{4.13}$$

Combining (2.1), (4.11) and (4.13), we obtain

$$C_2\Big[1+\frac{C_2^2(b_1-b_3)^2}{(b_3-b_2)^2(b_1-b_2)^2}\Big]=0,\quad C_1\Big[1+\frac{C_1^2(b_2-b_3)^2}{(b_3-b_1)^2(b_2-b_1)^2}\Big]=0.$$

Thus $C_1=C_2=0$, that is $C=0$.

Next we consider Case 2, $E\neq 0$. We can rechoose an orthonormal basis $\{E_3,\cdots,E_n\}$ in V_3 such that $E_3=\frac{E}{|E|}$. Under the orthonormal basis $\{E_1,E_2,E_3,\cdots,E_n\}$, the equations (4.1), (4.5) and (4.6) still hold. Moreover,

$$B_{12,3}\neq 0,\quad B_{12,4}=0,\quad \cdots,\quad B_{12,n}=0 \tag{4.14}$$

and

$$\begin{cases}\omega_{12}=\dfrac{C_2}{b_2-b_1}\omega_1+\dfrac{C_1}{b_2-b_1}\omega_2+\dfrac{B_{12,3}}{b_1-b_2}\omega_3,\\ \omega_{13}=\dfrac{B_{12,3}}{b_1-b_3}\omega_2+\dfrac{C_1}{b_3-b_1}\omega_3,\quad \omega_{1a}=\dfrac{C_1}{b_3-b_1}\omega_a,\quad a\geq 4,\\ \omega_{23}=\dfrac{B_{12,3}}{b_2-b_3}\omega_1+\dfrac{C_2}{b_3-b_2}\omega_3,\quad \omega_{2a}=\dfrac{C_2}{b_3-b_2}\omega_a,\quad a\geq 4.\end{cases} \tag{4.15}$$

Using $d\omega_{ij}-\sum\limits_m\omega_{im}\wedge\omega_{mj}=-\sum\limits_{k<l}R_{ijkl}\omega_k\wedge\omega_l$ and (4.15), we have

$$\begin{cases}-\sum\limits_{k<l}R_{13kl}\omega_k\wedge\omega_l\\ =\Big[\dfrac{-2C_1B_{12,3}}{(b_1-b_3)(b_1-b_2)}-\dfrac{C_1B_{12,3}}{(b_1-b_3)^2}\Big]\omega_1\wedge\omega_2-\dfrac{C_1C_2}{(b_1-b_3)(b_1-b_2)}\omega_2\wedge\omega_3\\ \quad+\Big[\dfrac{C_1^2}{(b_1-b_3)^2}+\dfrac{C_2^2}{(b_2-b_1)(b_2-b_3)}-\dfrac{2B_{12,3}^2}{(b_2-b_1)(b_2-b_3)}\Big]\omega_1\wedge\omega_3,\\ -\sum\limits_{k<l}R_{23kl}\omega_k\wedge\omega_l\\ =\Big[\dfrac{2C_2B_{12,3}}{(b_2-b_3)(b_2-b_1)}+\dfrac{C_2B_{12,3}}{(b_2-b_3)^2}\Big]\omega_1\wedge\omega_2-\dfrac{C_1C_2}{(b_3-b_2)(b_1-b_2)}\omega_1\wedge\omega_3\\ \quad+\Big[\dfrac{C_1^2}{(b_3-b_1)(b_2-b_1)}+\dfrac{C_2^2}{(b_2-b_3)^2}-\dfrac{2B_{12,3}^2}{(b_1-b_2)(b_1-b_3)}\Big]\omega_2\wedge\omega_3,\end{cases} \tag{4.16}$$

$$\begin{cases}-\sum\limits_{k<l}R_{1akl}\omega_k\wedge\omega_l\\ =\Big[\dfrac{C_1^2}{(b_1-b_3)^2}+\dfrac{C_2^2}{(b_2-b_1)(b_2-b_3)}\Big]\omega_1\wedge\omega_a-\dfrac{C_1C_2}{(b_1-b_3)(b_1-b_2)}\omega_2\wedge\omega_a\\ \quad-\dfrac{C_2B_{12,3}}{(b_2-b_1)(b_2-b_3)}\omega_3\wedge\omega_a-\dfrac{B_{12,3}}{b_1-b_3}\omega_2\wedge\omega_{3a},\quad a\geq 4,\\ -\sum\limits_{k<l}R_{2akl}\omega_k\wedge\omega_l\\ =\dfrac{-C_1C_2}{(b_2-b_3)(b_2-b_1)}\omega_1\wedge\omega_a+\Big[\dfrac{C_1^2}{(b_2-b_1)(b_3-b_1)}+\dfrac{C_2^2}{(b_2-b_3)^2}\Big]\omega_2\wedge\omega_a\\ \quad-\dfrac{C_1B_{12,3}}{(b_1-b_3)(b_1-b_2)}\omega_3\wedge\omega_a-\dfrac{B_{12,3}}{b_2-b_3}\omega_1\wedge\omega_{3a},\quad a\geq 4.\end{cases} \tag{4.17}$$

From (2.4) and (4.16), we have

$$R_{1313} = b_1b_3 + A_{11} + A_{33} = -\Big[\frac{C_1^2}{(b_1-b_3)^2} + \frac{C_2^2}{(b_2-b_1)(b_2-b_3)} - \frac{2B_{12,3}^2}{(b_2-b_1)(b_2-b_3)}\Big],$$
$$R_{2323} = b_2b_3 + A_{22} + A_{33} = -\Big[\frac{C_1^2}{(b_3-b_1)(b_2-b_1)} + \frac{C_2^2}{(b_2-b_3)^2} - \frac{2B_{12,3}^2}{(b_1-b_2)(b_1-b_3)}\Big].$$

Thus we have

$$\begin{aligned}&(b_1-b_2)b_3 + A_{11} - A_{22}\\ &= \frac{C_1^2(b_3-b_2)}{(b_3-b_1)^2(b_2-b_1)} + \frac{C_2^2(b_3-b_1)}{(b_3-b_2)^2(b_2-b_1)} + \frac{2B_{12,3}^2(2b_3-b_2-b_1)}{(b_3-b_1)(b_2-b_1)(b_2-b_3)}.\end{aligned} \tag{4.18}$$

Similarly from (2.4) and (4.17), we have

$$(b_1-b_2)b_3 + A_{11} - A_{22} = \frac{C_1^2(b_3-b_2)}{(b_3-b_1)^2(b_2-b_1)} + \frac{C_2^2(b_3-b_1)}{(b_3-b_2)^2(b_2-b_1)}. \tag{4.19}$$

Noting that $B_{12,3} \neq 0$, and comparing (4.18) and (4.19), we obtain

$$2b_3 - b_2 - b_1 = 0.$$

Since $\mathrm{tr}(B) = 0$, we have

$$b_1 + b_2 = 0, \quad b_3 = 0. \tag{4.20}$$

Again we use (2.4), (4.16)–(4.17) to obtain

$$\begin{cases} A_{23} = R_{1312} = \dfrac{2C_1B_{12,3}}{(b_1-b_3)(b_1-b_2)} + \dfrac{C_1B_{12,3}}{(b_1-b_3)^2},\\ A_{13} = -R_{2312} = \dfrac{2C_2B_{12,3}}{(b_2-b_3)(b_2-b_1)} + \dfrac{C_2B_{12,3}}{(b_2-b_3)^2},\\ A_{13} = R_{1a3a} = \dfrac{C_2B_{12,3}}{(b_2-b_1)(b_2-b_3)},\\ A_{23} = R_{2a3a} = \dfrac{C_1B_{12,3}}{(b_1-b_3)(b_1-b_2)}. \end{cases} \tag{4.21}$$

From (4.21), we have

$$(2b_1 - b_2 - b_3)C_2 = 0, \quad (2b_2 - b_1 - b_3)C_1 = 0.$$

Since $|B|^2 = \frac{n-1}{n}$, combining (4.20), we deduce that $C_1 = C_2 = 0$. Thus $C = 0$, and we finish the proof of Lemma 4.2.

Lemma 4.3 *Let $x : M^n \mapsto \mathbb{S}^{n+1}$ $(n \geq 4)$ be a Möbius homogeneous hypersurface with three distinct principal curvatures. If $m_1 = 1,\ m_2 \geq 2,\ m_3 \geq 2$, then the Möbius form of x vanishes, i.e., $C = 0$.*

Proof In the last of the section, we make the following indices convention:

$$1 \leq i,j,k,m \leq n, \quad 2 \leq a,b \leq m_2+1, \quad m_2+2 \leq s,t \leq n. \tag{4.22}$$

The distributions $V_1 = \mathrm{span}\{E_1\}$, $V_2 = \mathrm{span}\{E_2, \cdots, E_{m_2+1}\}$, $V_3 = \mathrm{span}\{E_{m_2+2}, \cdots, E_n\}$, determined by eigenvectors of the Möbius second fundamental form, are invariant under the subgroup Π. Since $\dim V_1 = 1$, the eigenvectors E_1 are invariant under the subgroup Π. Therefore the data, $A_{11} = A(E_1, E_1)$, C_1, are constants. Since $m_2, m_3 \geq 2$, we can choose $i, j \in [b_2]$ (or $i, j \in [b_3]$) such that $i \neq j$. From (4.4), we have

$$B_{ii,j} = 0, \quad B_{ij,m} = 0, \quad 1 \leq m \leq n.$$

By (2.3),

$$0 = B_{ii,j} - B_{ij,i} = C_j.$$

Thus $C_a = 0$ and $C_s = 0$. Combining (4.4) and (2.3) again, we obtain

$$\begin{cases} B_{ij,k} = 0, \quad 2 \leq i, j, k \leq n, \\ B_{1a,1} = B_{1s,1} = 0, \quad B_{ii,j} = 0, \\ B_{1a,b} = B_{ab,1} = B_{st,1} = 0, \quad a \neq b, \\ B_{1a,a} = B_{1s,s} = -C_1 \end{cases} \tag{4.23}$$

and

$$\begin{cases} \omega_{1a} = \dfrac{C_1}{b_2 - b_1}\omega_a + \displaystyle\sum_t \dfrac{B_{1a,t}}{b_1 - b_2}\omega_s, \\ \omega_{1s} = \displaystyle\sum_b \dfrac{B_{1s,b}}{b_1 - b_3}\omega_b + \dfrac{C_1}{b_3 - b_1}\omega_s, \\ \omega_{as} = \dfrac{B_{as,1}}{b_2 - b_3}\omega_1. \end{cases} \tag{4.24}$$

Since C_1 is constant and $C_a = C_s = 0$, using $\mathrm{d}C_i + \sum_j C_j\omega_{ji} = \sum_j C_{i,j}\omega_j$, we have

$$C_{1,i} = 0, \quad \sum_j C_{a,j}\omega_j = C_1\omega_{1a}, \quad \sum_j C_{s,j}\omega_j = C_1\omega_{1s}. \tag{4.25}$$

Combining (4.24) and (4.25), we have

$$C_{a,1} = C_{s,1} = 0, \quad C_{a,s} = \frac{C_1 B_{1a,s}}{b_1 - b_2}, \quad C_{s,a} = \frac{C_1 B_{1s,a}}{b_1 - b_3}. \tag{4.26}$$

From (2.2), we obtain

$$A_{1a} = A_{1s} = 0, \quad A_{as} = \frac{C_1 B_{1a,s}}{(b_1 - b_2)(b_1 - b_3)}. \tag{4.27}$$

Using $\mathrm{d}B_{ij,k} + \sum_m B_{mj,k}\omega_{mi} + \sum_m B_{im,k}\omega_{mj} + \sum_m B_{ij,m}\omega_{mk} = \sum_m B_{ij,km}\omega_m$, (4.23)–(4.24), we obtain

$$\begin{cases} B_{1a,1a} = 2\displaystyle\sum_t \dfrac{B_{1a,t}^2}{b_3 - b_1} + \dfrac{C_1^2}{b_2 - b_1}, \quad B_{1a,a1} = 2\displaystyle\sum_t \dfrac{B_{1a,t}^2}{b_3 - b_2}, \\ B_{1s,1s} = 2\displaystyle\sum_b \dfrac{B_{1b,s}^2}{b_2 - b_1} + \dfrac{C_1^2}{b_3 - b_1}, \quad B_{1s,s1} = 2\displaystyle\sum_b \dfrac{B_{1b,s}^2}{b_2 - b_3}, \\ B_{as,as} = \dfrac{2B_{1a,s}^2}{b_1 - b_2} + \dfrac{C_1^2}{b_1 - b_3}, \qquad B_{as,sa} = \dfrac{2B_{1a,s}^2}{b_1 - b_3} + \dfrac{C_1^2}{b_1 - b_2} \end{cases} \tag{4.28}$$

and

$$\begin{cases} B_{as,sb}=2\dfrac{B_{1s,a}B_{1s,b}}{b_1-b_3}, & B_{as,bs}=2\dfrac{B_{1s,a}B_{1s,b}}{b_1-b_2},\\ B_{as,at}=2\dfrac{B_{1a,s}B_{1a,t}}{b_1-b_2}, & B_{as,ta}=2\dfrac{B_{1a,s}B_{1a,t}}{b_1-b_3}. \end{cases} \tag{4.29}$$

Combining (4.28)–(4.29) and $B_{ij,kl}-B_{ij,lk}=\sum_m B_{mj}R_{mikl}+\sum_m B_{im}R_{mjkl}$, we have

$$\begin{cases} R_{1a1a}=\displaystyle\sum_t \frac{2B_{1a,t}^2}{(b_3-b_1)(b_3-b_2)}-\frac{C_1^2}{(b_1-b_2)^2},\\ R_{1s1s}=\displaystyle\sum_b \frac{2B_{1b,s}^2}{(b_2-b_1)(b_2-b_3)}-\frac{C_1^2}{(b_1-b_3)^2}, \quad R_{asas}=\frac{2B_{1a,s}^2-C_1^2}{(b_1-b_2)(b_1-b_3)},\\ R_{asbs}=A_{ab}=\dfrac{2B_{1s,a}B_{1s,b}}{(b_1-b_2)(b_1-b_3)}, \quad a\neq b, \quad A_{st}=\dfrac{2B_{1a,s}B_{1a,t}}{(b_1-b_2)(b_1-b_3)}, \quad s\neq t. \end{cases} \tag{4.30}$$

Repeat the above derivation as (4.28), we have

$$\begin{cases} B_{as,tb}=\dfrac{1}{b_1-b_3}[B_{1a,s}B_{1b,t}+B_{1a,t}B_{1b,s}], & a\neq b,\ s\neq t,\\ B_{as,bt}=\dfrac{1}{b_1-b_2}[B_{1a,s}B_{1b,t}+B_{1a,t}B_{1b,s}], & a\neq b,\ s\neq t. \end{cases} \tag{4.31}$$

The equations (2.4) and (4.1) imply

$$R_{asbt}=0, \quad a\neq b,\ s\neq t.$$

Combining Ricci identity and (4.31) yields

$$B_{1a,s}B_{1b,t}+B_{1a,t}B_{1b,s}=0, \quad a\neq b,\ s\neq t. \tag{4.32}$$

Since $B|_{V_2}=b_2\mathrm{id}$ and $B|_{V_3}=b_3\mathrm{id}$, we can rechoose basis $\{E_2,\cdots,E_{m_2+1}\}$ in V_2 and $\{E_{m_2+2},\cdots,E_n\}$ in V_3 such that

$$A_{ab}=0, \quad a\neq b, \quad A_{st}=0, \quad s\neq t. \tag{4.33}$$

From the third formula of (4.30), we have

$$B_{1s,a}B_{1s,b}=0, \quad a\neq b, \quad B_{1a,s}B_{1a,t}=0, \quad s\neq t. \tag{4.34}$$

Combining (4.32) and (4.34), it follows that there exists at most one non-zero element in matrix $(B_{1a,s})$, $2\le a\le m_2+1$, $m_2+2\le s\le n$. Thus we can assume that

$$B_{1a,s}=0, \quad a\neq 2 \ \text{ or } \ s\neq n. \tag{4.35}$$

From (2.4), we have $R_{1s1s}-R_{asas}=(b_1-b_2)b_3+A_{11}-A_{aa}$. Combining (4.30), we can deduce

$$\begin{aligned} A_{11}-A_{aa}=&\sum_b \frac{2B_{1b,s}^2}{(b_2-b_1)(b_2-b_3)}-\frac{2B_{1a,s}^2}{(b_1-b_2)(b_1-b_3)}\\ &+\frac{C_1^2(b_2-b_3)}{(b_1-b_2)(b_1-b_3)^2}+(b_2-b_1)b_3, \quad m_2+2\le s\le n. \end{aligned} \tag{4.36}$$

Now we take $a = 3$ in (4.36), and noting (4.35), we have

$$A_{11} - A_{33} = \frac{2B_{12,s}^2}{(b_2 - b_1)(b_2 - b_3)} + \frac{C_1^2(b_2 - b_3)}{(b_1 - b_2)(b_1 - b_3)^2} + (b_2 - b_1)b_3. \tag{4.37}$$

We take $s = m_2 + 2$ and $s = n$ in (4.37), respectively and obtain the following two equations:

$$A_{11} - A_{33} = \frac{C_1^2(b_2 - b_3)}{(b_1 - b_2)(b_1 - b_3)^2} + (b_2 - b_1)b_3,$$
$$A_{11} - A_{33} = \frac{2B_{12,n}^2}{(b_2 - b_1)(b_2 - b_3)} + \frac{C_1^2(b_2 - b_3)}{(b_1 - b_2)(b_1 - b_3)^2} + (b_2 - b_1)b_3.$$

Comparing the two equations, we have $B_{12,n} = 0$. Thus, the matrix $(B_{1a,s}),\ 2 \le a \le m_2 + 1,\ m_2 + 2 \le s \le n$ is zero. From (4.27), we deduce that

$$(A_{ij}) = \mathrm{diag}(A_{11}, \cdots, A_{nn}). \tag{4.38}$$

Since x is a Möbius homogeneous hypersurface, the eigenvalues $A_{ii}, 1 \le i \le n$ of the tensor A are constant. Using $\mathrm{d}A_{ij} + \sum\limits_m A_{mj}\omega_{mi} + \sum\limits_m A_{im}\omega_{mj} = \sum\limits_m A_{ij,m}\omega_m$, we obtain

$$A_{ii,j} = 0, \quad A_{1a,a} = (A_{11} - A_{aa})\frac{C_1}{b_2 - b_1}. \tag{4.39}$$

Combining (2.1), (4.37) and (4.39), we deduce

$$\begin{aligned} -b_2C_1 = A_{aa,1} - A_{1a,a} = -A_{1a,a} &= -(A_{11} - A_{aa})\frac{C_1}{b_2 - b_1} \\ &= -\Big[\frac{C_1^2(b_2 - b_3)}{(b_1 - b_2)(b_1 - b_3)^2} + (b_2 - b_1)b_3\Big]\frac{C_1}{b_2 - b_1}, \end{aligned}$$

which implies

$$C_1\Big[1 + \frac{C_1^2(b_2 - b_3)}{(b_1 - b_2)^2(b_1 - b_3)^2}\Big] = 0,$$

and $C_1 = 0$. Thus $C = 0$, and we finish the proof of Lemma 4.3.

Since the dimension of x $n \ge 4$, From Lemmas 4.1–4.3, we obtain Proposition 4.1.

5 The Proof of the Main Theorem 1.1

When the dimension of the hypersurfaces $n \ge 4$. From Proposition 4.1, we know that the Möbius form of the Möbius homogeneous hypersurfaces vanishes when x has three distinct principal curvatures. On the other hand, the Möbius principal curvatures of the Möbius homogeneous hypersurfaces are constant. Thus the Möbius homogeneous hypersurfaces with three distinct principal curvatures are Möbius isoparametric hypersurfaces. In [5], the authors classified the Möbius isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ with three distinct principal curvatures.

Theorem 5.1 (see [5]) *Let $x : M^n \to \mathbb{S}^{n+1}$ be a Möbius isoparametric hypersurface with three distinct principal curvatures. Then x is Möbius equivalent to an open part of one of the following hypersurfaces:*

(1) *The image of σ of the warped product embedding*

$$\widetilde{x} : S^p(a) \times S^q(\sqrt{1 - a^2}) \times R^+ \times R^{n-p-q-1} \to \mathbb{R}^{n+1},$$

with $p \geq 1,\ q \geq 1,\ p+q \leq n-1$ *defined by*

$$\widetilde{x}(u,v,t,w) = (tu, tv, w), \quad u \in S^p(a),\ v \in S^q(\sqrt{1-a^2}),\ t \in R^+,\ w \in \mathbb{R}^{n-p-q-1}.$$

(2) *The image of* σ *of the cone* $\widetilde{x} : M^{n-1} \times R^+ \to \mathbb{R}^{n+1}$ *defined by* $\widetilde{x}(x,t) = tx$, *where* $t \in R^+$ *and* $x : M^{n-1} \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ *is the Cartan's minimal isoparametric hypersurface in* $\mathbb{S}^n$ *with three distinct principal curvatures.*

(3) *The Euclidean isoparametric hypersurfaces in* $\mathbb{S}^{n+1}$ *with three distinct principal curvatures.*

Using Proposition 1.1, we know that the isoparametric hypersurfaces in Theorem 5.1 is Möbius homogeneous. Thus when the Möbius homogeneous hypersurfaces have three distinct principal curvatures, the main Theorem 1.1 holds. Combining the result in Section 3, we finish the proof of the main Theorem 1.1.

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References

[1] Akivis, M. A. and Goldberg, V. V., A conformal differential invariants and the conformal rigidity of hypersurfaces, *Proc. Amer. Math. Soc.*, **125**, 1997, 2415–2424.

[2] Cartan, E., Sur des familes remarquables d'hypersurfaces isoparamétriques dans les espace sphériques, *Math. Z.*, **45**, 1939, 335–367.

[3] Cecil, T. E., Lie Sphere Geometry: With Applications to Submanifolds, Springer-Verlag, New York, 1992.

[4] Guo, Z., Li, H. and Wang, C. P., The Möbius characterizations of Willmore tori and Veronese submanifolds in unit sphere, *Pacific J. Math.*, **241**, 2009, 227–242.

[5] Hu, Z. J. and Zhai, S. J., Möbius isoparametric hypersurfaces with three distinct principal curvatures, II, *Pacific J. Math.*, **249**, 2011, 343–370.

[6] Li, T. Z., Ma, X. and Wang, C. P., Möbius homogeneous hypersurfaces with two distinct principal curvatures in $\mathbb{S}^{n+1}$, *Ark. Mat.*, **51**, 2013, 315–328.

[7] Li, X. X. and Zhang, F. Y., On the Blaschke isoparametric hypersurfaces in the unit sphere, *Acta Math. Sin.* (*Engl. Ser.*), **25**, 2009, 657–678.

[8] Liu, H., Wang, C. P. and Zhao, G. S., Möbius isotropic submanifolds in S^n, *Tohoku Math. J.*, **53**, 2001, 553–569.

[9] O'Neil, B., Semi-Riemannian Geometry, Academic Press, New York, 1983.

[10] Sulanke, R., Möbius geometry V: Homogeneous surfaces in the Möbius space S^3, Topics in Differential Geometry, Vol. I, II, Debrecen, 1984, 1141–1154, Colloq. Math. Soc. János Bolyai, 46, North-Holland, Amsterdam, 1988.

[11] Wang, C. P., Möbius geometry of submanifolds in S^n, *Manuscripta Math.*, **96**, 1998, 517–534.

[12] Wang, C. P., Möbius geometry for hypersurfaces in S^4, *Nagoya Math. J.*, **139**, 1995, 1–20.