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Dimension of Slices Through Fractals with Initial Cubic Pattern*

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Abstract In this paper, the Hausdorff dimension of the intersection of self-similar fractals in Euclidean space $\mathbb{R}^n$ generated from an initial cube pattern with an $(n-m)$-dimensional hyperplane V in a fixed direction is discussed. The authors give a sufficient condition which ensures that the Hausdorff dimensions of the slices of the fractal sets generated by "multi-rules" take the value in Marstrand's theorem, i.e., the dimension of the self-similar sets minus one. For the self-similar fractals generated with initial cube pattern, this sufficient condition also ensures that the projection measure μ_V is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^m$. When $\mu_V \ll \mathcal{L}^m$, the connection of the local dimension of μ_V and the box dimension of slices is given.

Keywords Slice, Self-similar set, Dimension, Fractal
2000 MR Subject Classification 28A80, 37C45

1 Introduction

1.1 Dimension of slices

The intersections of Borel sets in $\mathbb{R}^n$ with $(n-m)$-dimensional subspace in random directions are studied in many publications. The following Marstrand's theorem (see [12–13]) is well known: Suppose that $A \subset \mathbb{R}^n$ is a Borel set with $0 < \mathcal{H}^s(A) < \infty$ and $m < s < n$, then for $\gamma_{n,n-m}$-almost all $(n-m)$-dimensional subspace V and $\mathcal{H}^s$-almost all $x \in A$,

$$\dim_H[A \cap (V + x)] = s - m.$$

For a fixed V, let $\Lambda_V = \{a \in V^\perp : A \cap (V + a) \neq \varnothing\}$. Wen and Xi [19] studied the slices of scaling self-similar set

$$E = \bigcup_i (r_i E + b_i) \quad (r_i \in (0,1)\ \forall i)$$

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in $\mathbb{R}^n$, and obtained that for a fixed $(n-m)$-dimensional subspace V, there are constants $c_1 \le c_2 \le c_3$ such that for $\mathcal{H}^m$-almost all $a \in \Lambda_V$,

$$\dim_H[E \cap (V+a)] = c_1, \quad \underline{\dim}_B[E \cap (V+a)] = c_2, \quad \overline{\dim}_B[E \cap (V+a)] = c_3.$$

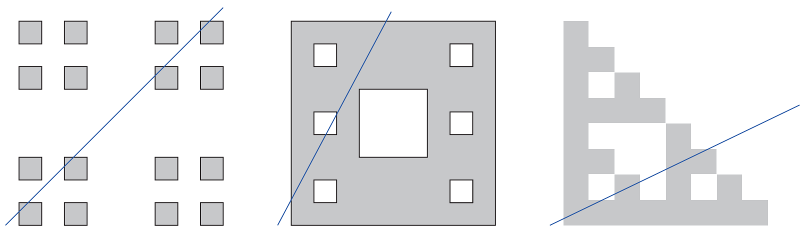
The following examples are intersections of planar sets with lines in the fixed direction.

Example 1.1 Suppose that C is the Cantor ternary set, then $\dim_H C \times C = \frac{\log 4}{\log 3}$ (see Figure 1). Let $L_{k,b} = \{(x,y) : y = kx + b\}$ be a line of slope k and $J_k = \{b \in \mathbb{R} : L_{k,b} \cap (C \times C) \neq \varnothing\}$. According to Marstrand's theorem, for $\mathcal{H}^1$-almost all $k \in \mathbb{R}$,

$$\dim_H[L_{k,b} \cap (C \times C)] = \frac{\log 4}{\log 3} - 1 \quad \text{for } \mathcal{H}^1 \text{ a.e. } b \in J_k. \tag{1.1}$$

However, (1.1) does not hold even for $k = 1$, since Hawkes, J. showed in [6] that for $\mathcal{H}^1$ a.e. $t \in [-1, 1]$,

$$\dim_H C \cap (C + t) = \frac{\log 2}{3 \log 3} < \frac{\log 4}{\log 3} - 1.$$



$\mathcal{C} \times \mathcal{C}$. The Sierpinski carpet. The right-angle Sierpinski gasket.

Figure 1 Example 1.1.

Example 1.2 Suppose that E is the Sierpinski carpet (see figure 1) with $\dim_H E = \frac{\log 8}{\log 3}$. Let $L_{k,b} = \{(x,y) : y = kx + b\}$ be a line of slope k and $J_k = \{b \in \mathbb{R} : L_{k,b} \cap E \neq \varnothing\}$. According to Marstrand's theorem, for $\mathcal{H}^1$-almost all $k \in \mathbb{R}$,

$$\dim_H E \cap L_{k,b} = \frac{\log 8}{\log 3} - 1 \quad \text{for } \mathcal{H}^1 \text{ a.e. } b \in J_k. \tag{1.2}$$

However, the equality (1.2) does not hold for $k \in \mathbb{Q}$, since Manning and Simon showed in [11] that for all $k \in \mathbb{Q}$,

$$\dim_H E \cap L_{k,b} < \frac{\log 8}{\log 3} - 1 \quad \text{for } \mathcal{H}^1 \text{ a.e. } b \in J_k.$$

Example 1.3 Suppose that Δ is the right-angle Sierpinski gasket (see Figure 1) with $\dim_H \Delta = \frac{\log 3}{\log 2}$. Let $L_{k,b} = \{(x,y) : y = kx + b\}$ be a line of slope k and $J_k = \{b \in \mathbb{R} : L_{k,b} \cap \Delta \neq \varnothing\}$. According to Marstrand's theorem, for $\mathcal{H}^1$-almost all $k \in \mathbb{R}$,

$$\dim_H \Delta \cap L_{k,b} = \frac{\log 3}{\log 2} - 1 \quad \text{for } \mathcal{H}^1 \text{ a.e. } b \in J_k. \tag{1.3}$$

Again, the equality (1.3) does not hold for $k \in \mathbb{Q}$. Recently, Báraány, Ferguson and Simon proved in [1] that for all $k \in \mathbb{Q}$,

$$\dim_H \Delta \cap L_{k,b} < \frac{\log 3}{\log 2} - 1 \quad \text{for } \mathcal{H}^1 \text{ a.e. } b \in J_k.$$

Remark 1.1 When the slices of a fractal take the value in Marstrand's theorem? The above examples show us that this is not an easy question.

1.2 Slices of self-similar fractals

Suppose that $p \in \mathbb{N}$ with $p \geq 2$, $\Omega = \{0, 1, (p-1)\}^n$ and the self-similar set satisfies

$$E = \bigcup_{v \in \Omega} \frac{E + v}{p}. \tag{1.4}$$

Or, we can see that

$$E = \Big\{ \sum_{i=1}^{\infty} \frac{v_i}{p^i} : v_i \in \Omega \quad \text{for all } i \Big\}. \tag{1.5}$$

Let V be an $(n-m)$-dimensional subspace of $\mathbb{R}^n$ such that its orthogonal complement

$$V^{\perp} = \operatorname{span}\{\alpha_1, \cdots, \alpha_m\} \quad \text{with } \alpha_i \in \mathbb{Q}^n \quad \text{for all } i. \tag{1.6}$$

Without loss of generality, we may assume that

$$\alpha_i \in \mathbb{Z}^n \quad \text{for all } i \text{ and the inner product } (\alpha_i, \alpha_j) = 0 \quad \text{for all } i \neq j. \tag{1.7}$$

Therefore, there are vectors $\beta_1, \cdots, \beta_{n-m} \in \mathbb{Z}^n$ such that $(\beta_i, \beta_j) = 0$ for all $i \neq j$. For $x \in \mathbb{R}^n$, write

$$(x, \boldsymbol{\alpha}) = ((x, \alpha_1), (x, \alpha_2), \cdots, (x, \alpha_m)).$$

Let $\alpha_i = (a_1^{(i)}, a_2^{(i)}, \cdots, a_n^{(i)}) \in \mathbb{Z}^n \backslash \{0\}$. Write

$$A_i^- = \sum_{a_j^{(i)} < 0} a_j^{(i)}, \quad A_i^+ = \sum_{a_j^{(i)} > 0} a_j^{(i)},$$

and set $A_i^- = 0$ if $\{j : a_j^{(i)} < 0\} = \varnothing$ and $A_i^+ = 0$ if $\{j : a_j^{(i)} > 0\} = \varnothing$.

Given $L = \mathbb{Z}^m \cap \big(\prod_{i=1}^{m} [A_i^-, A_i^+)\big)$ and $t \in \{0, 1, \cdots, (p-1)\}^m := \Theta$, let $M_t = (c_{l,l'}^t)_{l,l' \in L}$ be defined by

$$c_{l,l'}^t = \#\{v \in \Omega : pl + t - l' = (v, \boldsymbol{\alpha})\}. \tag{1.8}$$

Then the Lyapunov exponent for symmetric independent random product $\{M_\tau\}_{\tau \in \Theta}$ is a constant λ such that

$$\lim_{k \to \infty} \frac{\log \|M_{\tau_1} M_{\tau_2} \cdots M_{\tau_k}\|}{k} = \lambda \quad \text{for } \mu_0\text{-almost all } \tau_1 \tau_2 \cdots \in \Theta^{\infty},$$

where μ_0 is the symmetric Bernoulli measure on Θ^{∞} given by $\{\frac{1}{p^m}, \cdots, \frac{1}{p^m}\}^{\mathbb{N}}$.

Let $\Lambda = \{a \in V^{\perp} : E \cap (V + a) \neq \varnothing\}$.

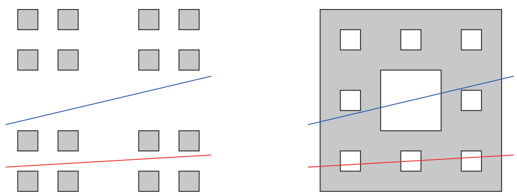
We will prove the following result.

Theorem 1.1 *Suppose that E is a self-similar set satisfying* (1.4), *and V is an $(n-m)$-dimensional subspace satisfying* (1.6). *Then for $\mathcal{H}^m$-almost all $a \in \Lambda$,*

$$\dim_B[E \cap (V+a)] = \dim_H[E \cap (V+a)] = \frac{\lambda}{\log p} \leq \dim_H E - m,$$

where λ is the Lyapunov exponent for the symmetric independent random product of $\{M_t\}_{t\in\Theta}$ defined in (1.8).

Remark 1.2 The box dimension of the intersection of the Sierpinski carpet with lines of rational slopes can be computed using the nature cover of the Sierpinski carpet directly, since if the intersection of the square $[0,1]\times[0,1]$ with a line is not empty, then the intersection of the Sierpinski carpet with the same line is not empty (see Figure 2). However, this is not true for general self-similar sets. Figure 2 shows that there exist lines that pass through the unit square but do not meet any point of the self-similar set. Therefore, we can not use the nature cover of the self-similar set directly, which complicates the computation of the box dimension of the slices.



$\mathcal{C}\times\mathcal{C}$. The Sierpinski carpet.

Figure 2

1.3 The case of absolute continuous $\boldsymbol{\mu_V}$

Let μ be the natural measure of E, i.e., $\mu = \frac{\mathcal{H}^d|_E}{\mathcal{H}^d(E)}$ where $d = \dim_H E$ and $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure. When E is a self-similar set satisfying (1.4), μ is the natural self-similar measure supported on E. Let $\mu_V = \mu \circ \mathrm{proj}_V^{-1}$ be the projection measure of μ by direction V, where $\mathrm{proj}_V(x) = (x, \boldsymbol{\alpha})$. It is easy to see that $\mathrm{spt}\mu \subset \Lambda$. The local dimension of this self-similar measure is defined as follows:

$$\dim_{\mathrm{loc}} \mu_V(x) = \lim_{\delta\to 0} \frac{\log \mu_V(B(x,\delta))}{\log \delta}, \tag{1.9}$$

where the existence of the limit on the right of the above equation is proved in [3, Theorem 2.12] by Feng and Hu. They also showed that the local-dimension is almost everywhere constant. Moreover, Young proved in [22] that this constant is the Hausdorff dimension of the measure, i.e., for μ_V almost all x,

$$\dim_{\mathrm{loc}} \mu_V(x) = \dim_H \mu_V = \inf\{\dim_H A : \mu_V(A) = 1\}.$$

Let $\mathcal{L}^m$ denote the m-dimensional Lebesgue measure.

Proposition 1.1 *Suppose that E is a self-similar set satisfying* (1.4) *and V is an $(n-m)$-dimensional subspace satisfying* (1.6)*. If $\mu_V \ll \mathcal{L}^m$, then for μ_V almost all $a \in \Lambda$, we have*

$$\dim_{\mathrm{loc}} \mu_V(a) = \dim_H E - \dim_B[E \cap (V+a)].$$

Remark 1.3 By Proposition 1.1 and [3, Theorem 2.12], if V is an $(n-m)$-dimensional subspace satisfying (1.6) and $\mu_V \ll \mathcal{L}^m$, then for μ_V almost all $a \in \Lambda$, we have

$$\dim_B[E \cap (V+a)] = \dim_H E - \dim_{\mathrm{loc}} \mu_V(a) \geq \dim_H E - m. \tag{1.10}$$

Our second result will show that under some suitable conditions, the equality in (1.10) will hold. First, we give the following s-star condition, which was introduced in [21] (see also [18]).

Definition 1.1 *A set $\Omega \subset \{0,1,\cdots,p-1\}^n$ $(n \geq 2)$ is said to satisfy s-star condition if there exists an integer $s > 1$, such that for all $\boldsymbol{t} \in \{0,1,\cdots,p-1\}^m$,*

$$\#\{v \in \Omega : (v, \boldsymbol{\alpha}) \equiv \boldsymbol{t} \mod p\} = s. \tag{$\star$}$$

Example 1.4 Suppose $n=2$, $m=1$ and $p=3$. Let $\boldsymbol{\alpha} = (-3,4)$ and $\Omega = \{(0,0),(1,1),(2,2),(2,0),(0,1),(0,2)\}$. Note that

$$\boldsymbol{\alpha} \cdot \Omega = \{0,1,2,-6,4,8\}.$$

Hence Ω satisfies the 2-star condition (see Figure 3).

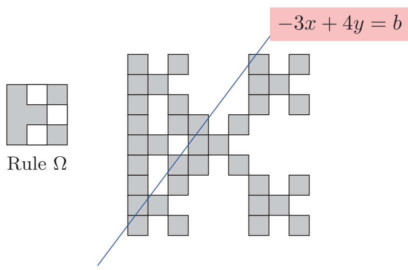


Figure 3 The intersection of the limit set generated by a rule satisfying the 2-star condition with the line $-3x+4y=b$ for some $b \in [0,1]$.

Theorem 1.2 *Suppose that E is a self-similar set satisfying* (1.4)*. If V satisfies $\mu_V \ll \mathcal{L}^m$, then for $\mathcal{L}^m$-almost all $a \in \Lambda$,*

$$\dim_B[E \cap (V+a)] = \dim_H E - m. \tag{1.11}$$

In addition, if the conditions in Theorem 1.1 *hold and Ω in* (1.4) *satisfies the s-star condition, then $\mu_V \ll \mathcal{L}^m$.*

Remark 1.4 Theorem 1.2 shows that the projection defined by $\mathrm{proj}_V(x) = (x, \boldsymbol{\alpha})$ is dimension conserving (see [5, Definition 1.1]). Precisely, let $\delta := \dim_B[E \cap (V+a)]$, then

$$\delta + \dim_H\{a \in \Lambda : \dim_H[E \cap (V+a)] \geq \delta\} \geq \dim_H E.$$

Namely,

$$\begin{aligned}&\dim_H\{a\in\Lambda:\dim_H[E\cap(V+a)]\geq\delta\}\\ \geq&\dim_H\{a\in\Lambda:\dim_B[E\cap(V+a)]=\dim_H[E\cap(V+a)]\geq\delta\}\\ \geq&\dim_H\mu_V=\dim_H E-\delta.\end{aligned}$$

Remark 1.5 In fact, the converse of the first part of Theorem 1.2 is true. That is, in case of self-similar sets satisfying (1.4), the slices take the typical value in Marstrand's theorem if and only if the projection measure is absolutely continuous with respect to the Lebesgue measure. This result is proved by Feng [4] in a more general case recently. When we turn to the Sierpinski carpet, which does not satisfy the s-star condition, Niu and Xi proved in [14] that the projection of the self-similar measure on Sierpinski carpet onto a line with rational slope is singular.

The s-star condition is easy to verify. However, the following example shows that it is not a necessary condition to ensure the continuity of the projection measure.

Example 1.5 Consider a planar set intersect with a line, i.e., $n=2$ and $m=1$. Let $p=6$, $\alpha=(5,1)$, $V^{\perp}=\text{span}\{\alpha\}$ and $\Omega=\{(0,0),(0,1),(0,3),(0,4),(1,1),(1,2),(1,4),(1,5),(2,2),(2,3),(3,0),(3,1)\}$. Then

$$\begin{aligned}\{(v,\alpha):v\in\Omega\}&=\{0,1,3,4,6,7,9,10,12,13,15,16\}\\ &\equiv\{0,1,3,4\}\bmod 6,\end{aligned}$$

which implies that Ω does not satisfy the s-star condition. We will prove in Section 6 that μ_V is absolutely continuous with respect to the 1-dimensional Lebesgue measure $\mathcal{L}^1$. Figure 4 is the first two steps in generating the self-similar set satisfying (1.4).

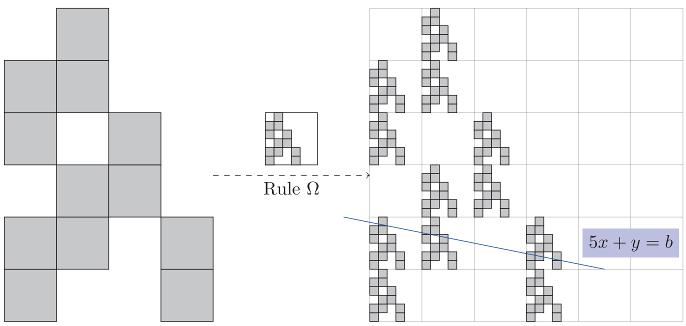


Figure 4 The case that the s-star condition does not hold.

1.4 Fractals generated by rules

Let $p\in\mathbb{N}$ with $p\geq 2$ and Q be the unit cube $[0,1]^n$ in $\mathbb{R}^n$. We regard $\Omega\subset\{0,1,\cdots,(p-1)\}^n$ as a "rule" for defining a subset of Q.

Definition 1.2 (see [2]) *For a rule Ω, we replace any cube $S(Q)$ with $\#\Omega$ smaller cubes $\bigcup\limits_{v\in\Omega} S(p^{-1}(Q+v))$, where $S: x\mapsto ax+b$ $(0<a<1)$ is a similitude.*

The way that we construct a fractal by a sequence of rules $\Omega_1\Omega_2\Omega_3\cdots$ is as follows:

1. Start with Q and replace Q by $\#\Omega$ smaller cubes $\bigcup\limits_{v\in\Omega} p^{-1}(Q+v)$ of side length p^{-1}.

2. Inductively, for $k\geq 1$, replace each small cube of side length p^{-k} obtained in the k-th step by $\#\Omega$ smaller cubes of side length $p^{-(k+1)}$ using the rule Ω_{k+1}.

The above procedure leads to the limit set—a fractal. The precise definition is showed as follows.

Definition 1.3 *For any rule sequence $\omega=\Omega_1\Omega_2\Omega_3\cdots$, denote by $\omega|_k$ the prefix of ω of length k, i.e., $\omega|_k=\Omega_1\Omega_2\cdots\Omega_k$. For any $v\in\{0,1,\cdots,(p-1)\}^n$, let $f_v: x\mapsto \frac{x+v}{p}$ be a similitude on $\mathbb{R}^n$. For $k\geq 1$,*

$$E_{\omega|_k}=\bigcup_{v_1\cdots v_k\in\omega|_k} f_{v_1\cdots v_k}(Q) \text{ and } E_\omega=\bigcap_{k\geq 1} E_{\omega|_k},$$

where $f_{v_1\cdots v_k}=f_{v_k}\circ\cdots\circ f_{v_1}$.

Remark 1.6 The set $E_{\omega|_{k+1}}$ is generated by $E_{\omega|_k}$ with respect to the rule Ω_{k+1}, that is

$$E_{\omega|_{k+1}}=\bigcup_{v\in\Omega_{k+1}} f_v(E_{\omega|_k}).$$

Remark 1.7 If the rule sequence ω is chosen to be a constant sequence, say $\Omega\Omega\cdots$, then the limit set E_ω is a self-similar set satisfying the open set condition. In this case, for the sake of simplicity, we denote the sets E_ω and $E_{\omega|_k}$ by E and E_k respectively.

Example 1.6 Let $\Omega=\{0,2\}^2$ and $\omega=\Omega\Omega\cdots$, then the limit set E_ω is the self-similar set $C\times C$, where C is the Cantor ternary set (see Figure 5).

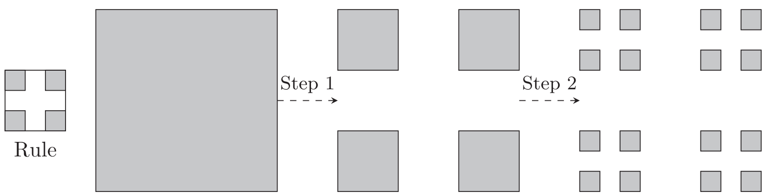


Figure 5 $\mathcal{C}\times\mathcal{C}$.

Example 1.7 Let $\Omega=\{0,1,2\}^2\backslash\{(1,1)\}$. The Sierpinski carpet $E\subset\mathbb{R}^2$ is generated by only one rule Ω (see Figure 6).

Example 1.8 Let $\Omega=\{0,2\}^2$, $\Omega_0=\{(0,1),(1,0),(2,2)\}$ and $\omega=\Omega\Omega_0\Omega\cdots$. Figure 7 illustrates the first two steps in the construction of E_ω.

1.5 Slices of fractals generated by multi-rules

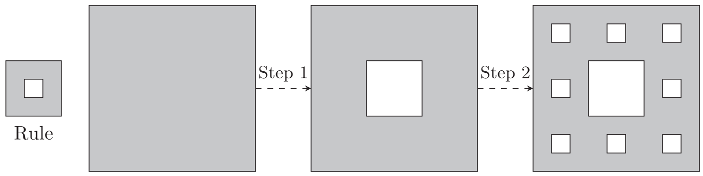


Figure 6 The Sierpinski Carpet.

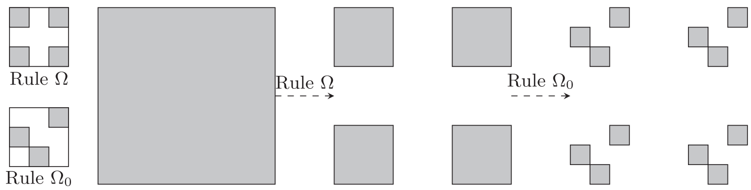


Figure 7 The limit set generated by the rule sequence $\Omega\Omega_0\Omega\cdots$.

Further, we discuss the box dimension for the following fractal sets which are generated by multi-rules. For any fixed $s \in [1, p^{n-1}]$, let

$$\mathcal{O}_s := \{\Omega \subset \{0, 1, \cdots, p-1\}^n : \Omega \text{ satisfies } s\text{-star condition}\}.$$

It is easy to see that $\#\mathcal{O} < \infty$. Then $\mathcal{O}_s$ can be represented as follows:

$$\mathcal{O}_s = \{\Omega^{(1)}, \cdots, \Omega^{(\#\mathcal{O})}\}.$$

Consider the infinite product space $(\mathcal{O}_s^\infty, P)$ equipped with a probability measure P. For any $\omega = \Omega_1\Omega_2\cdots \in \mathcal{O}_s^\infty$, let

$$F_\omega = \Big\{x = \sum_{i=1}^{\infty} \frac{v_i}{p^i} : v_i \in \Omega_i \quad \text{for all } i\Big\}. \tag{1.12}$$

Example 1.9 Let $\boldsymbol{\alpha} = (1, 3)$, $\Omega^{(1)} = \{0, 1, 2\}^2 \backslash \{(0, 0), (1, 2), (2, 0)\}$ and $\Omega^{(2)} = \{0, 1, 2\}^2 \backslash \{(0, 2), (1, 0), (2, 2)\}$. It is easy to verify that both $\Omega^{(1)}$ and $\Omega^{(2)}$ satisfy the 2-star condition. Suppose $\omega = \Omega^{(1)}\Omega^{(2)}\Omega^{(1)}\cdots$. The first two steps of E_ω are showed in Figure 8.

For $b \in \mathbb{R}^m$, let $\Pi_b = \{x \in \mathbb{R}^n : (x, \boldsymbol{\alpha}) = b\}$ be an $(n-m)$-dimensional hyperplane, and $\Lambda_{F_\omega} = \{b \in \mathbb{R}^m : F_\omega \cap \Pi_b \neq \varnothing\}$.

Theorem 1.3 *$\mathcal{H}^m(\Lambda_{F_\omega}) \geq 1$ and for every $b \in \mathbb{R}^m$,*

$$\dim_B \Big(\bigcup_{z \in b+\mathbb{Z}^m} F_\omega \cap \Pi_z\Big) = \frac{\log s}{\log p} = \dim_H F_\omega - m.$$

Remark 1.8 The projection of the Sierpinski carpet E in any direction is an interval. This good property ensures that if the intersection of the E_k with a line, say L, is not empty, then

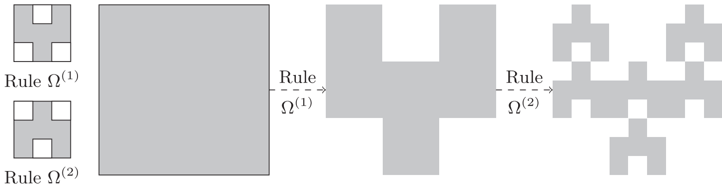


Figure 8 Example 1.9.

the intersection of E with the line is also not empty. However, the limit set F generated by rules may not hold this good property. That is to say, even if $L \cap F_k \neq \varnothing$, we can not deduce that $L \cap F \neq \varnothing$. In fact, for some line L and the limit set F, although $L \cap F_k \neq \varnothing$, the intersection $L \cap F_{k+1}$ may be empty (see Figure 9).

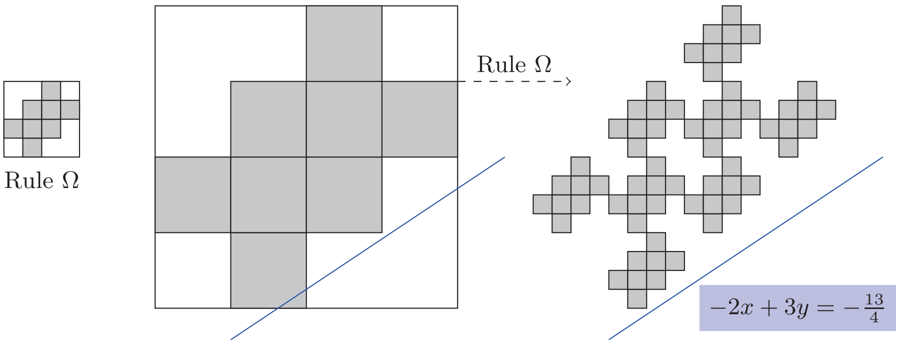


Figure 9 The intersection of the limit set E generated by the rule Ω with the line $-2x + 3y = -\frac{13}{4}$, where $\Omega = \{(0,1),(1,0),(1,1),(1,2),(2,1),(2,2),(2,3),(3,2)\}$.

Combining Theorem 1.1 and Theorem 1.3, we have the following corollary which tells us that when the s-star condition holds, then the Hausdorff dimension of slices of self-similar sets take the Mastrand's value.

Corollary 1.1 *Suppose that E is a self-similar set satisfying* (1.4) *and V is an $(n-m)$-dimensional subspace satisfying* (1.6). *If Ω in* (1.4) *satisfies the s-star condition, then $\lambda = \log s$ and for $\mathcal{H}^m$-almost all $a \in \Lambda$,*

$$\dim_B[E \cap (V+a)] = \dim_H[E \cap (V+a)] = \frac{\log s}{\log p} = \dim_H E - m.$$

This paper is organized as follows. Section 2 gives some preliminary information such as definitions and basic lemmas. The box dimension of slices is discussed in Section 3. The equivalence of the box dimension and the Hausdorff dimension of the slices and Theorem 1.1

are proved in Section 4. In Section 5, we discuss slices of the fractal generated by multi-rules satisfying the s-star condition, and prove Theorem 1.3. In the last section, we focus on the case that the projection measure is absolutely continuous with respect to the Lebesgue measure, and prove Theorem 1.2 and other remaining results.

2 Preliminaries

Fix a V such that $V^{\perp} = \mathrm{span}\{\alpha_1, \cdots, \alpha_m\}$ satisfying (1.7). Let $\alpha_i = (a_1^{(i)}, a_2^{(i)}, \cdots, a_n^{(i)}) \in \mathbb{Z}^n \backslash \{0\}$. Write

$$A_i^- = \sum_{a_j^{(i)}<0} a_j^{(i)}, \quad A_i^+ = \sum_{a_j^{(i)}>0} a_j^{(i)},$$

and set $A_i^- = 0$ if $\{j : a_j^{(i)} < 0\} = \varnothing$ and $A_i^+ = 0$ if $\{j : a_j^{(i)} > 0\} = \varnothing$. Write

$$J = \prod_{i=1}^{m} [A_i^-, A_i^+] \ (\subset \mathbb{R}^m).$$

For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, denote $\|x\|_1 = \sum\limits_{i=1}^{n} |x_i|$ and write

$$(x, \boldsymbol{\alpha}) = ((x, \alpha_1), (x, \alpha_2), \cdots, (x, \alpha_m)).$$

For $b \in \mathbb{R}^m$, let Π_b be an $(n-m)$-dimensional hyperplane in $\mathbb{R}^n$ defined by

$$\Pi_b = \{x \in \mathbb{R}^n : (x, \boldsymbol{\alpha}) = b\}.$$

An easy observation shows that

$$[0,1]^n \cap \Pi_b \neq \varnothing \quad \text{if and only if } b \in J. \tag{2.1}$$

For $E = \bigcup\limits_{v \in \Omega} \frac{E+v}{p}$, we denote the slices

$$E_b = E \cap \Pi_b.$$

Given $v \in \{0, 1, \cdots, (p-1)\}^m$, let

$$f_v(x) = \frac{x+v}{p}.$$

We define $T_v : \mathbb{R}^m \to \mathbb{R}^m$ by

$$T_v(x) = px - (v, \boldsymbol{\alpha}),$$

and $S_v(x) = T_v^{-1}(x) = \frac{x + (v, \boldsymbol{\alpha})}{p}$. Note that

$$f_v^{-1}(\Pi_b) = \Pi_{T_v(b)}.$$

Using the above formula, we can check that for any set $A \subset \mathbb{R}^n$,

$$f_{v_1 \cdots v_k}(A) \cap \Pi_z = f_{v_1 \cdots v_k}(A \cap \Pi_{T_{v_k \cdots v_1}(z)}), \tag{2.2}$$

where $f_{v_1\cdots v_k} = f_{v_1} \circ \cdots \circ f_{v_k}$ and $T_{v_k\cdots v_1} = T_{v_k} \circ \cdots \circ T_{v_1}$.

When $v \in \{0, 1, \cdots, (p-1)\}^n$ and $b = (b_1, \cdots, b_m) \in J$, we have

$$(v, \alpha_i) \in [(p-1)A_i^-, (p-1)A_i^+]$$

and $b_i \in [A_i^-, A_i^+]$. Hence for any $b \in J$,

$$(S_v(b))_i = \frac{b_i + (v, \alpha_i)}{p} \in [A_i^-, A_i^+],$$

which implies

$$S_v(J) \subset J. \tag{2.3}$$

Noting that $T_v = S_v^{-1}$, we obtain that

$$T_v(J^c) \subset J^c. \tag{2.4}$$

For any $b \in \mathbb{R}^m$, let

$$\Gamma_b = J \cap (b + \mathbb{Z}^m)\ (\subset J). \tag{2.5}$$

Then $\Gamma_b = \Gamma_{b'}$ if $b \equiv b'$ (mod 1). For any $b \in \mathbb{R}^m$, we have

$$\prod_{i=1}^m (A_i^+ - A_i^-) \leq \#\Gamma_b \leq \prod_{i=1}^m (A_i^+ - A_i^- + 1).$$

Since $(v, \boldsymbol{\alpha}) \in \mathbb{Z}^m$, we obtain the following lemma.

Lemma 2.1 *If $z \in \Gamma_b$ and $T_v(z) \in J$, then $T_v(z) \in \Gamma_{pb}$.*

For $b \in \mathbb{R}$, the integer matrix $M(b) = (c_{\mathbf{i},\mathbf{j}})_{\mathbf{i}\in\Gamma_b, \mathbf{j}\in\Gamma_{pb}}$ is defined by

$$c_{\mathbf{i},\mathbf{j}} = \#\{v \in \Omega : T_v(\mathbf{i}) = \mathbf{j}\}, \tag{2.6}$$

where $c_{\mathbf{i},\mathbf{j}}$ is the number of reduced copies of $E_{\mathbf{j}}$ contained in $E_{\mathbf{i}}$ as in (2.11). Let

$$D = \{b = (b_1, \cdots, b_m) \in \mathbb{R}^m : p^k b_i \notin \mathbb{Z} \quad \text{for all integers } k \geq 0 \text{ and } i \leq m\},$$

and $L = \mathbb{Z}^m \cap \big(\prod_{i=1}^m [A_i^-, A_i^+]\big)$. Then for any $b \in J \cap D$, there exists $l = (l_1, \cdots, l_m) \in L$ so that

$$b \in \prod_{i=1}^m (l_i, l_i + 1) =: I_l.$$

It is easy to see that

$$\mathcal{H}^m(D^c) = 0, \tag{2.7}$$

and $pD \subset D$, $pD^c \subset D^c$.

For $b \in D$, we have

$$\#\Gamma_b = \prod_{i=1}^m (A_i^+ - A_i^-) = \#L,$$

and we can list the elements of Γ_b, i.e.,

$$\Gamma_b = \{\Gamma_b(l)\}_{l\in L} \quad \text{with } \Gamma_b(l) \in I_l.$$

Now $M(b) = (c_{l,l'})_{l,l'\in L}$ is a $\#L \times \#L$ integer matrix for any $b \in D$, where

$$c_{l,l'} = \#\{v \in \Omega : T_v(\Gamma_b(l)) = \Gamma_{pb}(l')\}. \tag{2.8}$$

Since

$$M(b) = M(b') \quad \text{if } b \equiv b' \pmod 1 \quad \text{with } b, b' \in D, \tag{2.9}$$

without loss of generality, we only focus on $M(b)$ for $b \in [0,1]^m$.

For all $t = (t_1, \cdots, t_m) \in \{0, 1, \cdots, (p-1)\}^m$, let

$$c(t) = \prod_{i=1}^m \Big(\frac{t_i}{p}, \frac{t_i+1}{p}\Big).$$

Recall that $I_l = \prod\limits_{i=1}^m (l_i, l_i+1) = l + (0,1)^m$.

Lemma 2.2 *Given $t \in \{0, 1, \cdots, (p-1)\}^m$, for all $l \in L$,*

$$M(b) \textit{ is constant on } b \in D \cap c(t).$$

Further, for $b \in D \cap c(t)$, we have $M_t := M(b) = (c^t_{l,l'})_{l,l'\in L}$, where

$$c^t_{l,l'} = \#\{v \in \Omega : pl + t - l' = (v, \boldsymbol{\alpha})\}. \tag{2.10}$$

Proof In fact, by (2.8), we have

$$\begin{aligned} c^t_{l,l'} &= \#\{v \in \Omega : T_v(l + c(t)) = I_{l'}) \\ &= \#\Big\{v \in \Omega : T_v\Big(l + \frac{t}{p} + \frac{(0,1)^m}{p}\Big) = l' + (0,1)^m\Big\} \\ &= \#\Big\{v \in \Omega : T_v\Big(l + \frac{t}{p}\Big) = l'\Big\} \\ &= \#\{v \in \Omega : pl + t - l' = (v, \boldsymbol{\alpha})\}. \end{aligned}$$

By Lemma 2.2, we get an equivalent definition of the integer matrix $M_t = (c^t_{l,l'})_{l,l'\in L}$ as follows:

$$c^t_{l,l'} = \#\{v \in \Omega : T_v(I^t_l) = I_{l'}\},$$

where $I^t_l := l + c(t) = \prod\limits_{i=1}^m \big(l_i + \frac{t_i}{p}, l_i + \frac{t_i+1}{p}\big)$.

We will use the nested structure of the slices $\{E_b\}_{b\in J}$, which is characterized in the following lemma.

Lemma 2.3 *For any $b \in J$,*

$$E_b = \bigcup_{v\in\Omega} f_v(E_{T_v(b)}) = \bigcup_{v\in\Omega} \frac{E_{T_v(b)} + v}{p}. \tag{2.11}$$

Further, for $k \geq 1$,

$$E_b = \bigcup_{v_1\cdots v_k \in \Omega^k} f_{v_1\cdots v_k}(E_{T_{v_k}\cdots v_1(b)}) = \bigcup_{T_{v_k\cdots v_1}(b)\in\Gamma_{p^k b}} f_{v_1\cdots v_k}(E_{T_{v_k\cdots v_1}(b)}). \tag{2.12}$$

Proof Notice that $f_v^{-1}(\Pi_b) = \Pi_{T_v(b)}$. Hence for any $b \in J$, we have

$$\begin{aligned} E_b = E \cap \Pi_b &= \Big[\bigcup_{v\in\Omega} f_v(E) \Big] \cap \Pi_b \\ &= \bigcup_{v\in\Omega} [f_v(E) \cap \Pi_b] = \bigcup_{v\in\Omega} f_v(E \cap f_v^{-1}(\Pi_b)) \\ &= \bigcup_{v\in\Omega} f_v(E \cap \Pi_{T_v(b)}) = \bigcup_{v\in\Omega} f_v(E_{T_v(b)}). \end{aligned}$$

By induction, we have

$$E_b = \bigcup_{v_1\cdots v_k \in \Omega^k} f_{v_1\cdots v_k}(E \cap \Pi_{T_{v_k\cdots v_1}(b)}).$$

Further, if $T_{v_k\cdots v_1}(b) \in J^c$, then $[0,1]^n \cap \Pi_{T_{v_k\cdots v_1}(b)} = \varnothing$ by (2.1), and thus

$$E \cap \Pi_{T_{v_k\cdots v_1}(b)} = \varnothing,$$

which implies $E_b = \bigcup\limits_{T_{v_k\cdots v_1}(b)\in\Gamma_{p^k b}} f_{v_1\cdots v_k}(E_{T_{v_k\cdots v_1}(b)})$.

Remark 2.1 Some $E_{T_v(b)}$ in (2.11) may be empty.

When is the slice E_b non-empty?

Lemma 2.4 *For any $b \in J$,*

$$E_b \neq \varnothing \iff \exists\ v_1 v_2 \cdots v_k \cdots \in \Omega^\infty,\ \textit{s.t.}\ T_{v_k\cdots v_1}(b) \in J \quad \textit{for all } k \geq 1.$$

Proof The sufficiency. By Lemma 2.3, the non-empty set

$$E_b = \bigcup_{v\in\Omega} f_v(E \cap \Pi_{T_v(b)}).$$

Then there exists $v_1 \in \Omega$ such that $E \cap \Pi_{T_{v_1}(b)} \neq \varnothing$. Since $E \subset [0,1]^n$, we have $[0,1]^n \cap \Pi_{T_v(b)} \neq \varnothing$, which implies $T_{v_1}(b) \in J$ due to (2.1).

Inductively, if we get $v_1 \cdots v_k \in \Omega^k$ such that $E \cap \Pi_{T_{v_k\cdots v_1}(b)} \neq \varnothing$, then the non-empty set

$$E \cap \Pi_{T_{v_k\cdots v_1}(b)} = \bigcup_{v\in\Omega} f_v(E \cap \Pi_{T_{v v_k\cdots v_1}(b)}).$$

Then there exists $v_{k+1} \in \Omega$ such that $E \cap \prod\limits_{T_{v_{k+1} v_k\cdots v_1}(b)} \neq \varnothing$. In the same way, we have $T_{v_{k+1}v_k\cdots v_1}(b) \in J$. Then we can obtain an infinite word in Ω^∞.

The necessity. Suppose $v_1 v_2 \cdots v_k \cdots \in \Omega^\infty$, s.t. $T_{v_k\cdots v_1}(b) \in J$ for all $k \geq 1$. Then we claim that

$$\sum_{i=1}^\infty \frac{v_i}{p^i} \in E = E \cap \Pi_b.$$

Certainly, $\sum\limits_{i=1}^\infty \frac{v_i}{p^i} \in E$, we only need to show that $\sum\limits_{i=1}^\infty \frac{v_i}{p^i} \in \Pi_b$, i.e.,

$$b = \Big(\sum_{i=1}^\infty \frac{v_{i+k-1}}{p^i}, \boldsymbol{\alpha} \Big). \tag{2.13}$$

Notice that for any k,

$$\begin{aligned}\Big\|b-\Big(\sum_{i=1}^{k}\frac{v_i}{p^i},\boldsymbol{\alpha}\Big)\Big\|_1&=\Big\|\frac{pb-(v_k,\boldsymbol{\alpha})}{p}-\Big(\sum_{i=2}^{k}\frac{v_i}{p^i},\alpha\Big)\Big\|_1\\&=\Big\|\frac{1}{p}T_{v_k}(b)-\Big(\sum_{i=2}^{k}\frac{v_i}{p^i},\alpha\Big)\Big\|_1\\&=\cdots=\Big\|\frac{T_{v_k\cdots v_1}(b)}{p^k}\Big\|_1\\&\le\frac{\sum\limits_{i=1}^{m}\max\{A_i^+,|A_i^-|\}}{p^k}\to 0\quad\text{as } k\to\infty.\end{aligned}$$

Therefore we obtain (2.13).

3 Box Dimension of Slices

For $b\in D$, let $N_k(b)$ be the number of p-adic cubes of side length p^{-k} which intersect $\bigcup\limits_{z\in\Gamma_b}[E\cap\Pi_z]$. Denote

$$\begin{aligned}V_k(b)&=\sum_{z\in\Gamma_b}\#\{v_1\cdots v_k\in\Omega^k: f_{v_1\cdots v_k}(E)\cap\Pi_z\neq\varnothing\},\\U_k(b)&=\sum_{z\in\Gamma_b}\#\{v_1\cdots v_k\in\Omega^k: f_{v_1\cdots v_k}([0,1]^n)\cap\Pi_z\neq\varnothing\}.\end{aligned}$$

It is easy to see that

$$V_k(b)\le N_k(b). \tag{3.1}$$

We also have

$$U_k(b)=\sum_{z\in\Gamma_b}\#\{v_1\cdots v_k\in\Omega^k: T_{v_k\cdots v_1}(z)\in J\}, \tag{3.2}$$

since

$$\begin{aligned}f_{v_1\cdots v_k}([0,1]^n)\cap\Pi_z\neq\varnothing&\iff [0,1]^n\cap\Pi_{T_{v_k\cdots v_1}(z)}\neq\varnothing\\&\iff T_{v_k\cdots v_1}(z)\in J\end{aligned}$$

due to (2.1)–(2.2).

Moreover, we have the following auxiliary lemma about $U_k(b)$ which will be used in estimating the upper bound of the box dimension of the slices.

Lemma 3.1

$$\begin{aligned}U_k(b)&=\sum_{z\in\Gamma_b}\sum_{v_1\cdots v_k\in\Omega^k}\mathbf{1}_{S_{v_1\cdots v_k}(J)}(z),\\\frac{1}{\mathcal{L}^m(J)}\int_J U_k(b)\mathrm{d}\mathcal{L}^m&=c\Big(\frac{\#\Omega}{p^m}\Big)^k,\end{aligned}$$

where $c=\prod\limits_{i=1}^{m}(A_i^+-A_i^-)$ *is a constant.*

Proof By (3.2), we have

$$\begin{aligned}
U_k(b) &= \sum_{z\in\Gamma_b} \#\{v_1\cdots v_k \in \Omega^k : T_{v_k\cdots v_1}(z) \in J\} \\
&= \sum_{z\in\Gamma_b} \#\{v_1\cdots v_k \in \Omega^k : z \in S_{v_1\cdots v_k}(J)\} \\
&= \sum_{z\in\Gamma_b} \sum_{v_1\cdots v_k\in\Omega^k} \mathbf{1}_{S_{v_1\cdots v_k}(J)}(z)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\mathcal{L}^m(J)}\int_J U_k(b)\mathrm{d}\mathcal{L}^m &= \frac{1}{\mathcal{L}^m(J)} \sum_{v_1\cdots v_k\in\Omega^k} \int_J \sum_{z\in\Gamma_b} \mathbf{1}_{S_{v_1\cdots v_k}(J)}(z)\mathrm{d}\mathcal{L}^m \\
&= \frac{1}{\mathcal{L}^m(J)} \sum_{v_1\cdots v_k\in\Omega^k} \Big\{ \int_{J\cap D} \sum_{z\in\Gamma_b} \mathbf{1}_{S_{v_1\cdots v_k}(J)}(z)\mathrm{d}\mathcal{L}^m \\
&\quad + \int_{J\setminus D} \sum_{z\in\Gamma_b} \mathbf{1}_{S_{v_1\cdots v_k}(J)}(z)\mathrm{d}\mathcal{L}^m \Big\} \\
&= \frac{1}{\mathcal{L}^m(J)} \sum_{v_1\cdots v_k\in\Omega^k} \Big\{ \Big(\prod_{i=1}^m (A_i^+ - A_i^-)\Big) \frac{\mathcal{L}^m(J)}{p^{mk}} + 0 \Big\} \\
&= \Big(\prod_{i=1}^m (A_i^+ - A_i^-)\Big)\Big(\frac{\#\Omega}{p^m}\Big)^k.
\end{aligned}$$

Lemma 3.2 $N_k(b) \le 3^n U_k(b)$.

Proof Given a p-adic cube of side length p^{-k} intersecting $E\cap\Pi_z$, we denote it by B, then $B\cap E\cap\Pi_z \neq \varnothing$, i.e.,

$$B \cap \Big(\bigcup_{v_1\cdots v_k\in\Omega^k} f_{v_1\cdots v_k}(E)\Big) \cap \Pi_z \neq \varnothing.$$

Then for some $v_1\cdots v_k \in \Omega^k$,

$$\begin{aligned}
& B\cap f_{v_1\cdots v_k}(E)\cap \Pi_z \neq \varnothing \\
\Longrightarrow\ & B\cap f_{v_1\cdots v_k}(E) \neq \varnothing \text{ and } f_{v_1\cdots v_k}(E)\cap\Pi_z \neq \varnothing \\
\Longrightarrow\ & B\cap f_{v_1\cdots v_k}([0,1]^n) \neq \varnothing \text{ and } f_{v_1\cdots v_k}([0,1]^n)\cap\Pi_z \neq \varnothing,
\end{aligned}$$

which implies $N_k(b) \le 3^n U_k(b)$.

For a matrix $M = (d_{l,l'})_{l,l'\in L}$, denote its norm $\|M\|_1 = \sum\limits_{l,l'\in L} |d_{l,l'}|$.

The following lemma shows us how to compute $U_k(b)$.

Lemma 3.3 *For any integers $k \ge 1$, we have*

$$U_k(b) = \|M(b)M(pb)\cdots M(p^{k-1}b)\|_1.$$

Proof First, by (3.2), we only need to show that

$$\|M(b)M(pb)\cdots M(p^{k-1}b)\|_1 = \sum_{z\in\Gamma_b} \#\{v_1\cdots v_k \in \Omega^k : T_{v_k\cdots v_1}(z) \in J\}.$$

In fact, we obtain that

$$
\begin{aligned}
&\|M(b)M(pb)\cdots M(p^{k-1}b)\|_1\\
=&\sum_{(\mathbf{i}_1,\cdots,\mathbf{i}_k)\in\Gamma_b\times\cdots\times\Gamma_{p^{k-1}b}}\Big(\prod_{h=1}^{t-1}\#\{v_h\in\Omega : T_{v_h}(\mathbf{i}_h)=\mathbf{i}_{h+1}\}\Big)\\
=&\sum_{(\mathbf{i}_1,\cdots,\mathbf{i}_k)\in\Gamma_b\times\cdots\times\Gamma_{p^{k-1}b}}\#\{v_1\cdots v_k\in\Omega^k : T_{v_h}(\mathbf{i}_h)=\mathbf{i}_{h+1}\ \forall\ h\}\\
=&\sum_{z\in\Gamma_b}\#\{v_1\cdots v_k\in\Omega^k : T_{v_1}(z)\in\Gamma_{pb},\ T_{v_2v_1}(z)\in\Gamma_{p^2b},\cdots,T_{v_k\cdots v_1}(z)\in\Gamma_{p^kb}\}\\
=&\sum_{z\in\Gamma_b}\#\{v_1\cdots v_k\in\Omega^k : T_{v_k\cdots v_1}(z)\in\Gamma_{p^kb}\},
\end{aligned}
$$

where the last equality follows from (2.4).

Lemma 3.4 *For $\mathcal{L}^m$ almost all $b\in J$,*

$$\lim_{k\to\infty}\frac{\log U_k(b)}{k}=\lambda\le\log\Big(\frac{\#\Omega}{p^m}\Big).$$

Proof By the definition of the set D, we know that D mod 1 is a set of full measure contained in $(\mathbb{R}^n/(\text{mod }1),\mathcal{L}^m)$ which is an ergodic dynamic system under the transformation $x\mapsto px$. Moreover,

$$
\begin{aligned}
\log U_{k+l}(b)&=\log\|M(b)M(pb)\cdots M(p^{k+l-1}b)\|_1\\
&\le\log(\|M(b)M(pb)\cdots M(p^{k-1}b)\|_1\|M(b^{k+l})M(pb)\cdots M(p^{k+l-1}b)\|_1)\\
&\le\log\|M(b)M(pb)\cdots M(p^{k-1}b)\|_1\\
&\quad+\log\|M(p^kb)M(p^{k+1}b)\cdots M(p^{k+l-1}b)\|_1\\
&=\log U_k(b)+\log U_l(p^kb).
\end{aligned}
$$

According to the sub-additive ergodic theorem (see [17, Theorem 10.1]), there exists a constant λ such that

$$
\begin{aligned}
\lambda&=\lim_{k\to\infty}\frac{\log U_k(b)}{k}\\
&=\lim_{k\to\infty}\frac{1}{k}\int\log U_k(b)\mathrm{d}\mathcal{L}^m(b)\\
&\le\lim_{k\to\infty}\frac{1}{k}\log\int U_k(b)\mathrm{d}\mathcal{L}^m(b)\quad\text{by the convexity of }\log(x)
\end{aligned}
$$

for $\mathcal{L}^m$ almost all $b\in$ mod 1. Therefore for $\mathcal{L}^m$ almost all $b\in J$,

$$
\begin{aligned}
\lambda&\le\lim_{k\to\infty}\frac{1}{k}\log\int U_k(b)\mathrm{d}\mathcal{L}^m(b)\\
&=\lim_{k\to\infty}\frac{1}{k}\log\Big[\frac{1}{\mathcal{L}^m(J)}\int_J U_k(b)\mathrm{d}\mathcal{L}^m(b)\Big]\\
&=\lim_{k\to\infty}\frac{1}{k}\log\Big[\Big(\prod_{i=1}^m(A_i^+-A_i^-)\Big)\Big(\frac{\#\Omega}{p^m}\Big)^k\Big]\quad\text{(by Lemma 3.1)}\\
&=\log\Big(\frac{\#\Omega}{p^m}\Big).
\end{aligned}
$$

Let $K = \{(x, \alpha) : x \in E\}$. We claim that

$$\mathcal{H}^m(\Lambda) > 0 \quad \text{if and only if } \mathcal{H}^m(K) > 0, \tag{3.3}$$

where $\Lambda = \{a \in V^\perp : E \cap (V + a) \neq \varnothing\}$. In fact, for $a \in V^\perp$,

$$a = \sum_{i=1}^{m} \frac{(a, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i,$$

then $f(\Lambda) = K$, where

$$f(a) = (a, \alpha)$$

is a bi-Lipschitz mapping from $V^\perp$ to $\mathbb{R}^m$ which ensures that claim (3.3).

Without loss of generality, we assume that

$$\mathcal{H}^m(K) > 0.$$

We also assume that the Lyapunov exponent

$$\lambda > 0.$$

Otherwise, we assume that $\lambda = 0$, then by the definition of the box dimension, we have

$$\begin{aligned} \overline{\dim}_B\Big(\bigcup_{z \in \Gamma_b} E_z\Big) &= \limsup_{k \to \infty} \frac{\log N_k(b)}{k \log p} \\ &\le \limsup_{k \to \infty} \frac{\log 3^n U_k(b)}{k \log p} \quad \text{(by Lemma 3.2)} \\ &= \limsup_{k \to \infty} \frac{\log U_k(b)}{k \log p}. \end{aligned}$$

We notice that for $\mathcal{H}^m$-almost all $b \in \mathbb{R}^m$,

$$\lim_{k \to \infty} \frac{\log U_k(b)}{k} = \lim_{k \to \infty} \frac{\log \|M(b)M(pb) \cdots M(p^{k-1}b)\|_1}{k} = \lambda = 0. \tag{3.4}$$

Therefore, for $\mathcal{H}^m$-almost all $b \in \mathbb{R}^m$,

$$0 \le \dim_H\Big(\bigcup_{z \in \Gamma_b} E_z\Big) = \overline{\dim}_B\Big(\bigcup_{z \in \Gamma_b} E_z\Big) \le 0.$$

Then Theorem 1.1 follows in this case.

Lemma 3.5 *If the Lyapunov exponent $\lambda > 0$, then*

$$\mathcal{H}^m\Big((0,1)^m \backslash \Big(\bigcup_{j \in L} (K - j)\Big)\Big) = 0.$$

Furthermore, for $\mathcal{H}^m$-almost all $b \in (0,1)^m$,

$$\bigcup_{j \in L} E_{b+j} = \bigcup_{z \in \Gamma_b} E_z \neq \varnothing.$$

Proof Fix $\varepsilon \in (0, \lambda)$. Then by (3.2) and Lemma 3.3, for $\mathcal{H}^m$-almost all $b \in (0,1)^m$, there exists $k(b)$ such that

$$U_k(b) = \sum_{z\in\Gamma_b} \#\{v_1 \cdots v_k \in \Omega^k : T_{v_k\cdots v_1}(z) \in J\} \geq (\lambda - \varepsilon)^k \quad \text{for any } k \geq k(b). \tag{3.5}$$

On the other hand, it follows from Lemma 2.4 that

$$\begin{aligned} \bigcup_{z\in\Gamma_b} E_z = \varnothing \iff & \sum_{z\in\Gamma_b} \#\{v_1 \cdots v_k \in \Omega^k : T_{v_k\cdots v_1}(z) \in J\} = 0 \quad \text{for some } k \geq 1, \\ \iff & U_k(b) = 0 \quad \text{for some } k \geq 1, \end{aligned}$$

which contradicts (3.5).

For $b \in D \cap [0,1]^m$, we suppose

$$b = \frac{t_1}{p} + \frac{t_2}{p^2} + \cdots + \frac{t_k}{p^k} + \frac{t_{k+1}}{p^{k+1}} + \cdots,$$

where $t_i \in \{0, 1, \cdots, (p-1)\}^m$ for all i. Let

$$\{p^k b\} = \frac{t_{k+1}}{p} + \frac{t_{k+2}}{p^2} + \cdots.$$

Assume that

$$T_{v_k\cdots v_1}(b+i) = \{p^k b\} + j \quad \text{with } i, j \in L. \tag{3.6}$$

Then

$$p^k\Big(\Big(\frac{t_1}{p} + \cdots + \frac{t_k}{p^k}\Big) + i\Big) + \{p^k b\} - \sum_{i=1}^{k} p^{k-i}(v_i, \boldsymbol{\alpha}) = \{p^k b\} + j,$$

i.e.,

$$p^k\Big(\Big(\frac{t_1}{p} + \cdots + \frac{t_k}{p^k}\Big) + i\Big) - j = \sum_{i=1}^{k} p^{k-i}(v_i, \boldsymbol{\alpha}) \in \mathbb{Z}^m.$$

Given $b \in D \cap [0,1]^m$, for $i, j \in L$, let

$$\begin{aligned} M_{i,j}(b,k) &= \#\{v_1 \cdots v_k : T_{v_k\cdots v_1}(b+i) = \{p^k b\} + j\}, \\ N_{i,j}(b,k) &= \#\{v_1 \cdots v_k : T_{v_k\cdots v_1}(b+i) = \{p^k b\} + j \text{ and } f_{v_1\cdots v_k}(E) \cap \Pi_{b+i} \neq \varnothing\}. \end{aligned}$$

Then for all $b \in D \cap [0,1]^m$,

$$\begin{aligned} U_k(b) &= \sum_{i,j\in L} M_{i,j}(b,k) = \|M(b)M(pb)\cdots M(p^{k-1}b)\|_1, \\ N_k(b) \geq V_k(b) &= \sum_{i,j\in L} N_{i,j}(b,k). \end{aligned}$$

For notational convenience, we also write

$$\sum_{i\in L} N_{i,j}(b,k) = N_j(b,k) \quad \text{and} \quad \sum_{i\in L} M_{i,j}(b,k) = M_j(b,k).$$

Let $\mathbf{1}$ be the row vector in $\mathbb{R}^{\#L}$ with its every coordinate 1.

For the union $\bigcup_{z\in\Gamma_b} E_z$ of slices, we have the following result.

Proposition 3.1 *For $\mathcal{H}^m$-almost all $b \in (0,1)^m$, we have*

$$\dim_B\Big(\bigcup_{z\in\Gamma_b} E_z\Big) = \frac{\lambda}{\log p} \le \frac{\log \#\Omega}{\log p} - m = \dim_H E - m.$$

Proof It follows from Lemma 3.5 that for almost all $b \in [0,1]^m$,

$$b \in \bigcup_{j\in L}(K-j),$$

i.e.,

$$\mathcal{H}^m\Big([0,1]^m \cap \Big(\bigcup_{j\in L}(K-j)\Big)\Big) = 1. \tag{3.7}$$

By (3.7), we can select a subset $\Xi \subset L$ such that

$$\mathcal{H}^m\Big([0,1]^m \cap \Big(\bigcap_{j\in\Xi}(K-j)\Big)\Big) > 0$$

and

$$\mathcal{H}^m\Big([0,1]^m \cap \Big(\bigcap_{j\in\Xi}(K-j)\Big) \cap (K-i)\Big) = 0 \quad \text{for any } i \notin \Xi.$$

Notice that

$$\{b : b \notin K-i\} = \bigcup_{k\ge 1} B_{k,i},$$

where $B_{k,i} = \{b : \sum\limits_{j\in L} M_{i,j}(b,k) = 0\}$ satisfying

$$B_{1,i} \subset \cdots \subset B_{k,i} \subset B_{k+1,i} \subset \cdots.$$

Then there exists an integer k_0 large enough so that

$$\rho_{k_0} = \mathcal{H}^m\Big([0,1]^m \cap \Big(\bigcap_{j\in\Xi}(K-j)\Big) \cap \Big(\bigcap_{i\notin\Xi} B_{k_0,i}\Big)\Big) > 0. \tag{3.8}$$

Note that the following self-mapping of $[0,1]^m$,

$$b \to \{pb\} \quad \text{for } b \in (0,1)^m$$

is ergodic. Applying ergodic theorem to (3.8), we obtain that for $\mathcal{H}^m$-almost all $b \in (0,1)^m \cap D$, there is a sequence

$$n_1(b) < n_2(b) < \cdots < n_q(b) < n_{q+1}(b) < \cdots$$

satisfying

$$\lim_{k\to\infty} \frac{\#\{i : n_i(b) \le k\}}{k} = \rho_{k_0},$$

where $\{n_i(b)\}_i = \big\{k : \{p^k b\} \in \big(\bigcap\limits_{j\in\Xi}(K-j)\big) \cap \big(\bigcap\limits_{i\notin\Xi} B_{k_0,i}\big)\big\}$.

Taking a subsequence $m_i(b) = n_{2ik_0}(b)$, then we have for every $k = m_q$,

$$\begin{aligned} &\{p^k b\} \in K - j \quad \text{for any } j \in \Xi, \\ &\{p^k b\} \in B_{k_0,i} \quad \text{for any } i \notin \Xi, \end{aligned}$$

and

$$m_{i+1}(b) - m_i(b) \geq 2k_0.$$

We also have

$$\lim_{k\to\infty} \frac{\#\{i : m_i(b) \leq k\}}{k} = \frac{\rho_{k_0}}{2k_0},$$

which implies

$$\lim_{q\to\infty} \frac{m_{i+1}(b) - m_i(b)}{m_{i+1}(b)} = 0. \tag{3.9}$$

As we know, given $j \in \Xi$, if $T_{v_k\cdots v_1}(b + j') = \{p^k b\} + j$, then

$$E \cap \Pi_{T_{v_k\cdots v_1}(b+j')} = E \cap \Pi_{\{p^k b\}+j} \neq \varnothing$$

due to $\{p^k b\} + j \in K$, and thus

$$f_{v_1\cdots v_k}(E) \cap \Pi_{b+j'} \neq \varnothing.$$

As a result, we have

$$M_{j',j}(b, m_q(b)) = N_{j',j}(b, m_q(b)) \quad \text{for any } j \in \Xi \text{ and } j' \in L,$$

which implies

$$M_j(b, m_q(b)) = N_j(b, m_q(b)) \quad \text{for any } j \in \Xi. \tag{3.10}$$

Now, we obtain that

$$U_{m_q(b)}(b) = \sum_{j\in L} M_j(b, m_q(b)) \geq \sum_{j\in\Xi} N_j(b, m_q(b)). \tag{3.11}$$

On the other hand, we obtain that

$$\begin{aligned} U_{m_{q+1}}(b) &= \|\mathbf{1} M(b) \cdots M(p^{m_{q+1}(b)-1} b)\|_1 \\ &= \|(\mathbf{1} M(b) \cdots M(p^{m_q(b)-1} b))(M(p^{m_q(b)} b) \cdots M(p^{m_{q+1}(b)-1} b))\|_1 \\ &= \|(M_1(b, m_q(b)), \cdots, M_{\#L}(b, m_q(b))) \cdot T_q\|_1, \end{aligned} \tag{3.12}$$

where $T_q = M(p^{m_q(b)} b) \cdots M(p^{m_{q+1}(b)-1} b)$.

We note that there is a constant $C > 0$ such that for every k,

$$\|M(p^k b)\|_1 \leq C, \tag{3.13}$$

which implies

$$\|T_q\|_1 \leq C^{m_{q+1}(b)-m_q(b)}. \tag{3.14}$$

Since $\{p^k b\} \in B_{k_0,i}$ and $|m_{q+1}(b) - m_q(b)| > k_0$, we have the following claim.

Claim 3.1 Every entry of T_q in the row respect to $i \notin \Xi$ is zero.

Therefore, by (3.10), (3.12) and (3.14), we have

$$\begin{aligned} U_{m_{q+1}(b)}(b) &\le C^{m_{q+1}(b)-m_q(b)} \sum_{j\in\Xi} M_j(b, m_q(b)) \\ &= C^{m_{q+1}(b)-m_q(b)} \sum_{j\in\Xi} N_j(b, m_q(b)). \end{aligned} \tag{3.15}$$

Recall that for $\mathcal{H}^m$-almost all $b \in (0,1)^m$,

$$\frac{\log U_{m_{q+1}(b)}(b)}{m_{q+1}(b)}, \frac{\log U_{m_q(b)}(b)}{m_q(b)} \to \lambda \quad \text{as } q \to \infty, \tag{3.16}$$

where $\lim\limits_{q\to\infty} \frac{m_{i+1}(b)-m_i(b)}{m_{i+1}(b)} = 0$.

It follows from (3.11) and (3.15)–(3.16) that

$$\frac{\log \sum\limits_{j\in\Xi} N_j(b, m_q(b))}{m_q(b)} \to \lambda \quad \text{as } q \to \infty.$$

Noticing that

$$\frac{\log \sum\limits_{j\in\Xi} N_j(b, m_q(b))}{m_q(b)} \le \frac{\log N_{m_q(b)}(b)}{m_q(b)} \le \frac{\log 3^n U_{m_q(b)}(b)}{m_q(b)},$$

we have

$$\lim_{q\to\infty} \frac{\log N_{m_q(b)}(b)}{m_q(b)} = \lambda. \tag{3.17}$$

Since $\lim\limits_{q\to\infty} \frac{m_{i+1}(b)-m_i(b)}{m_{i+1}(b)} = 0$, it follows from (3.17) that

$$\lim_{k\to\infty} \frac{\log N_k(b)}{k} = \lambda.$$

That means for $\mathcal{H}^m$-almost all $b \in (0,1)^m$,

$$\dim\Big(\bigcup_{z\in\Gamma_b} E_z\Big) = \lim_{k\to\infty} \frac{\log N_k(b)}{k\log p} = \frac{\lambda}{\log p}.$$

According to Lemma 3.4,

$$\dim_B\Big(\bigcup_{z\in\Gamma_b} E_z\Big) = \frac{\lambda}{\log p} \le \frac{\log \#\Omega}{\log p} - m = \dim_H E - m.$$

4 Sections in Torus

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus and $P : \mathbb{R}^n \to \mathbb{T}^n$ the map defined by

$$P(x_1, \cdots, x_n) = (y_1, \cdots, y_n) \in \mathbb{T}^n,$$

where $y_i = \{x_i\}$ the fractional part of x_i for every i. For $y, y' \in \mathbb{T}^n$, the metric d on $\mathbb{T}^n$ is defined as follows:

$$\mathrm{d}(y, y') = \min_{\substack{P(x)=y \\ P(x')=y'}} |x - x'|.$$

Let $\tau : \mathbb{T}^n \to \mathbb{T}^n$ be the map $\tau(y) = py$.

Suppose $E = \bigcup\limits_{v\in\Omega} \frac{E+v}{p}$. Set

$$\Delta = P\Big(\bigcup_{v\in\Omega} f_v([0,1]^n)\Big) \subset \mathbb{T}^n.$$

Suppose $K = \{y \in \mathbb{T}^n : \tau^k(y) \in \Delta, \forall k \geq 0\}$, then

$$K = P(E).$$

Then K is τ-invariant. For $b \in \mathbb{R}^m$, write

$$K_b = K \cap \{y \in \mathbb{T}^n : (y, \boldsymbol{\alpha}) \equiv b \ (\mathrm{mod}\ 1)\}.$$

Lemma 4.1 *For any $b \in \mathbb{R}^m$, we have*

$$P\Big(\bigcup_{z\in\Gamma_b} E_z\Big) = K_b.$$

Proof Suppose $x \in \bigcup\limits_{z\in\Gamma_b} E_z$. We have $(x, \boldsymbol{\alpha}) \equiv b$ (mod 1) since $\alpha_i \in \mathbb{Z}^m$ for every i. Let $y = P(x)$. Then $y \in K$ and $(y, \boldsymbol{\alpha}) \equiv b$ (mod 1). Hence $P\big(\bigcup\limits_{z\in\Gamma_b} E_z\big) \subset K_b$.

On the other hand, suppose $y \in K_b$ with $(y, \boldsymbol{\alpha}) \equiv b$ (mod 1). Since $P(E) = K$, there exists $x \in E$ such that $P(x) = y$. Then $z = (x, \boldsymbol{\alpha}) \in b + \mathbb{Z}^m$. Note that $x \in E \subset [0,1]^n$, we have $\Pi_z \cap [0,1]^n \neq \varnothing$, which implies $z \in J$ due to (2.1), i.e., $z \in \Gamma_b = J \cap (b + \mathbb{Z}^m)$. Hence $y = P(x)$, where $x \in E \cap \Pi_z = E_z$ with $z \in \Gamma_b$. Therefore $K_b \subset P\big(\bigcup\limits_{z\in\Gamma_b} E_z\big)$.

Notice that $E \subset [0,1]^n$ and there exists a constant $\delta > 0$ such that

$$\mathrm{d}(P(x), P(x')) = |x - x'|$$

whenever $|x - x'| < \delta$. Therefore, we have the following result.

Lemma 4.2 *For any $b \in \mathbb{R}^m$, we have*

$$\dim\Big(\bigcup_{z\in\Gamma_b} E_z\Big) = \dim K_b,$$

where $\dim$ *stands for any one of* $\dim_H$, $\overline{\dim}_B$.

Now, we will show that the Hausdorff dimension of the slice equals its box dimension almost everywhere. During the proof, we will use the following result provided by Ledrappier (see [8, Proposition 2.6]).

Lemma 4.3 (Ledrappier) *Let T_p denote the endomorphism $T_p x = px$ (mod 1) of the $(n-m)$-dimensional torus $\mathbb{T}^{n-m}$, and let S be a continuous transformation of a metric space Y.*

Assume that $\Lambda \subset \mathbb{T}^{n-1} \times Y$ is compact and invariant under the map $T_p \times S$, and that ν is an S-invariant probability measure on Y. Then for ν-a.e. y, we have

$$\dim_H[\pi^{-1}(y)] = \overline{\dim}_B[\pi^{-1}(y)],$$

where $\pi : \Lambda \to Y$ is the projection onto the second coordinate.

Remark 4.1 This lemma implies for ν-a.e. y,

$$\dim_B[\pi^{-1}(y)] = \underline{\dim}_B[\pi^{-1}(y)] = \overline{\dim}_B[\pi^{-1}(y)].$$

Proof of Theorem 1.1 It follows from the result of [19] that there are constants $c_1 \leq c_2 \leq c_3$ such that for $\mathcal{H}^m$-almost all b with $E_b \neq \varnothing$,

$$\dim_H E_b = c_1, \quad \underline{\dim}_B E_b = c_2, \quad \overline{\dim}_B E_b = c_3.$$

By Proposition 3.1, we only need to show that for $\mathcal{H}^m$-almost all $b \in [0,1]^m$,

$$\dim_H\Big(\bigcup_{z\in\Gamma_b} E_b\Big) = \overline{\dim}_B\Big(\bigcup_{z\in\Gamma_b} E_b\Big). \tag{4.1}$$

Let T_p denote the endomorphism $T_p x = px \pmod 1$ of the $(n-m)$-dimensional torus $\mathbb{T}^{n-m}$, $S(x) = px \pmod 1$ the map on m-dimensional torus $\mathbb{T}^m$, and $g : \mathbb{T}^n \to \mathbb{T}^n$ the map

$$g(x) = ((x,\beta_1),\cdots,(x,\beta_{n-m}),(x,\alpha_1),\cdots,(x,\alpha_m)) \pmod 1.$$

Then $\tau = T_p \times S$. Both K and $g(K)$ are τ-invariant, i.e., $K = \tau(K)$ and

$$\tau(g(K)) = g(\tau(K)) = g(K)$$

since $g \circ \tau = \tau \circ g$.

Since $\mathbb{R}^n = \mathrm{span}\{\beta_1,\cdots,\beta_{n-m},\alpha_1,\cdots,\alpha_m\}$, we obtain that g is a local bi-Lipschitz map on the compact set $\mathbb{T}^n$, which implies

$$\dim g(K_b) = \dim K_b, \tag{4.2}$$

where dim stands any one of $\dim_H, \overline{\dim}_B$.

Now, let $Y = \mathbb{T}^m = [0,1]^m/\mathbb{Z}^m$ equipped with Lebesgue measure ν on $(0,1)^m$. Since

$$\pi^{-1}[b \pmod 1] = g(K_b),$$

then by the previous lemma, for ν-almost all $b \in \mathbb{T}$,

$$\dim_H g(K_b) = \overline{\dim}_B g(K_b).$$

Therefore, it follows from (4.2) that for $\mathcal{H}^m$-almost all $b \in [0,1]^m$,

$$\dim_H K_b = \overline{\dim}_B K_b.$$

By Lemma 4.2, we obtain (4.1).

5 Sections of Fractal Generated by Multi-rules

In this section, we will prove Theorem 1.3.

Fix a sequence $\{\Omega_i\}_{i\geq 1}$ satisfying s-star condition. We will discuss the slices of following sets

$$F^k = \Big\{ \sum_{i=k}^{\infty} \frac{v_i}{p^{i-k+1}} : v_i \in \Omega_i \text{ for all } i \geq k \Big\}.$$

Then $F = F^1$ and

$$F^k = \bigcup_{v\in\Omega_k} \frac{F^{k+1}+v}{p} = \bigcup_{v\in\Omega_k} f_v(F^{k+1}). \tag{5.1}$$

Denote $F_{k,b} = F^k \cap \Pi_b$.

Given integers $k' \geq k \geq 1$, let

$$\Theta_{k,k'} = \{v_k v_{k+1} \cdots v_{k'-1} :\ v_i \in \Omega_i \text{ for all } k \leq i \leq k'-1\},$$

and $\Theta_{k,\infty} = \{v_k v_{k+1} \cdots :\ v_i \in \Omega_i \text{ for all } i \geq k\}$.

The following two lemmas are similar to Lemmas 2.3–2.4. The first one is again the nested structure of the slices $\{F_{k,b}\}_{b\in J}$.

Lemma 5.1 *For any $b \in J$ and $k \geq 1$,*

$$F_{k,b} = \bigcup_{v\in\Omega_k} f_v(F_{k+1,T_v(b)}) = \bigcup_{v\in\Omega_k} \frac{F_{k+1,T_v(b)}+v}{p}. \tag{5.2}$$

Further, for any $k' \geq k$,

$$F_{k,b} = \bigcup_{\mathbf{v}\in\Theta_{k,k'}} f_{\mathbf{v}}(F_{k',T_{\mathbf{v}}(b)}) = \bigcup_{\substack{\mathbf{v}\in\Theta_{k,k'} \\ T_{\mathbf{v}}(b)\in\Gamma_{p^{k'-k}(b)}}} f_{\mathbf{v}}(F_{k',T_{\mathbf{v}}(b)}). \tag{5.3}$$

The second one tells us when the slice $F_{k,b}$ is not empty.

Lemma 5.2 *For any $b \in J$,*

$$F_{k,b} \neq \varnothing \iff \exists\ v_k v_{k+1} \cdots \in \Theta_{k,\infty},\ \textit{s.t. } T_{v_{k'}\cdots v_k}(b) \in J \quad \textit{for all } k' \geq k.$$

We record the number of copies with a non-negative integer matrix.

Given $b \in \mathbb{R}^m$ and $k \geq 1$, the integer matrix $M_k(b) = (c^k_{\mathbf{i},\mathbf{j}})_{\mathbf{i}\in\boldsymbol{\Gamma}_b,\mathbf{j}\in\Gamma_{pb}}$ is defined by

$$c^k_{\mathbf{i},\mathbf{j}} = \#\{v \in \Omega_k : T_v(\mathbf{i}) = \mathbf{j}\}. \tag{5.4}$$

Then $M_k(b)$ is a $\#\Gamma_b \times \#\Gamma_{pb}$ non-negative integer matrix.

Let $\mathbf{1}_b = (1, \cdots, 1)$ be a vector in $\mathbb{R}^{\#\Gamma_b}$ with every coordinate 1.

Lemma 5.3 *For any $b \in \mathbb{R}^m$, every column sum of the matrix $M_k(b)$ equals to s, i.e.,*

$$\mathbf{1}_b M_k(b) = s\mathbf{1}_{pb}. \tag{5.5}$$

Proof We need to show that for every $\mathbf{j} \in \Gamma_{pb}$,

$$\sum_{\mathbf{i}\in\Gamma_b} c^k_{\mathbf{i},\mathbf{j}} = s.$$

Let

$$\mathbf{j} = pb + q \quad \text{with } q \in \mathbb{Z}^m.$$

By (5.4), we have

$$\begin{aligned}\sum_{\mathbf{i}\in\Gamma_b} c^k_{\mathbf{i},\mathbf{j}} &= \sum_{\mathbf{i}\in\Gamma_b} \#\{v \in \Omega_k : T_v(\mathbf{i}) = \mathbf{j}\} \\ &= \#\{v \in \Omega_k : \exists\ z \in \Gamma_b,\ \text{s.t. } T_v(z) = pb + q\}.\end{aligned}$$

Suppose that $z = b + q'$ with $q' \in \mathbb{Z}^m$. Then

$$\begin{aligned}T_v(z) = pb + q &\implies p(b + q') - (v, \boldsymbol{\alpha}) = pb + q \\ &\implies (v, \boldsymbol{\alpha}) \equiv -q \pmod p.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{\mathbf{i}\in\Gamma_b} c^k_{\mathbf{i},\mathbf{j}} &= \#\{v \in \Omega_k : \exists\ z \in \Gamma_b,\ \text{s.t. } T_v(z) = pb + q\} \\ &\le \#\{v \in \Omega_k : (v, \boldsymbol{\alpha}) \equiv -q \pmod p\} = s.\end{aligned}$$

Conversely, for any $v \in \Omega_k$ with $(v, \boldsymbol{\alpha}) \equiv -q \pmod p$, we have

$$S_v(\mathbf{j}) = S_v(pb + q) = b + \frac{q + (v, \boldsymbol{\alpha})}{p} \in b + \mathbb{Z}^m.$$

By (2.3), we know that $S_v(\mathbf{j}) \subset S_v(J) \subset J$. That means

$$S_v(\mathbf{j}) \in (b + \mathbb{Z}^m) \cap J = \Gamma_b,$$

i.e., there exists $z \in \Gamma_b$, such that $T_v(z) = \mathbf{j}$. Therefore,

$$\begin{aligned}s &= \#\{v \in \Omega_k : (v, \boldsymbol{\alpha}) \equiv -q \pmod p\} \\ &\le \#\{v \in \Omega_k : \exists\ z \in \Gamma_b,\ \text{s.t. } T_v(z) = \mathbf{j}\} = \sum_{\mathbf{i}\in\Gamma_b} c^k_{\mathbf{i},\mathbf{j}}.\end{aligned}$$

This completes the proof.

Given $k \ge 1$ and $t \ge 1$, we denote

$$N_{k,t}(b) = \sum_{z\in\Gamma_b} \#\{v_k \cdots v_{k+t-1} \in \Theta_{k,k+t} : f_{v_k\cdots v_{k+t-1}}([0,1]^n) \cap \Pi_z \ne \varnothing\}. \tag{5.6}$$

We also have

$$N_{k,t}(b) = \sum_{z\in\Gamma_b} \#\{v_k \cdots v_{k+t-1} \in \Theta_{k,k+t} : T_{v_{k+t-1}\cdots v_k}(z) \in J\}, \tag{5.7}$$

since

$$\begin{aligned}f_{v_k\cdots v_{k+t-1}}([0,1]^n) \cap \Pi_z \ne \varnothing &\iff [0,1]^n \cap \Pi_{T_{v_{k+t-1}\cdots v_k}(z)} \ne \varnothing \\ &\iff T_{v_{k+t-1}\cdots v_k}(z) \in J\end{aligned}$$

due to (2.2) and (2.1).

The following proposition shows us how to compute $N_{k,t}(b)$.

Lemma 5.4 *For any integers $k, t \geq 1$, we have*

$$N_{k,t}(b) = \|\mathbf{1}_b M_k(b) M_{k+1}(pb) \cdots M_{k+t-1}(p^{t-1}b)\|_1. \tag{5.8}$$

Proof First, by (5.7), for any $k > 0$, we obtain that

$$N_{k,t}(b) = \sum_{z \in \Gamma_b} \#\{v_k \cdots v_{k+t-1} \in \Theta_{k,k+t} : T_{v_{k+t-1} \cdots v_k}(z) \in \Gamma_{p^t b}\}. \tag{5.9}$$

Now we will show that

$$\begin{aligned} &\|\mathbf{1}_b M_k(b) M_{k+1}(pb) \cdots M_{k+t-1}(p^{t-1}b)\|_1 \\ &= \sum_{z \in \Gamma_b} \#\{v_k \cdots v_{k+t-1} \in \Theta_{k,k+t} : T_{v_{k+t-1} \cdots v_k}(z) \in \Gamma_{p^t b}\}. \end{aligned}$$

We obtain that

$$\begin{aligned} &\|\mathbf{1}_b M_k(b) M_{k+1}(pb) \cdots M_{k+t-1}(p^{t-1}b)\|_1 \\ &= \sum_{(\mathbf{i}_k, \cdots, \mathbf{i}_{k+t}) \in \Gamma_b \times \cdots \times \Gamma_{p^t b}} \Big(\prod_{h=k}^{k+t-1} \#\{v_h \in \Omega_h : T_{v_h}(\mathbf{i}_h) = \mathbf{i}_{h+1}\} \Big) \\ &= \sum_{(\mathbf{i}_k, \cdots, \mathbf{i}_{k+t}) \in \Gamma_b \times \cdots \times \Gamma_{p^t b}} \#\{v_k \cdots v_{k+t-1} \in \Theta_{k,k+t} : T_{v_h}(\mathbf{i}_h) = \mathbf{i}_{h+1} \ \forall h\} \\ &= \sum_{z \in \Gamma_b} \#\{v_k \cdots v_{k+t-1} \in \Theta_{k,k+t} : T_{v_k}(z) \in \Gamma_{pb}, \ \cdots, T_{v_{k+t-1} \cdots v_k}(z) \in \Gamma_{p^t b}\} \\ &= \sum_{z \in \Gamma_b} \#\{v_k \cdots v_{k+t-1} \in \Theta_{k,k+t} : T_{v_{k+t-1} \cdots v_k}(z) \in \Gamma_{p^t b}\}, \end{aligned}$$

where the last equality follows from (2.4).

Proposition 5.1 *Every column sum of $M_k(b) M_{k+1}(pb) \cdots M_{k+t-1}(p^{t-1}b)$ is s^t. Further,*

$$N_{k,t}(b) = (\#\Gamma_{p^t b}) s^t.$$

Proof It follows from Lemma 5.3 that

$$\begin{aligned} (\mathbf{1}_b M_k(b)) M_{k+1}(pb) \cdots M_{k+t-1}(p^{t-1}b) &= s(\mathbf{1}_{pb} M_k(pb)) \cdots M_{k+t-1}(p^{t-1}b) \\ &= \cdots = s^t \mathbf{1}_{p^t b}. \end{aligned}$$

By Lemma 5.4, we have

$$N_{k,t}(b) = s^t \|\mathbf{1}_{p^t b}\|_1 = (\#\Gamma_{p^t b}) s^t.$$

Corollary 5.1 *For any $b \in \mathbb{R}^m$ and any integer $k \geq 1$, we have*

$$\bigcup_{z \in \Gamma_b} F_{k,z} \neq \varnothing.$$

Proof Suppose on the contrary that $\bigcup_{z \in \Gamma_b} F_{k,z} = \varnothing$. Then by Lemma 5.2, there exists an integer t such that

$$T_{v_{k+t-1} \cdots v_k}(z) \notin J \quad \text{for all } z \in \Gamma_b \text{ and any } v_k \cdots v_{k+t-1} \in \Theta_{k,k+t}.$$

It follows from (5.7) that $N_{k,t}(b) = 0$. However, by Proposition 5.1, $N_{k,t}(b) = (\#\Gamma_{p^t b})s^t > 0$. It is a contradiction.

Let

$$V_{k,t}(b) = \sum_{z\in\Gamma_b} \#\{v_k \cdots v_{k+t-1} \in \Theta_{k,k+t} : f_{v_k\cdots v_{k+t-1}}(F^{k+t}) \cap \Pi_z \neq \varnothing\}.$$

Corollary 5.2 *For any $b \in \mathbb{R}^m$ and integers $k, t \geq 1$,*

$$V_{k,t}(b) \geq s^t.$$

Proof The last corollary implies that there exists $z \in \Gamma_b$ such that $F^k \cap \Pi_z \neq \varnothing$. In the same way, there exists $z^* \in \Gamma_{p^t b}$ such that $F^{k+t} \cap \Pi_{z^*} \neq \varnothing$.

By Lemma 5.1, we obtain that

$$\begin{aligned}\bigcup_{z\in\Gamma_b} F^k \cap \Pi_z &= \bigcup_{z\in\Gamma_b} \bigcup_{v_k\cdots v_{k+t-1}\in\Theta_{k,k+t}} f_{v_k\cdots v_{k+t-1}}(F^{k+t} \cap \Pi_{T_{v_{k+t-1}\cdots v_k}(z)}) \\ &= \bigcup_{z'\in\Gamma_{p^t b}} \bigcup_{z\in\Gamma_b} \bigcup_{T_{v_{k+t-1}\cdots v_k}(z)=z'} f_{v_k\cdots v_{k+t-1}}(F^{k+t} \cap \Pi_{z'}).\end{aligned}$$

Then

$$f_{v_k\cdots v_{k+t-1}}(F^{k+t} \cap \Pi_{z^*}) = f_{v_1\cdots v_k}(F^{k+t}) \cap \Pi_z \neq \varnothing$$

for any $T_{v_{k+t-1}\cdots v_k}(z) = z^*$ with $z \in \Gamma_b$. Denote

$$\begin{aligned}\delta_t = \#\{&v_k \cdots v_{k+t-1} \in \Theta_{k,k+t} : f_{v_k\cdots v_{k+t-1}}(F^{k+t} \cap \Pi_{z^*}) \neq \varnothing \text{ with} \\ &T_{v_{k+t-1}\cdots v_k}(z) = z^* \text{ for some } z \in \Gamma_b\}.\end{aligned}$$

Hence $V_{k,t}(b) \geq \delta_t$.

For the matrix $M_k(b)M_{k+1}(pb)\cdots M_{k+t-1}(p^{t-1}b)$, when we consider its column sum with respect to z^*, by Proposition 5.1 we have $\delta_t \geq s^t$, which implies $V_{k,t}(b) \geq s^t$.

By the result in [20], this fractal set F^k have dimension $\frac{\log sp^m}{\log p} = \frac{\log s}{\log p} + m$.

Consider the union $\bigcup_{z\in\Gamma_b} [F^k \cap \Pi_z]$ of slices. We have the following proposition.

Proposition 5.2 *For all $b \in \mathbb{R}^m$ and integer $k \geq 1$,*

$$\dim_B \Big(\bigcup_{z\in\Gamma_b} F_{k,z} \Big) = \frac{\log s}{\log p} = \dim_H F^k - m.$$

Proof Let $U_{k,t}(b)$ be the number of p-adic cubes of side length p^{-k} which intersect $\bigcup_{z\in\Gamma_b} [F^k \cap \Pi_z]$. By the definition of the box dimension, we have

$$\begin{aligned}\overline{\dim}_B\Big(\bigcup_{z\in\Gamma_b} [F^k \cap \Pi_z] \Big) &= \limsup_{t\to\infty} \frac{\log U_{k,t}(b)}{t \log p}, \\ \underline{\dim}_B\Big(\bigcup_{z\in\Gamma_b} [F^k \cap \Pi_z] \Big) &= \liminf_{t\to\infty} \frac{\log U_{k,t}(b)}{t \log p}.\end{aligned}$$

We notice that

$$U_{k,t}(b) \geq V_{k,t}(b) \geq s^t. \tag{5.10}$$

On the other hand, it suffices to verify

$$U_{k,t}(b) \le 3^n N_{k,t}(b) \le 3^n \Big(\prod_{i=1}^{m} (A_i^+ - A_i^- + 1) \Big) s^t. \tag{5.11}$$

In fact, by (5.10)–(5.11), we have

$$\dim_B \Big(\bigcup_{z \in \Gamma_b} [F^k \cap \Pi_z] \Big) = \lim_{t \to \infty} \frac{\log U_{k,t}(b)}{t \log p} = \frac{\log s}{\log p}.$$

To verify (5.11), given a p-adic cube of side length $p^{-(k+t)}$ intersecting $F^k \cap \Pi_z$, we denote it by B, then $B \cap F^k \cap \Pi_z \neq \varnothing$, i.e.,

$$B \cap \Big(\bigcup_{v_k \cdots v_{k+t-1} \in \Theta_{k,k+t}} f_{v_k \cdots v_{k+t-1}}(F^{k+t}) \Big) \cap \Pi_z \neq \varnothing.$$

Then for some $v_k \cdots v_{k+t-1} \in \Theta_{k,k+t}$,

$$\begin{aligned} & B \cap f_{v_k \cdots v_{k+t-1}}(F^{k+t}) \cap \Pi_z \neq \varnothing \\ \Longrightarrow\ & B \cap f_{v_k \cdots v_{k+t-1}}(F^{k+t}) \neq \varnothing \text{ and } f_{v_k \cdots v_{k+t-1}}(F^{k+t}) \cap \Pi_z \neq \varnothing \\ \Longrightarrow\ & B \cap f_{v_k \cdots v_{k+t-1}}([0,1]^n) \neq \varnothing \text{ and } f_{v_k \cdots v_{k+t-1}}([0,1]^n) \cap \Pi_z \neq \varnothing, \end{aligned}$$

which implies

$$U_{k,t}(b) \le 3^n N_{k,t}(b).$$

Hence (5.11) holds.

Let

$$\Lambda_F = \{ b \in J : F \cap \Pi_b \neq \varnothing \}.$$

Suppose $\Psi = \mathbb{Z}^m \cap \prod_{t=1}^{m} [A_i^-, A_i^+)$ with $\#\Psi = N$. For any $\mathbf{i} = (i_1, \cdots, i_m)$ in Ψ, we denote

$$I_{\mathbf{i}} = \prod_{t=1}^{m} (i_t, i_t + 1).$$

Then $\mathcal{H}^m(J) = \sum_{\mathbf{i} \in \Psi} \mathcal{H}^m(I_{\mathbf{i}})$.

Proposition 5.3 $\mathcal{H}^m(\Lambda_F) \ge 1$.

Proof Then $\mathcal{H}^m(\Lambda_F) = \sum_{\mathbf{i} \in \Psi} \mathcal{H}^m(I_{\mathbf{i}} \cap \Lambda_F)$.

By Corollary 1, we have $\bigcup_{z \in \Gamma_b} F_z = \bigcup_{z \in \Gamma_b} F_{1,z} \neq \varnothing$ for any $b \in J$. That means

$$\begin{aligned} N = \mathcal{H}^m(J) &\le \sum_{\mathbf{i} \in \Psi} \sum_{\substack{p \in \mathbb{Z}^m \\ I_{\mathbf{i}} + p \subset [0,1]^m}} \mathcal{H}^m(I_{\mathbf{i}} \cap \Lambda_F) \\ &\le \sum_{\mathbf{i} \in \Psi} N \mathcal{H}^m(I_{\mathbf{i}} \cap \Lambda_F) \\ &= N \sum_{\mathbf{i} \in \Psi} \mathcal{H}^m(I_{\mathbf{i}} \cap \Lambda_F) \\ &= N \mathcal{H}^m(\Lambda_F), \end{aligned}$$

which implies $\mathcal{H}^m(\Lambda_F) \ge 1$.

Proof of Theorem 1.3 Theorem 1.3 follows from Propositions 5.2–5.3.

6 The Case of Continuous Projection Measure

6.1 Proof of Proposition 1.1

6.1.1 The local dimension of μ_V

Suppose that E is the attractor of the IFS $\{f_v\}_{v\in\Omega}$. Write $\Psi : \Omega^{\mathbb{N}} \to E$ for the natural projection

$$\Psi(w) := \lim_{n\to\infty} f_{w_1\cdots w_n}(\mathbf{0})$$

and the natural measure μ on E can be also defined as the push forward measure of the Bernoulli measure on $\Omega^{\mathbb{N}}$ given by $\{\frac{1}{\#\Omega}, \cdots, \frac{1}{\#\Omega}\}^{\mathbb{N}}$. Recall that $\mu_V = \mu \circ \mathrm{proj}_V^{-1}$ satisfying

$$\mu_V = \sum_{v\in\Omega} \frac{1}{\#\Omega} \mu_V \circ S_v^{-1}. \tag{6.1}$$

Let $B_V \subset \Lambda$ be the set of all 'bad' points in Λ that the number of appearance of digits 1 and $(p-1)$ in the p-adic expansion is finite, i.e.,

$$B_V = \{x = (x_1, \cdots, x_m) \in \Lambda : \exists\ N_0,\ \text{s.t. } x_i^{(N_0)} x_i^{(N_0+1)} \cdots \in \{2, \cdots, (p-2)\}^{\mathbb{N}}\},$$

where for $i = 1, \cdots, m$, $x_i = x_i^{(1)} x_i^{(2)} \cdots$ is the p-adic expansion. Roughly speaking, 'bad' points are located near the boundary of squares. It is easy to see that $\mathcal{L}^m(\Lambda \backslash B_V) = 0$.

Proof of Proposition 1.1 For any $x = (x_1, \cdots, x_m) \in B_V$, supposet that the p-adic expansion is $x_i = x_i^{(1)} x_i^{(2)} \cdots$. Then, denote the p-adic square of level j that contains x by $\square_j(x)$, i.e.,

$$\square_j(x) = \prod_{i=1}^{m} \Big(\sum_{l=1}^{j} \frac{x_i^{(l)}}{p^l}, \sum_{l=1}^{j} \frac{x_i^{(l)}}{p^l} + \frac{1}{p^j} \Big).$$

Then by the definition of B_V, for $j > N_0$, we have

$$\square_{j+1}(x) \subset B(x, p^{-j}) \subset \square_{j-1}(x).$$

Hence

$$\mu_V(\square_{j+1}(x)) \le \mu_V(B(x, p^{-j})) \le \mu_V(\square_{j-1}(x)).$$

Since $\mu_V = \mu \circ \mathrm{proj}_V^{-1}$, then

$$\begin{aligned}\mu_V(\square_j(x)) &= \mu\{y \in E : \mathrm{proj}_V(y) \in \square_j(x)\} \\ &= \frac{1}{\#\Omega^j} \#\{v_1 \cdots v_j \in \Omega^j : f_{v_1\cdots v_j}([0,1]^n) \cap \mathrm{proj}_V^{-1}(\square_j(x)) \ne \varnothing\} \\ &= \frac{1}{\#\Omega^j} \mathbf{1}_x M(x) M(px) \cdots M(p^{j-1}x) e_t, \end{aligned} \tag{6.2}$$

where $e_t = (0, \cdots, 0, 1, 0, \cdots, 0)^{\mathrm{T}}$ with the t-th coordinate 1 and t is dependent on x.

According to the equation (6.2), we have for all $x \in B_V$,

$$\frac{\log \#\Omega}{\log p} - o(j) \le \frac{\log N_j(x)}{\log p^j} + \frac{\log \mu_V(B(x, p^{-j}))}{\log p^{-j}} \le \frac{\log \#\Omega}{\log p} + o(j). \tag{6.3}$$

Similarly with Lemma 3.3, for $x \in \mathrm{spt}\mu_V$,

$$\dim_B[E \cap (V+x)] = \lim_{j\to\infty} \frac{\log[\mathbf{1}_x M(x)M(px)\cdots M(p^{j-1}x)e_t]}{\log p^j}. \tag{6.4}$$

According to (6.3)–(6.4), letting $j \to \infty$, we obtain

$$\dim_H E = \dim_B[E \cap (V+x)] + \dim_{\mathrm{loc}} \mu_V(x).$$

6.2 Proof of Theorem 1.2

Recall that $E = \bigcup_{v\in\Omega} \frac{E+v}{p}$, and μ_V is a self-similar measure satisfying

$$\mu_V = \sum_{v\in\Omega} \frac{1}{\#\Omega} \mu_V \circ S_v^{-1}.$$

Lemma 6.1 *If $\mu_V \ll \mathcal{L}^m$, then for $\mathcal{L}^m$ almost all $x \in \mathrm{spt}\mu_V \cap \Lambda$, $\dim_{\mathrm{loc}} \mu_V(x) = m$.*

Proof Notice that μ_V is a self-similar measure and μ_V is absolutely continuous with respect to the Lebesgue measure. By [16, Proposition 3.1], which tells us that a self-similar measure is either equivalent to the Lebesgue or singular, we have $\mu_V \sim \mathcal{L}^m$.

Since for all $x \in \mathrm{spt}\mu_V$, $\mu_V(B(x,r)) > 0$ for sufficiently small r (if $\mu_V(B(x,r)) = 0$, then $\mu_V((\mathrm{spt}\mu_V)^c \cup B(x,r)) = 0$, which contradicts the definition of the support of a measure), the local dimension of μ_V is

$$\dim_{\mathrm{loc}} \mu_V(x) = \lim_{r\to 0} \frac{\log \mu_V(B(x,r))}{\log r} = \lim_{r\to 0} \frac{\log \mathcal{L}^m(B(x,r))}{\log r} = m.$$

According to Proposition 1.1 and Lemma 6.1, we obtain (1.11).

Now we will prove the rest of Theorem 1.2.

The self-similar measure μ_V satisfies (6.1). Applying Fourier transform to the both sides of (6.1), we obtain

$$\widehat{\mu}_V(\xi) = \phi\Big(\frac{\xi}{p}\Big)\widehat{\mu}_V\Big(\frac{\xi}{p}\Big), \tag{6.5}$$

where

$$\phi(\xi) = \sum_{v\in\Omega} \frac{1}{\#\Omega} \mathrm{e}^{-\mathrm{i}\xi(v,\boldsymbol{\alpha})} \tag{6.6}$$

is the characteristic function of (6.1). Iterating the equation (6.5) j times ($j \geq 2$), we obtain

$$\widehat{\mu}_V(\xi) = \phi\Big(\frac{\xi}{p}\Big)\cdots\phi\Big(\frac{\xi}{p^j}\Big)\widehat{\mu}_V\Big(\frac{\xi}{p^j}\Big). \tag{6.7}$$

To show that $\mu_V \ll \mathcal{L}^m$, we need the following lemma which is a higher-dimensional version of [15, Lemma 2]. We give the proof here for consistency.

Lemma 6.2 *$\mu_V \ll \mathcal{L}^m$ if and only if the following equation holds:*

$$\widehat{\mu}_V(2\pi\boldsymbol{k}) = \delta_{\boldsymbol{k}}, \quad \boldsymbol{k} \in \mathbb{Z}^m,$$

where $\delta_{\boldsymbol{k}} = 1$ if $\boldsymbol{k} = (0,\cdots,0)$ and $\delta_{\boldsymbol{k}} = 0$ otherwise.

Proof The necessity. By (6.7), we have

$$\widehat{\mu}_V(2\pi \boldsymbol{k} p^j) = (\phi(\mathbf{0}))^j \widehat{\mu}_V(2\pi \boldsymbol{k}) = \widehat{\mu}_V(2\pi \boldsymbol{k}).$$

Since μ_V is absolutely continuous with respect to $\mathcal{L}^m$ by the assumption, applying the Riemann-Lebesgue lemma, we have

$$\widehat{\mu}_V(2\pi \boldsymbol{k} p^j) \to 0 \quad \text{as } j \to \infty.$$

Therefore, $\widehat{\mu}_V(2\pi \boldsymbol{k}) = 0$ for every $\boldsymbol{k} \neq \mathbf{0}$. The necessity is proved.

The Sufficiency. Denote by $\mathcal{S}(\mathbb{R}^m)$ the Schwartz space of all indefinitely differentiable rapidly decreasing functions f on $\mathbb{R}^m$:

$$\mathcal{S}(\mathbb{R}^m) = \{f \in C^\infty(\mathbb{R}^m) : \|f\|_{\alpha,\beta} < \infty, \forall \alpha, \beta\},$$

where α, β are multi-indices and $\|f\|_{\alpha,\beta} = \sup\limits_{x \in \mathbb{R}^m} |x^\alpha (\frac{\partial}{\partial x})^\beta f(x)|$. And denote by $\mathcal{E}_r$ the space of infinitely differentiable functions with period 1:

$$\mathcal{E}_r = \{g \in C^\infty(\mathbb{R}^m) : g(x + e_j) \equiv g(x), \text{ for } j = 1, \cdots, m\}.$$

Every function $g \in \mathcal{E}_r$ can be decomposed into a Fourier series that converges to g in the space $\mathcal{E}_r$:

$$g(x) = \sum_{\boldsymbol{k} \in \mathbb{Z}^m} a_{\boldsymbol{k}}(g) \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot x}, \quad a_{\boldsymbol{k}}(g) = \int_{Q_1} g(x) \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \cdot x} \mathrm{d}x,$$

where Q_1 denote the closed cube

$$Q_1 = \{x \in \mathbb{R}^m : 0 \le x_j \le 1, \quad \text{for } j = 1, \cdots, m\}.$$

Now we consider the periodization operator taking a function $f(x) \in \mathcal{S}$ to the function $\widetilde{f}(x) = \sum\limits_{\boldsymbol{k} \in \mathbb{Z}^m} f(x + \boldsymbol{k})$. It is a continuous operator from $\mathcal{S}$ to $\mathcal{E}_r$.

For any cube

$$\begin{aligned} Q &= \{x \in \mathbb{R}^m : a_j \le x_j \le b_j, \quad \text{for } j = 1, \cdots, m\}, \\ Q_\varepsilon &= \{x \in \mathbb{R}^m : a_j - \varepsilon < x_j < b_j + \varepsilon, \quad \text{for } j = 1, \cdots, m\}, \end{aligned}$$

where $b_j - a_j < \frac{1}{2}$ for $j = 1, \cdots, m$ and $0 < \varepsilon < \frac{1}{4}$, consider a function $f_\varepsilon \in \mathcal{S}(\mathbb{R}^m)$ such that $f_\varepsilon(x) = 1$ on Q and $f_\varepsilon(x) \equiv 0$ on $\mathbb{R}^m \backslash Q_\varepsilon$ and $0 < f_\varepsilon(x) < 1$ otherwise. Then

$$\begin{aligned} \sum_{\boldsymbol{k} \in \mathbb{Z}^m} \int_{\mathbb{R}^m} f_\varepsilon(x + \boldsymbol{k}) \mathrm{d}\mu_V &= \int_{\mathbb{R}^m} \widetilde{f}_\varepsilon(x) \mathrm{d}\mu_V = \int_{\mathbb{R}^m} \sum_{\boldsymbol{k} \in \mathbb{Z}^m} a_{\boldsymbol{k}}(\widetilde{f}_\varepsilon) \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot x} \mathrm{d}\mu_V \\ &= \sum_{\boldsymbol{k} \in \mathbb{Z}^m} a_{\boldsymbol{k}}(\widetilde{f}_\varepsilon) \int_{\mathbb{R}^m} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot x} \mathrm{d}\mu_V = \sum_{\boldsymbol{k} \in \mathbb{Z}^m} a_{\boldsymbol{k}}(\widetilde{f}_\varepsilon) \widehat{\mu}_V(-2\pi \boldsymbol{k}) \\ &= \sum_{\boldsymbol{k} \in \mathbb{Z}^m} a_{\boldsymbol{k}}(\widetilde{f}_\varepsilon) \delta_{-\boldsymbol{k}} = a_{\mathbf{0}}(\widetilde{f}_\varepsilon) = \int_{Q_1} \widetilde{f}_\varepsilon(x) \mathrm{d}x \\ &= \int_{\mathbb{R}^m} f_\varepsilon(x) \mathrm{d}x < \prod_{j=1}^m (b_j - a_j + 2\varepsilon). \end{aligned}$$

Since $\int_{\mathbb{R}^m} f_\varepsilon(x+\boldsymbol{k})\mathrm{d}\mu_V \geq 0$ for all $\boldsymbol{k} \in \mathbb{Z}^m$, we have

$$\int_{\mathbb{R}^m} f_\varepsilon(x)\mathrm{d}\mu_V < \prod_{j=1}^m (b_j - a_j + 2\varepsilon).$$

Consider the sequence of functions $\{f_{\frac{1}{n}}\}_{n\geq 5}$. Letting $n \to \infty$, we obtain

$$f_{\frac{1}{n}}(x) \to \mathbf{1}_Q(x), \quad \forall x \in \mathbb{R}^m,$$

where $\mathbf{1}_Q$ denotes the characteristic function of Q. By Fatou's lemma, we have

$$\begin{aligned}\int_Q \mathrm{d}\mu_V &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^m} f_{\frac{1}{n}}(x)\mathrm{d}\Phi \\ &\leq \liminf_{n\to\infty} \prod_{j=1}^m \Big(b_j - a_j + \frac{2}{n}\Big) \\ &= \prod_{j=1}^m (b_j - a_j),\end{aligned}$$

which implies $\mu_V \ll \mathcal{L}^m$.

Now we will show that $\mu_V \ll \mathcal{L}^m$ of Theorem 1.2.

Proof of continuity Take $\boldsymbol{k} \in \mathbb{Z}^m$. By (6.7), we have

$$\widehat{\mu}_V(2\pi\boldsymbol{k}) = (\phi(\mathbf{0}))^j \widehat{\mu}_V(2\pi\boldsymbol{k}) = \widehat{\mu}_V(2\pi\boldsymbol{k}p^j),$$

we only need to consider $\boldsymbol{k} \in \mathbb{Z}^m$ with the form $\boldsymbol{k} = p^t\boldsymbol{q}$ where $t \geq 0$, $\boldsymbol{q} = (q_1, \cdots, q_m)$ and $0 < |q_j| < p$ for $j = 1, \cdots, m$. Substituting $\xi = 2\pi\boldsymbol{k}$ into (6.7), we obtain

$$\widehat{\mu}_V(2\pi\boldsymbol{k}) = \phi(2\pi p^{t-1}\boldsymbol{q}) \cdots \phi(2\pi\boldsymbol{q})\phi\Big(\frac{2\pi\boldsymbol{q}}{p}\Big)\widehat{\mu}_V\Big(\frac{2\pi\boldsymbol{q}}{p}\Big) = (\phi(\mathbf{0}))^t\phi\Big(\frac{2\pi\boldsymbol{q}}{p}\Big)\widehat{\mu}_V\Big(\frac{2\pi\boldsymbol{q}}{p}\Big).$$

For $j = 1, \cdots, m$, let $z_j = \mathrm{e}^{-\mathrm{i}2\pi q_j/p} \neq 1$ then $z_j^p = 1$ and $1 + z_j + \cdots + z_j^{p-1} = 0$. Hence

$$\begin{aligned}\phi\Big(\frac{2\pi\boldsymbol{q}}{p}\Big) &= \sum_{v\in\Omega} \mathrm{e}^{-\frac{2\pi\mathrm{i}\boldsymbol{q}\cdot d_v}{p}} = \sum_{v\in\Omega} z_1^{d_v^{(1)}} \cdots z_m^{d_v^{(m)}} \\ &= s^m \sum_{\boldsymbol{l}\in\{0,1,\cdots,p-1\}^m} z_1^{l_1} \cdots z_m^{l_m} \quad \text{(by s-star condition)} \\ &= s^m \prod_{j=1}^m (1 + z_j + \cdots + z_j^{p-1}) = 0,\end{aligned}$$

which implies $\widehat{\mu}_V(2\pi\boldsymbol{k}) = 0$. Therefore, by Lemma 6.2, the proof is finished.

6.3 Proof of Example 1.5

To prove Example 1.5, we need some notations and results in [15].

A tree is said to be the tree of order p ($p \geq 2$), if it is constructed as follows:

1. Put 2π at the root;
2. Put the number $\frac{2\pi t}{p}$, $t = 1, \cdots, p-1$ at the vertices on the first level;
3. Inductively, let a number α be associated to a vertex on the l-th level, then the numbers

$$\frac{\alpha + 2\pi t}{p}, \quad t = 0, 1, \cdots, p-1$$

are associated to its neighbours on the $(l+1)$-th level.

Definition 6.1 (see [15]) *A subset $\mathcal{A}$ of vertices of the tree of order p is a blocking set if the following conditions are satisfied:*

(a) $2\pi \notin \mathcal{A}$;

(b) $\alpha \in \mathcal{A} \Leftrightarrow 2\pi - \alpha \in \mathcal{A}$;

(c) *every infinite path starting at the root of the tree includes exactly one element of $\mathcal{A}$.*

From [15, Theorem 1], we have the following lemma.

Lemma 6.3 (see [15]) *The solution of* (6.1) *is absolutely continuous if and only if there is a blocking set that consists of roots of* (6.6).

Proof of Example 1.5 According to Lemma 6.3, we only need to show that there is a blocking set consisting of the roots of the following equation:

$$\phi(\xi) = \sum_{v\in\Omega} \frac{1}{\#\Omega} \mathrm{e}^{-\mathrm{i}\xi(v,\boldsymbol{\alpha})}.$$

Let $z = \mathrm{e}^{-\mathrm{i}\xi}$, the above equation turns to

$$\begin{aligned} g(z) &= 1 + z + z^3 + z^4 + z^6 + z^7 + z^9 + z^{10} + z^{12} + z^{13} + z^{15} + z^{16} \\ &= (z+1)^2(z^2 - z + 1)(z^6 + z^3 + 1)(z^6 - z^3 + 1). \end{aligned}$$

By some simple calculation, we have the roots of equation $\phi(\xi)$ as follows:

$$\{\xi : \phi(\xi) = 0\} = 2\pi\Big[\Big\{\frac{1}{6}, \frac{3}{6}, \frac{5}{6}\Big\} \cup \Big\{\frac{\frac{2}{6}+t}{6}, \frac{\frac{4}{6}+t}{6} : t = 0, \cdots, p-1\Big\}\Big],$$

which is a blocking set.

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