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Quasi Normal Sectors and Orbits in Regular Critical Directions of Planar System*

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Abstract This paper deals with the qualitative behavior of orbits at degenerate singular point with the method of quasi normal sector, which is a generalization of Frommer's normal sectors. Several examples show that this method is more effective than the well-known methods of Z-sectors, normal sectors and generalized normal sector.

Keywords Normal sector, Generalized normal sector, Quasi normal sector
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1 Introduction

One of the classic problems in the qualitative theory of the differential system is to characterize the local phase portraits near an isolated singular point. Consider C^k planar differential system

$$\begin{cases}\dot{x} = X(x,y),\\ \dot{y} = Y(x,y),\end{cases} \tag{1.1}$$

where the dot denotes derivatives with respect to the variable t, k denotes a positive integer, $+\infty$ or w, C^w stands for an analytic function, and the origin $O(0,0)$ is the isolated singular point such that $X(0,0) = Y(0,0) = 0$.

Concerning the simple singular point (those both eigenvalues of the Jacobian matrix at the singular point are different from zero), the Hartman-Grobman theorem completely solved them except when the singularity is monodromic, that is, the solution of the differential system turns around the singular point. The semi-simple singular points (where one of the eigenvalues equal to zero) are also classified (see for instance [3]).

Regarding the degenerate singular points (where both eigenvalues of the Jacobian matrix at the point equal to zero), the situation is much more difficult. The Andreev theorem (see [2]) classifies the nilpotent singular points, whose associated Jacobian matrix is not identically zero, except the monodromic case. If the Jacobian matrix is identically null, the problem is open. In this case, the only possibility is studying each degenerate singular point case by case.

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In polar coordinates $x = r\cos\theta, y = r\sin\theta$, the system (1.1) becomes

$$\begin{cases} \dot{r} = \dfrac{1}{r}(\cos\theta X(r\cos\theta, r\sin\theta) + \sin\theta Y(r\cos\theta, r\sin\theta)), \\ \dot{\theta} = \dfrac{1}{r^2}(\cos\theta Y(r\cos\theta, r\sin\theta) - \sin\theta X(r\cos\theta, r\sin\theta)). \end{cases} \tag{1.2}$$

Following Frommer [7], a direction of $\theta = \theta_0$ at the origin is called a critical direction of system (1.2) if there exists a sequence of points $P_n : (r_n, \theta_n)$ such that $\theta_n \to \theta_0,\ r_n \to 0,\ \tan\alpha_n \to 0$ as $n \to \infty$, where α_n is the angle turning anti-clockwise from the direction θ_n to the vector field at P_n.

There are several methods to study the number of orbits tending to the singular point in the critical direction. The first method is the so-called Z-sector (see [8]). The small neighborhood $U(O, r)$ near the origin is divided into a finite number of sectors by the branches of $Z(x, y) = xX(x, y) + yY(x, y) = 0$, these sectors are called Z-sectors. Within a Z-sector, $\frac{\mathrm{d}r}{\mathrm{d}t}$ has a fixed sign and contains a finitely number of critical directions.

The second method is to analysis normal sectors (normal domain) (see [10, 14]). A circular sector $S : 0 \le r \le r_0,\ \theta_1 \le \theta \le \theta_2$ is called a normal sector if (i) neither θ_1 nor θ_2 is a critical direction, (ii) within S, $\frac{\mathrm{d}r}{\mathrm{d}t} \ne 0$, (iii) between θ_1 and θ_2 there has a finite (possible zero) number of critical directions. Normal sectors usually construct as a small neighborhood of a critical direction, so in each normal sector there has at most one critical direction.

The third method is generalized normal sectors (see [12–13]), which is developed from normal sectors, do not restrict edges of the sectors to the radial lines but even allow orbits, horizontal isoclines and vertical isoclines to be edges.

A Z-sector possibly has more than one critical direction, so we can not determine which critical direction an orbit in the sector will be tangent to. Moreover, sometimes Z-branches of $Z(x, y) = 0$ are hardly solved, so it is difficult to find all Z-sector branches. For a normal sector, sometimes it is not always constructible about an critical direction. Construction of generalized normal sector is a technical task, generally drawing horizontal isoclines and vertical isoclines is helpful for analysis. We note that the common point in the above three methods is to apply the classic polar coordinate to analysis.

In this paper we apply the method of quasi normal sectors, which is a generalization of normal sectors, to determine orbits in the critical direction.

The organization of this paper is as follows: In Section 2, we construct the method of quasi normal sector and statement some preliminary results. Section 3 contains the main results. In Section 4, we give two concrete examples to show that our method maybe more effective than the above three methods in some cases. In the last section, we introduce the method of Newton polyhedron.

2 Quasi Normal Sector

Given $p, q, d \in N$, we say that a function $F(x, y)$ is called (p, q)-quasi-homogeneous function of weight degree d if $F(\lambda^p x, \lambda^q y) = \lambda^d F(x, y)$. A vector field $(P(x, y), Q(x, y))$ is called (p, q)-quasi-homogeneous of weight degree d if $P(x, y)$ and $Q(x, y)$ are (p, q)-quasi-homogeneous functions of weight degree $d + p - 1$ and $d + q - 1$ respectively.

For a given C^k differential system (1.1), if k is sufficiently large, we can always choose conveniently a pair of positive integers (p, q) by the Newton polyhedron (see [4]), hence the system (1.1) can be written as the following differential system:

$$\begin{cases}\dot{x} = X(x,y) = X_m(x,y) + \Phi(x,y),\\ \dot{y} = Y(x,y) = Y_n(x,y) + \Psi(x,y),\end{cases} \tag{2.1}$$

where $X_m(\lambda^p x, \lambda^q y) = \lambda^{m+p-1} X_m(x,y)$, $Y_n(\lambda^p x, \lambda^q y) = \lambda^{n+q-1} Y_n(x,y)$, $\Phi(x,y) = o((px^{2q} + qy^{2p})^{\frac{m+p-1}{2pq}})$, and $\Psi(x,y) = o((px^{2q} + qy^{2p})^{\frac{n+q-1}{2pq}})$. This is a different way of writing system (1.1).

In order to simplify the proof in this paper, firstly we have the following lemma.

Lemma 2.1 *Suppose $p = kp_1$, $q = kq_1$, $(p_1, q_1) = 1$, then $X_m(x,y)$ and $Y_n(x,y)$ are (p_1, q_1)-quasi-homogeneous functions of weight degree $m_1 + p_1 - 1$ and $n_1 + q_1 - 1$ respectively, where $m_1 = \frac{m-1}{k} + 1$ and $n_1 = \frac{n-1}{k} + 1$. Moreover, $\Phi(x,y) = o((p_1 x^{2q_1} + q_1 y^{2p_1})^{\frac{m_1+p_1-1}{2p_1q_1}})$, $\Psi(x,y) = o((p_1 x^{2q_1} + q_1 y^{2p_1})^{\frac{n_1+q_1-1}{2p_1q_1}})$.*

Proof From [11], we know that m_1 and n_1 are positive integers. Straight-forward calculations show that

$$X_m(\lambda^{p_1} x, \lambda^{q_1} y) = X_m(\lambda^{\frac{p}{k}} x, \lambda^{\frac{q}{k}} y) = \lambda^{\frac{m+p-1}{k}} X_m(x,y) = \lambda^{m_1+p_1-1} X_m(x,y),$$
$$Y_n(\lambda^{p_1} x, \lambda^{q_1} y) = Y_n(\lambda^{\frac{p}{k}} x, \lambda^{\frac{q}{k}} y) = \lambda^{\frac{n+q-1}{k}} Y_n(x,y) = \lambda^{n_1+q_1-1} Y_n(x,y).$$

It is obvious that $X_m(x,y)$ and $Y_n(x,y)$ are (p_1, q_1)-quasi-homogeneous functions of weight degree $m_1 + p_1 - 1$ and $n_1 + q_1 - 1$ respectively. Moreover,

$$(px^{2q} + qy^{2p})^{\frac{m+p-1}{2pq}} = (kp_1 x^{2kq_1} + kq_1 y^{2kp_1})^{\frac{m_1+p_1-1}{2kp_1q_1}} < C(p_1 x^{2q_1} + q_1 y^{2p_1})^{\frac{m_1+p_1-1}{2p_1q_1}},$$

where $C = k^{\frac{m_1+p_1-1}{2kp_1q_1}}$. In our hypotheses that $\Phi(x,y) = o((px^{2q} + qy^{2p})^{\frac{m+p-1}{2pq}})$, then we have $\Phi(x,y) = o((p_1 x^{2q_1} + q_1 y^{2p_1})^{\frac{m_1+p_1-1}{2p_1q_1}})$. In a similar way, $\Psi(x,y) = o((p_1 x^{2q_1} + q_1 y^{2p_1})^{\frac{n_1+q_1-1}{2p_1q_1}})$.

Without loss of generality, we assume $p \leqslant q$, otherwise we can interchange the variables x and y.

In order to study the origin $O(0,0)$ of the differential system (2.1), following Lyapunov [9], we introduce the generalized (p,q)-trigonometric functions $x(\theta) = \mathrm{Cs}\,\theta$ and $y(\theta) = \mathrm{Sn}\,\theta$ as the unique solution of the following initial problem:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -y^{2p-1}, \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = x^{2q-1}, \quad x(0) = p^{-\frac{1}{2q}}, \quad y(0) = 0.$$

We observe that, in the particular case $p = q = 1$, $\mathrm{Sn}\,\theta = \sin\theta$, $\mathrm{Cs}\,\theta = \cos\theta$. Therefore, the previous definition gives the classic trigonometric functions. We also define

$$\mathrm{Tn}\,\theta = \frac{\mathrm{Sn}^p\theta}{\mathrm{Cs}^q\theta}, \quad \mathrm{Ctn}\theta = \frac{\mathrm{Cs}^q\theta}{\mathrm{Sn}^p\theta}.$$

Some properties of these functions are stated in the next lemma.

Lemma 2.2 (see [5]) *Function $\mathrm{Sn}\,\theta$ and $\mathrm{Cs}\,\theta$ satisfy the following properties.*
(i) $\mathrm{Sn}\,\theta$ *and* $\mathrm{Cs}\,\theta$ *are T-periodic functions, where*

$$T = 2p^{\frac{-1}{2q}} q^{\frac{-1}{2p}} \frac{\Gamma(\frac{1}{2p})\Gamma(\frac{1}{2q})}{\Gamma(\frac{1}{2p} + \frac{1}{2q})},$$

(ii) $p\mathrm{Cs}^{2q}\theta + q\mathrm{Sn}^{2p}\theta = 1$,

(iii) $\mathrm{Cs}(-\theta) = \mathrm{Cs}\,\theta$,
(iv) $\mathrm{Sn}(-\theta) = -\mathrm{Sn}\,\theta$,
(v) $\mathrm{Cs}\big(\frac{T}{2} - \theta\big) = -\mathrm{Cs}\,\theta$,
(vi) $\mathrm{Sn}\big(\frac{T}{2} - \theta\big) = \mathrm{Sn}\,\theta$,
(vii) $\mathrm{Cs}\big(\frac{T}{2} + \theta\big) = -\mathrm{Cs}\,\theta$,
(viii) $\mathrm{Sn}\big(\frac{T}{2} + \theta\big) = -\mathrm{Sn}\,\theta$.

In the generalized (p,q)-polar coordinates $x = r^p\mathrm{Cs}\,\theta,\ y = r^q\mathrm{Sn}\,\theta$, (2.1) becomes

$$\begin{cases}\dot r = \dfrac{r^q\mathrm{Cs}^{2q-1}X(r^p\mathrm{Cs}\,\theta, r^q\mathrm{Sn}\,\theta) + r^p\mathrm{Sn}^{2p-1}\theta Y(r^p\mathrm{Cs}\,\theta, r^q\mathrm{Sn}\,\theta)}{r^{p+q-1}(p\mathrm{Cs}^{2q}\theta + q\mathrm{Sn}^{2p}\theta)},\\ \dot\theta = \dfrac{pr^{p-1}\mathrm{Cs}\,\theta Y(r^p\mathrm{Cs}\,\theta, r^q\mathrm{Sn}\,\theta) - qr^{q-1}\mathrm{Sn}\,\theta X(r^p\mathrm{Cs}\,\theta, r^q\mathrm{Sn}\,\theta)}{r^{p+q-1}(p\mathrm{Cs}^{2q}\theta + q\mathrm{Sn}^{2p}\theta)}.\end{cases}$$

View θ as the new independent variable, then we can transform the above differential system into the equivalent differential equation

$$\begin{aligned} r\frac{\mathrm{d}\theta}{\mathrm{d}r} &= \frac{p\mathrm{Cs}\,\theta r^pY(r^p\mathrm{Cs}\,\theta, r^q\mathrm{Sn}\,\theta) - q\mathrm{Sn}\,\theta r^qX(r^p\mathrm{Cs}\,\theta, r^q\mathrm{Sn}\,\theta)}{\mathrm{Cs}^{2q-1}\theta r^qX(r^p\mathrm{Cs}\,\theta, r^q\mathrm{Sn}\,\theta) + \mathrm{Sn}^{2p-1}\theta r^pY(r^p\mathrm{Cs}\,\theta, r^q\mathrm{Sn}\,\theta)} \\ &= \frac{G(\theta) + \eta_1(r,\theta)}{H(\theta) + \eta_2(r,\theta)}, \end{aligned} \tag{2.2}$$

where $\eta_1(r,\theta) = o(1),\ \eta_2(r,\theta) = o(1)$ as $r = (px^{2q} + qy^{2p})^{\frac{1}{2pq}} \to 0$,

$$G(\theta) = \begin{cases} -q\mathrm{Sn}\,\theta X_m(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta), & \text{if } m < n,\\ p\mathrm{Cs}\,\theta Y_n(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) - q\mathrm{Sn}\,\theta X_m(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta), & \text{if } m = n,\\ p\mathrm{Cs}\,\theta Y_n(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta), & \text{if } m > n,\end{cases} \tag{2.3}$$

$$H(\theta) = \begin{cases} \mathrm{Cs}^{2q-1}\theta X_m(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta), & \text{if } m < n,\\ \mathrm{Cs}^{2q-1}\theta X_m(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) + \mathrm{Sn}^{2p-1}\theta Y_n(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta), & \text{if } m = n,\\ \mathrm{Sn}^{2p-1}\theta Y_n(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta), & \text{if } m > n.\end{cases} \tag{2.4}$$

Definition 2.1 *If $G(\theta_k) = 0$ and $H(\theta_k) \neq 0$, then we say $\theta = \theta_k$ is a regular critical direction.*

Proposition 2.1 *From* (2.3)–(2.4)*, the following statements hold:*
(i) *If $m < n$ and $X_m(1,0) \neq 0$, then 0 and $\frac{T}{2}$ are regular critical direction.*
(ii) *If $m > n$ and $Y_n(0,1) \neq 0$, then $\frac{T}{4}$ and $-\frac{T}{4}$ are regular critical direction.*
(iii) *Assume $m = n$.*

If p is even and q is odd, then θ_k is a regular critical direction if and only if $-\theta_k$ is a regular critical direction.

If p is odd and q is odd, then θ_k is a regular critical direction if and only if $\frac{T}{2} + \theta_k$ is a regular critical direction.

If p is odd and q is even, then θ_k is a regular critical direction if and only if $\frac{T}{2} - \theta_k$ is a regular critical direction.

Proof The statements (i)–(ii) are obvious from the definition of regular critical direction.

Now we prove the statement (iii). Suppose that p is even and q is odd, then we get that

$$\begin{aligned}G(-\theta) &= p\mathrm{Cs}(-\theta)Y_m(\mathrm{Cs}(-\theta),\mathrm{Sn}(-\theta)) - q\mathrm{Sn}(-\theta)X_m(\mathrm{Cs}(-\theta),\mathrm{Sn}(-\theta))\\ &= p\mathrm{Cs}\,\theta Y_m(\mathrm{Cs}\,\theta,-\mathrm{Sn}\,\theta) + q\mathrm{Sn}\,\theta X_m(\mathrm{Cs}\,\theta,-\mathrm{Sn}\,\theta)\\ &= p(-1)^p\mathrm{Cs}\,\theta Y_m((-1)^p\mathrm{Cs}\,\theta,(-1)^q\mathrm{Sn}\,\theta) - q(-1)^q\mathrm{Sn}\,\theta X_m((-1)^p\mathrm{Cs}\,\theta,(-1)^q\mathrm{Sn}\,\theta)\\ &= (-1)^{m+p+q-1}(p\mathrm{Cs}\,\theta Y_m(\mathrm{Cs}\,\theta,\mathrm{Sn}\,\theta) - q\mathrm{Sn}\,\theta X_m(\mathrm{Cs}\,\theta,\mathrm{Sn}\,\theta))\\ &= (-1)^m G(\theta)\end{aligned}$$

and

$$\begin{aligned}H(-\theta) &= \mathrm{Cs}^{2q-1}(-\theta)X_m(\mathrm{Cs}(-\theta),\mathrm{Sn}(-\theta)) + \mathrm{Sn}^{2p-1}(-\theta)Y_m(\mathrm{Cs}(-\theta),\mathrm{Sn}(-\theta))\\ &= (-1)^{m+1}(\mathrm{Cs}^{2q-1}\theta X_m(\mathrm{Cs}\,\theta,\mathrm{Sn}\,\theta) + \mathrm{Sn}^{2p-1}\theta Y_m(\mathrm{Cs}\,\theta,\mathrm{Sn}\,\theta))\\ &= (-1)^{m+1}H(\theta).\end{aligned}$$

It is obvious that θ_k is a regular critical direction if and only if $-\theta_k$ is a regular critical direction.

If p is odd and q is odd, then in a similar way, we have $G\big(\frac{T}{2}+\theta\big) = (-1)^{m+1}G(\theta)$ and $H\big(\frac{T}{2}+\theta\big) = (-1)^{m+1}H(\theta)$. It is obvious that θ_k is a regular critical direction if and only if $\frac{T}{2}+\theta_k$ is a regular critical direction.

If p is odd and q is even, then we have $G\big(\frac{T}{2}-\theta\big) = (-1)^m G(\theta)$ and $H\big(\frac{T}{2}-\theta\big) = (-1)^{m+1}H(\theta)$. It is obvious that θ_k is a regular critical direction if and only if $\frac{T}{2}-\theta_k$ is a regular critical direction.

Consider a sector: $\Delta\widehat{OAB}: \theta_1 \le \theta \le \theta_2,\ 0 \le r \le r_0$, where

$$\begin{aligned}&\widehat{OA} = \Big\{(x,y) \mid \frac{y^p}{x^q} = \mathrm{Tn}(\theta_1)\Big\}, \quad \widehat{OB} = \Big\{(x,y) \mid \frac{y^p}{x^q} = \mathrm{Tn}(\theta_2)\Big\},\\ &\widehat{AB} = \{(x,y) \mid px^{2q} + qy^{2p} = r^{2pq}\}.\end{aligned}$$

Definition 2.2 *A sector $\Delta\widehat{OAB}$ is called quasi normal sector (QNS) if*

(C1) *there are no singular point in closure $cl(\Delta\widehat{OAB})$ except at $O(0,0)$,*

(C2) $\frac{\mathrm{d}r}{\mathrm{d}t} \neq 0$, *in $cl(\Delta\widehat{OAB}) \setminus \{O\}$,*

(C3) *between θ_1 and θ_2 there is at most one regular critical direction, neither θ_1 nor θ_2 is a regular critical direction.*

Without loss of generality, assume that $\frac{\mathrm{d}r}{\mathrm{d}t} < 0$. We need to consider only the three essentially different type of orbit behavior which are distinguished by the behavior of orbits as they cross the boundary segments with increasing t, as shown in Figure 1. The case $\frac{\mathrm{d}r}{\mathrm{d}t} > 0$ is the same with reversed time $-t$.

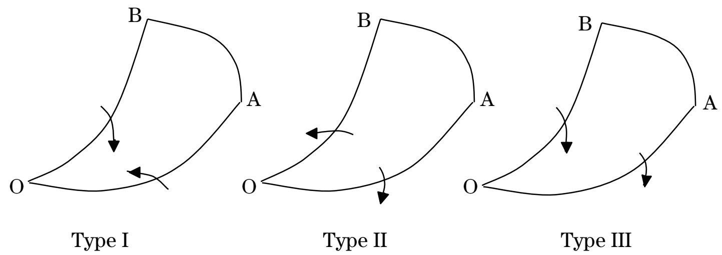


Figure 1 Three types of QNS.

The following four lemmas are generalization of the results in [14], and the method of proof these lemmas is similar.

Lemma 2.3 *If $\Delta\widehat{OAB}$ is a QNS of type* I*, then* (2.1) *has infinitely many orbits tending to the origin $O(0,0)$ in $\Delta\widehat{OAB}$ as $t \to +\infty$.*

Lemma 2.4 *If $\Delta\widehat{OAB}$ is a QNS of type* II*. then* (2.1) *has either a unique orbit or infinitely many orbits tending to the origin $O(0,0)$ in $\Delta\widehat{OAB}$ as $t \to +\infty$.*

Lemma 2.5 *If $\Delta\widehat{OAB}$ is a QNS of type* III*. then* (2.1) *has either no orbit or infinitely many orbits tending to the origin $O(0,0)$ in $\Delta\widehat{OAB}$ as $t \to +\infty$.*

Lemma 2.6 *If $G(\theta) \neq 0$ in $\Delta\widehat{OAB} : \theta_1 \leq \theta \leq \theta_2,\ 0 \leq r \leq r_1 \leq r_0$, where r_1 is sufficiently small, then there is no orbit tending to the origin $O(0,0)$ in $\Delta\widehat{OAB}$ as $t \to +\infty$. The orbits from one edge $\theta = \theta_1,\ 0 < r \leq r_1$ (resp. $\theta = \theta_2,\ 0 < r \leq r_1$) to another edge $\theta = \theta_2,\ 0 < r \leq r_1$ (resp. $\theta = \theta_1,\ 0 < r \leq r_1$).*

3 Main Results

For convenience, in this section we say the orbit $(x(t), y(t))$ of (2.1) tends to the origin if $(x(t), y(t)) \to (0,0)$ as $t \to +\infty$ or $t \to -\infty$.

We want to determine the number of orbits tending to the origin $O(0,0)$ in regular critical direction. According to the number of real roots of equation $G(\theta) = 0$, we can distinguish to the following three cases.

3.1 $G(\theta) = 0$ has no real root

Theorem 3.1 *If $G(\theta) = 0$ has no real roots, then $O(0,0)$ is a monodromic.*

Proof Let $\Delta\widehat{OAB} : 0 \leq \theta \leq T,\ 0 \leq r \leq r_1 \leq r_0$. There is no orbit tending to the origin $O(0,0)$ by Lemma 2.6, so $O(0,0)$ is a monodromic.

3.2 $G(\theta) \equiv 0$, the number of regular critical direction is infinite

In this case, it is obvious that $m = n$. Assume that $\Phi(x,y)$ and $\Psi(x,y)$ are analytic functions with respect to variables x and y near the origin $O(0,0)$.

$$\begin{aligned} G(\theta) &= p\mathrm{Cs}\,\theta Y_m(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) - q\mathrm{Sn}\,\theta X_m(\mathrm{Cs}\,\theta, \mathrm{Sn}\,\theta) \\ &= \frac{1}{r^{m+p+q-1}}[pxY_m(x,y) - qyX_m(x,y)] \equiv 0. \end{aligned}$$

Applying quasi-homogeneous directional blow-up (see [1]) in the positive (resp. negative) x direction $x = +u^p \geq 0,\ y = u^q v$ (resp. $x = -u^p \leq 0,\ y = u^q v$), (2.1) becomes

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{p\Psi^*_\pm(u,v) - qv\Phi^*_\pm(u,v)}{X_m(\pm 1, v) + u\Phi^*_\pm(u,v)}, \tag{3.1}$$

where

$$X_m(\pm 1, v) = \frac{X_m(\pm u^p, u^q v)}{u^{m+p-1}}, \quad \Phi^*_\pm(u,v) = \frac{\Phi(\pm u^p, u^q v)}{u^{m+p}}, \quad \Psi^*_\pm(u,v) = \frac{\pm\Psi(\pm u^p, u^q v)}{u^{m+q}}.$$

Note that if p is odd, then the positive x directional blow-up provides the information of the negative x directional blow-up. We only need employ the x directional blow-up $x = u^p \in \mathbb{R}$, $y = u^q v$.

In the following we only consider the positive x directional blow-up, the negative case is similar.

Case (a) $X_m(+1, v_k) \neq 0$.

Note that $(0, v_k)$ is a nonsingular point of system $(3.1)_+$, then there exists one and only one orbit cutting across the v-axis at $(0, v_k)$ in the (u, v)-plane (see Figure 2.1).

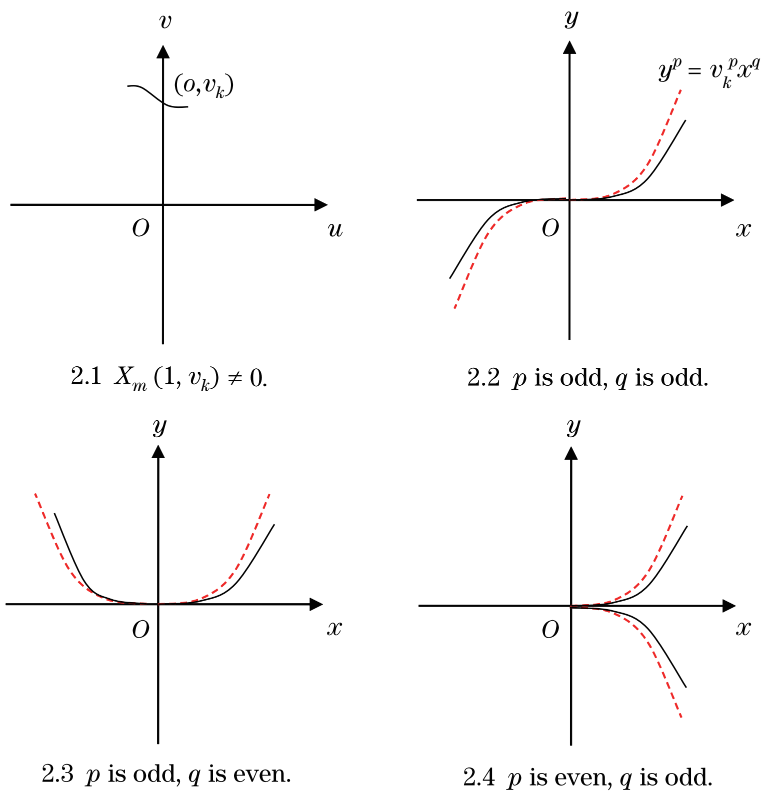


Figure 2 $X_m(1, v_k) \neq 0$.

Case (b) $X_m(+1, v_k) = 0$ and $p\Psi_+^*(0, v_k) - qv_k\Phi_+^*(0, v_k) \neq 0$.

In this case, consider the following system

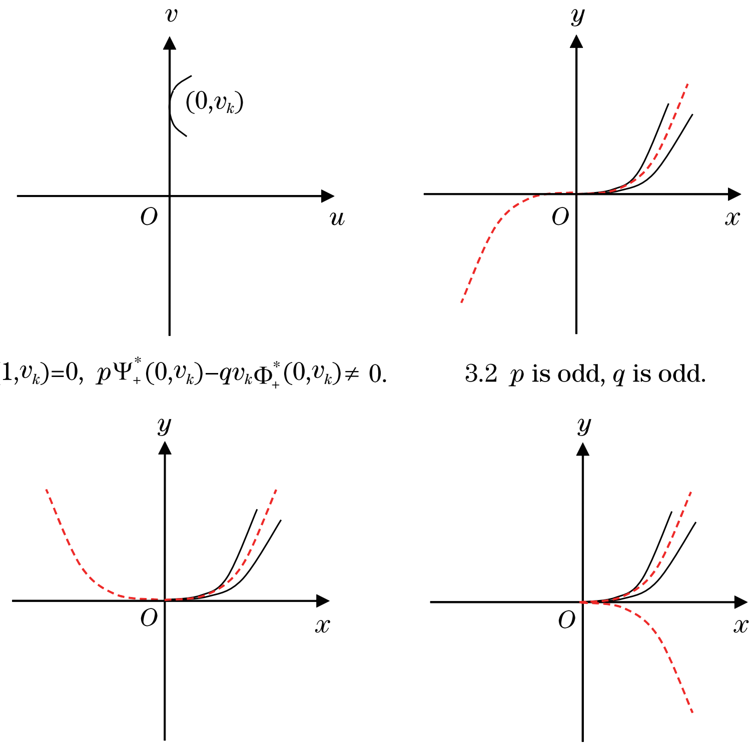
$$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{X_m(+1, v) + u\Phi_+^*(u, v)}{p\Psi_+^*(u, v) - qv\Phi_+^*(u, v)}. \tag{3.2}$$

It is easy to verify that $(0, v_k)$ is a nonsingular point of system (3.2) and the unique orbit through $(0, v_k)$ is tangent to the v-axis. Figure 3.1 denotes that the orbit remains in the same half-plane $u \geq 0$ and Figure 4.1 denotes that the orbit cuts across the v-axis at $(0, v_k)$.

Since $pxY_m(x, y) - qyX_m(x, y) \equiv 0$, then $X_m(+1, v)$ have at most $\left[\frac{m-1}{q}\right]$ real roots, where $[\zeta]$ denotes the integer part function of ζ. The fact follows from

$$\begin{aligned} pxY_m(x, y) - qyX_m(x, y) &= pu^pY_m(u^p, u^qv) - qu^qvX_m(u^p, u^qv) \\ &= u^{m+p+q-1}(pY_m(1, v) - qvX_m(1, v)), \end{aligned}$$

and $Y_m(1, v)$ is a polynomial in variable v with degree less than or equal to $\frac{m+q-1}{q}$. Therefore $X_m(1, v)$ is a polynomial in variable v with degree less than or equal to $\left[\frac{m-1}{q}\right]$.



3.1 $X_m(1,v_k)=0,\ p\Psi_+^*(0,v_k)-qv_k\Phi_+^*(0,v_k)\neq 0.$ 3.2 p is odd, q is odd.

3.3 p is odd, q is even. 3.4 p is even, q is odd.

Figure 3 $X_m(1, v_k) = 0,\ p\Psi_+^*(0, v_k) - qv_k\Phi_+^*(0, v_k) \neq 0.$
The orbit remains in the same half-plane $u \geq 0$.

Case (c) $X_m(+1, v_k) = 0$ and $p\Psi_+^*(0, v_k) - qv_k\Phi_+^*(0, v_k) = 0$.

In this case, $(0, v_k)$ is a singular point of system $(3.1)_+$, the behavior of solution near such a singular point may belong to any of the types discussed in the above cases. Since (2.1) is an analytic system, the desingularization theorem (see [6]) ensures that after a finite number of such blow-up, it is possible to solve this problem.

To study the behavior of orbits near the critical directions $\frac{T}{4}$ and $-\frac{T}{4}$, we employ the quasi-homogeneous directional blow-up in the positive (resp. negative) y direction $x = \widetilde{u}\widetilde{v}^p$, $y = +\widetilde{v}^q \geq 0$ (resp. $x = \widetilde{u}\widetilde{v}^p$, $y = -\widetilde{v}^q \leq 0$). (2.1) becomes

$$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}\widetilde{v}} = \frac{q\overline{\Phi}_\pm^*(\widetilde{u}, \widetilde{v}) - p\widetilde{u}\overline{\Psi}_\pm^*(\widetilde{u}, \widetilde{v})}{Y_m(\widetilde{u}, \pm 1) + \widetilde{v}\overline{\Psi}_\pm^*(\widetilde{u}, \widetilde{v})}, \tag{3.3}$$

where

$$Y_m(\widetilde{u}, \pm 1) = \frac{Y_m(\widetilde{u}\widetilde{v}^p, \pm\widetilde{v}^q)}{\widetilde{v}^{m+q-1}}, \quad \overline{\Phi}_\pm^*(\widetilde{u}, \widetilde{v}) = \frac{\pm\Phi(\widetilde{u}\widetilde{v}^p, \pm\widetilde{v}^q)}{\widetilde{v}^{m+p}}, \quad \overline{\Psi}_\pm^*(\widetilde{u}, \widetilde{v}) = \frac{\Psi(\widetilde{u}\widetilde{v}^p, \pm\widetilde{v}^q)}{\widetilde{v}^{m+q}}.$$

Note that if q is odd, then the positive y directional blow-up provides the information of negative y directional blow-up. So we only need to employ y directional blow-up $x = \widetilde{u}\widetilde{v}^p$, $y = \widetilde{v}^q \in \mathbb{R}$.

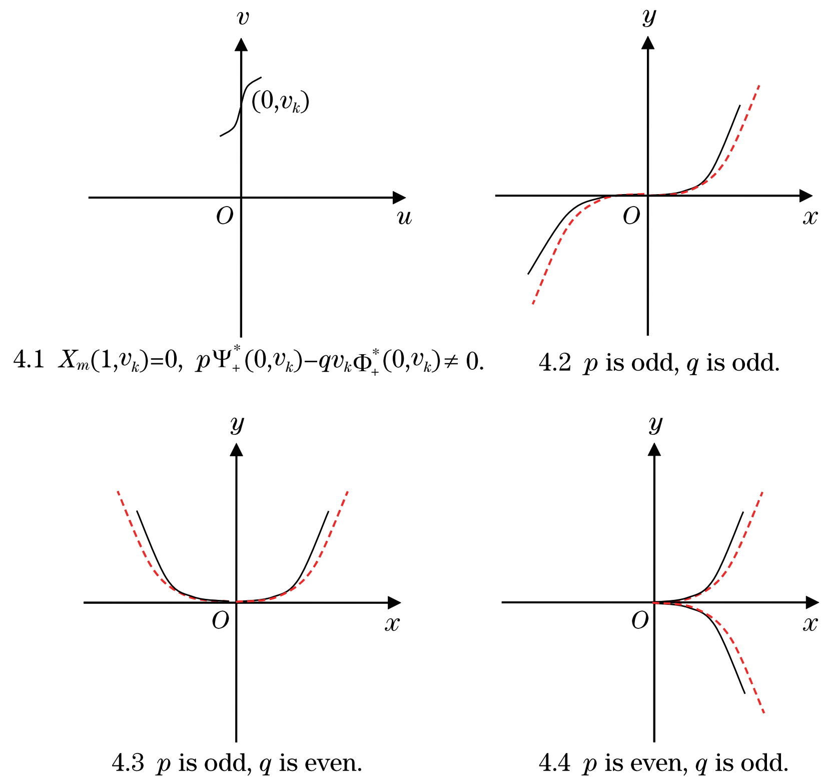


Figure 4 $X_m(1, v_k) = 0,\ p\Psi_+^*(0, v_k) - qv_k\Phi_+^*(0, v_k) \neq 0$. The orbit cuts across the v-axis.

In the following we also only consider the positive y directional blow-up.

Case (d) $Y_m(0, +1) \neq 0$.

There is a unique orbit of system $(3.3)_+$ through the origin $O(0,0)$ and cuts cross the $\widetilde{u}$-axis on the $(\widetilde{v}, \widetilde{u})$-plane (see Figure 5.1).

Case (e) $Y_m(0, +1) = 0,\ \overline{\Phi}_+^*(0, 0) \neq 0$.

In this case, consider the following system

$$\frac{\mathrm{d}\widetilde{v}}{\mathrm{d}\widetilde{u}} = \frac{Y_m(\widetilde{u}, +1) + \widetilde{v}\overline{\Psi}_+^*(\widetilde{u}, \widetilde{v})}{q\overline{\Phi}_+^*(\widetilde{u}, \widetilde{v}) - p\widetilde{u}\overline{\Psi}_+^*(\widetilde{u}, \widetilde{v})}. \tag{3.4}$$

We know that $(0,0)$ is a nonsingular point of system (3.4) and the unique orbit through $(0,0)$ is tangent to the $\widetilde{u}$-axis. Figures 6.1 and 7.1 denote that the orbit remains in the same half-plane $\widetilde{v} \geq 0$ and cuts across the $\widetilde{u}$-axis, respectively.

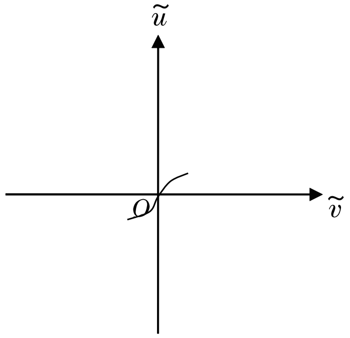
Case (f) $Y_m(0, +1) = 0$ and $\overline{\Phi}_+^*(0, 0) = 0$.

In this case, $O(0,0)$ is a singular point of $(3.3)_+$, finite number of such blow-up will solve this problem.

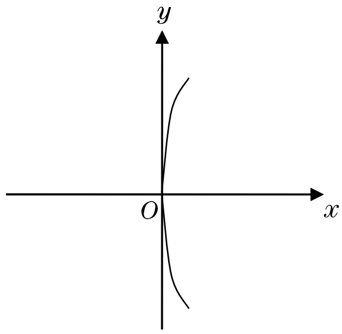
From the above analysis, we can obtain the following theorems.

Theorem 3.2 *Consider the analytic differential system* (2.1) *with* $G(\theta) \equiv 0$, *and assume that* p *and* q *are odd.*

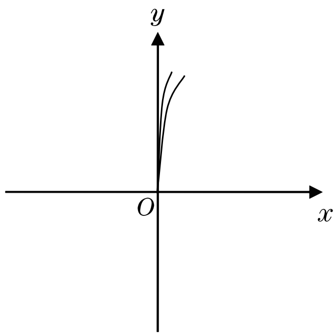
(i) *If* $X_m(1, v_k) \neq 0$, *then there is a unique orbit of* (2.1) *tending to the origin in the direction* θ_k *as well as a unique orbit tending to the origin in the direction* $\frac{T}{2} + \theta_k$ (*see Figure* 2.2), *where*



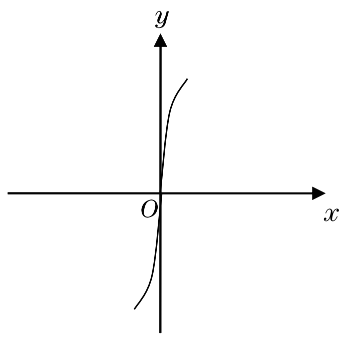
5.1 $Y_m(0,1)\neq 0$.



5.2 p is odd, q is odd.



5.3 p is odd, q is even.



5.4 p is even, q is odd.

Figure 5 $Y_m(0,1)\neq 0$.

$\operatorname{Tn}\theta_k = v_k^p, -\frac{T}{4} < \theta_k < \frac{T}{4}$.

(ii) *Let* $X_m(1,v_k)=0,\ p\Psi_+^*(0,v_k)-qv_k\Phi_+^*(0,v_k)\neq 0$.

If the orbit remains in the same half-plane near $(0,v_k)$, *say in the half-plane* $u\geq 0$ (*resp.* $u\leq 0$), *then there are exactly two orbits tending to the origin in the direction* θ_k (*resp.* $\frac{T}{2}+\theta_k$), *one on each side of the curve and there is no orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{2}+\theta_k$ (*resp.* θ_k) (*see Figure* 3.2).

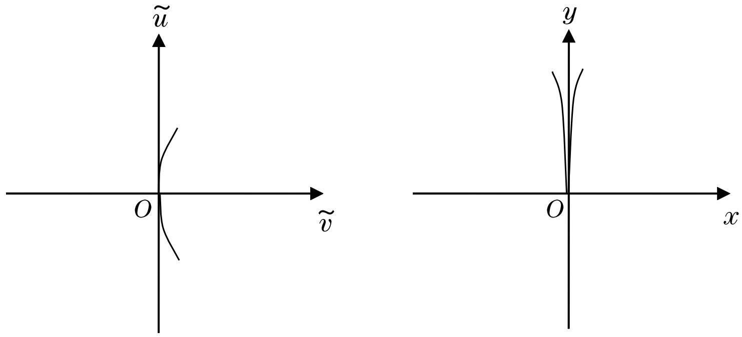
If the orbit cuts across the u-axis at $(0,v_k)$, *then there is a unique orbit of* (2.1) *tending to the origin in the direction* θ_k *as well as a unique orbit tending to the origin in the direction* $\frac{T}{2}+\theta_k$ (*see Figure* 4.2).

(iii) *If* $Y_m(0,1)\neq 0$, *then there is a unique orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{4}$ *as well as a unique orbit tending to the origin in the direction* $-\frac{T}{4}$ (*see Figure* 5.2).

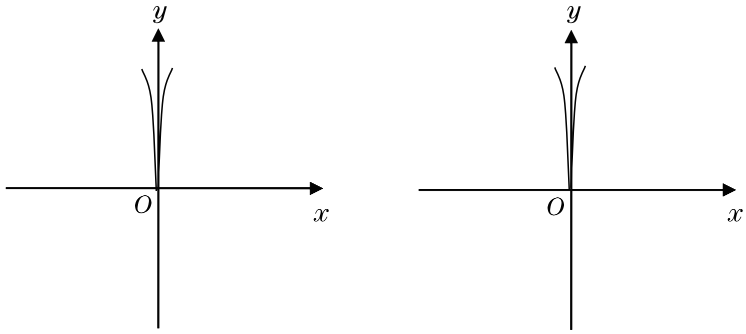
(iv) *Let* $Y_m(0,1)=0,\ \overline{\Phi}_+^*(0,0)\neq 0$.

If the orbit remains in the same half-plane $\widetilde{v}\geq 0$ (*resp.* $\widetilde{v}\leq 0$), *then there exist two orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{4}$ (*resp.* $-\frac{T}{4}$), *one on each side of the y-axis and there is no orbit of* (2.1) *tending to the origin in the direction* $-\frac{T}{4}$ (*resp.* $\frac{T}{4}$) (*see Figure* 6.2).

If the orbit cuts across the $\widetilde{u}$*-axis at* $(0,0)$, *then there is a unique orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{4}$ *as well as a unique orbit tending to the origin in the direction* $-\frac{T}{4}$ (*see Figure* 7.2).



6.1 $Y_m(0,1)=0,\ \overline{\Phi}_+^*(0,0)\neq 0.$ 6.2 p is odd, q is odd.



6.3 p is odd, q is even. 6.4 p is even, q is odd.

Figure 6 $Y_m(0,1)=0,\ \overline{\Phi}_+^*(0,0)\neq 0$. Orbits remains in the same half-plane $\widetilde{v}\geq 0$.

(v) *If* $X_m(1,v_k)=0,\ p\Psi_+^*(0,v_k)-qv_k\Phi_+^*(0,v_k)=0$ *or* $Y_m(0,1)=0,\ \overline{\Phi}_+^*(0,0)=0$*, then successive application of the above transformation can solve this problem.*

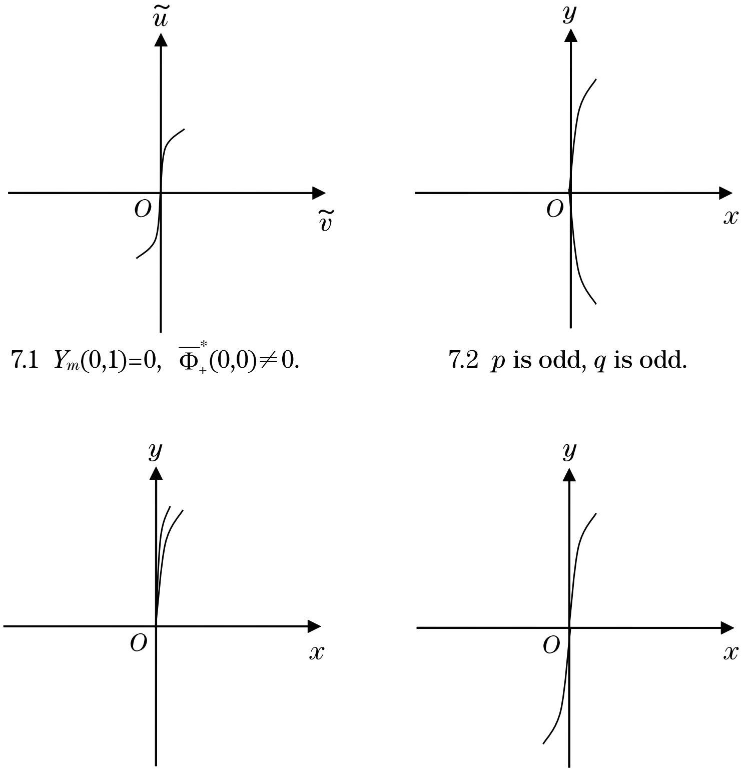
Proof Because p and q are odd, we only need employ the x directional blow-up $x=u^p\in\mathbb{R},\ y=u^qv$ and y directional blow-up $x=\widetilde{u}\widetilde{v}^p,\ y=\widetilde{v}^q\in\mathbb{R}$.

(i) In this case, there exist orbits of system (2.1) tending to the origin along the curve $y^p=v_k^px^q$. For every generalized polar direction θ_k in the (x,y)-plane such that $\mathrm{Tn}\,\theta_k=v_k^p,\ -\frac{T}{4}<\theta_k<\frac{T}{4}$, there is a unique orbit of (2.1) tending to the origin in the direction θ_k as well as a unique orbit tending to the origin in the direction $\frac{T}{2}+\theta_k$.

(ii) In the first subcase, if $u\geq 0$ (resp. $u\leq 0$), then $x=u^p\geq 0$ (resp. $x=u^p\leq 0$). So there are exactly two orbits tending to the origin in the direction θ_k (resp. $\frac{T}{2}+\theta_k$), one on each side of the curve and there is no orbit of (2.1) tending to the origin in the direction $\frac{T}{2}+\theta_k$ (resp. θ_k). The second subcase is similar to case (i).

(iii) If $Y_m(0,1)\neq 0$, then it is clearly that the orbit of system (3.3) cuts across the $\widetilde{u}$-axis. The statement follows from $y=\widetilde{v}^q$.

(iv) In the first subcase, if $\widetilde{v}\geq 0$ (resp. $\widetilde{v}\leq 0$), then $y=\widetilde{v}^q\geq 0$ (resp. $y=\widetilde{v}^q\leq 0$). So there are exactly two orbits tending to the origin in the direction $\frac{T}{4}$ (resp. $-\frac{T}{4}$), one on each side of the y-axis and there is no orbit of (2.1) tending to the origin in the direction $-\frac{T}{4}$ (resp. $\frac{T}{4}$). The second subcase is similar to case (iii).



7.1 $Y_m(0,1)=0$, $\overline{\Phi}_+^*(0,0)\neq 0$.

7.2 p is odd, q is odd.

7.3 p is odd, q is even.

7.4 p is even, q is odd.

Figure 7 $Y_m(0,1) = 0$, $\overline{\Phi}_+^*(0,0) \neq 0$. Orbits cuts across the $\widetilde{u}$-axis.

(v) It is obvious from cases (c) and (f).

Theorem 3.3 *Consider the analytic differential system* (2.1) *with* $G(\theta) \equiv 0$, *and assume that* p *and* q *are even.*

(i) *If* $X_m(1, v_k) \neq 0$, *then there is a unique orbit of* (2.1) *tending to the origin in the direction* θ_k *as well as a unique orbit tending to the origin in the direction* $\frac{T}{2} - \theta_k$ (*see Figure* 2.2).

(ii) *Let* $X_m(1, v_k) = 0$, $p\Psi_+^*(0, v_k) - qv_k\Phi_+^*(0, v_k) \neq 0$.

If the orbit remains in the same-half-plane near $(0, v_k)$, *say in the half-plane* $u \geq 0$ (*resp.* $u \leq 0$), *then there are exactly two orbits tending to the origin in the direction* θ_k (*resp.* $\frac{T}{2} - \theta_k$), *one on each side of the curve and there is no orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{2} - \theta_k$ (*resp.* θ_k) (*see Figure* 3.3).

If the orbit cuts across the u-*axis at* $(0, v_k)$, *then there is a unique orbit of* (2.1) *tending to the origin* $O(0,0)$ *in the direction* θ_k *as well as a unique orbit tending to the origin in the direction* $\frac{T}{2} - \theta_k$ (*see Figure* 4.3).

(iii) *If* $Y_m(0, +1) \neq 0$ (*resp.* $Y_m(0, -1) \neq 0$), *then there exist two orbits of* (2.1) *tending to the origin in the direction* $\frac{T}{4}$ (*resp.* $-\frac{T}{4}$) *and there is no orbit tending to the origin in the direction* $-\frac{T}{4}$ (*resp.* $\frac{T}{4}$), (*see Figure* 5.3).

(iv) *If* $Y_m(0, +1) = 0$, $\overline{\Phi}_+^*(0,0) \neq 0$, *then there exist two orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{4}$ *and there is no orbit of* (2.1) *tending to the origin in the direction* $-\frac{T}{4}$ (*see*

Figures 6.3 *and* 7.3).

(v) *If* $Y_m(0,-1)=0,\ \overline{\Phi}^*_-(0,0)\neq 0$, *then there exist two orbit of* (2.1) *tending to the origin in the direction* $-\frac{T}{4}$ *and there is no orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{4}$.

(vi) *If* $X_m(1,v_k)=0,\ p\Psi^*_+(0,v_k)-qv_k\Phi^*_+(0,v_k)=0$ *or* $Y_m(0,\pm1)=0,\ \overline{\Phi}^*_\pm(0,0)=0$, *then successive application of the above transformation can solve this problem.*

Proof Because p is odd, we only need to employ the x directional blow-up. Statements (i) and (ii) follow from the proof of Theorem 3.2 and the statement (vi) is obvious.

If $Y_m(0,+1)\neq 0$ or $Y_m(0,+1)=0,\ \overline{\Phi}^*_+(0,0)\neq 0$, employ the quasi-homogeneous directional blow-up in the positive y direction, then $y=\widetilde{v}^q\geq 0$, so the statements (iii) and (iv) are proved.

If $Y_m(0,-1)\neq 0$ or $Y_m(0,-1)=0,\ \overline{\Phi}^*_-(0,0)\neq 0$, employ the quasi-homogeneous directional blow-up in the negative y direction, then $y=\widetilde{v}^q\leq 0$, so there exist two orbits going into the origin in the direction $-\frac{T}{4}$. The statement (v) is proved.

Theorem 3.4 *Consider the analytic differential system* (2.1) *with* $G(\theta)\equiv 0$, *and assume that* p *and* q *are odd.*

(i) *If* $X_m(+1,v_k)\neq 0$ (*resp.* $X_m(-1,v_k)\neq 0$), *then there is a unique orbit of* (2.1) *tending to the origin in the direction* θ_k (*resp.* $\frac{T}{2}-\theta_k$) *as well as a unique orbit tending to the origin in the direction* $-\theta_k$ (*resp.* $\frac{T}{2}+\theta_k$) (*see Figure* 2.4).

(ii) *Let* $X_m(+1,v_k)=0,\ p\Psi^*_+(0,v_k)-qv_k\Phi^*_+(0,v_k)\neq 0$.

If the orbit remains in the same half-plane near $(0,v_k)$, *say in the half-plane* $u\geq 0$ (*resp.* $u\leq 0$), *then there are exactly two orbits tending to the origin in the direction* $\mathrm{sgn}(v_k)\theta_k$ (*resp.* $-\mathrm{sgn}(v_k)\theta_k$), *one on each side of the curve and there is no orbit of* (2.1) *tending to the origin in the direction* $-\mathrm{sgn}(v_k)\theta_k$ (*resp.* $\mathrm{sgn}(v_k)\theta_k$) (*see Figure* 3.4), *where* $\mathrm{sgn}(v_k)$ *denotes the sign function of* v_k.

If orbit cuts across the u*-axis at* $(0,v_k)$, *then there is a unique orbit of* (2.1) *tending to the origin in the direction* θ_k *as well as a unique orbit tending to the origin in the direction* $-\theta_k$ (*see Figure* 4.4).

(iii) *Let* $X_m(-1,v_k)=0,\ p\Psi^*_-(0,v_k)-qv_k\Phi^*_-(0,v_k)\neq 0$.

If the orbit remains in the same-half-plane near $(0,v_k)$, *say in the half-plane* $u\geq 0$ (*resp.* $u\leq 0$), *then there are exactly two orbits tending to the origin in the direction* $\frac{T}{2}+\mathrm{sgn}(v_k)\theta_k$ (*resp.* $\frac{T}{2}-\mathrm{sgn}(v_k)\theta_k$), *one on each side of the curve and there is no orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{2}-\mathrm{sgn}(v_k)\theta_k$ (*resp.* $\frac{T}{2}+\mathrm{sgn}(v_k)\theta_k$).

If the orbit cuts across the u*-axis at* $(0,v_k)$, *then there is a unique orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{2}+\theta_k$ *as well as a unique orbit tending to the origin in the direction* $\frac{T}{2}-\theta_k$.

(iv) *If* $Y_m(0,1)\neq 0$, *there is a unique orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{4}$ *as well as* $-\frac{T}{4}$ (*see Figure* 5.4).

(v) *Let* $Y_m(0,1)=0,\ \overline{\Phi}^*_+(0,0)\neq 0$.

If the orbit remains in the same half-plane $\widetilde{v}\geq 0$ (*resp.* $\widetilde{v}\leq 0$), *then there exist two orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{4}$ (*resp.* $-\frac{T}{4}$), *one on each side of the* y*-axis and there is no orbit of* (2.1) *tending to the origin in the direction* $-\frac{T}{4}$ (*resp.* $\frac{T}{4}$) (*see Figure* 6.4).

If the orbit cuts across the $\widetilde{u}$*-axis at* $(0,0)$, *then there is a unique orbit of* (2.1) *tending to the origin in the direction* $\frac{T}{4}$ *as well as a unique orbit tending to the origin in the direction* $-\frac{T}{4}$ (*see Figure* 7.4).

(vi) *If* $X_m(\pm1,v_k)=0,\ p\Phi^*_\pm(0,v_k)-qv_k\Psi^*_\pm(0,v_k)=0$ *or* $Y_m(0,1)=0,\ \overline{\Phi}^*_+(0,0)=0$, *then successive application of the above transformation can solve this problem.*

Proof Because q is odd, we only need to employ the y directional quasi-homogeneous blow-up, so the statements (iv), (v) and (vi) are obvious from the proof of Theorem 3.2.

If $X_m(+1, v_k) = 0,\ p\Psi^*_+(0, v_k) - qv_k\Phi^*_+(0, v_k) \neq 0$, employ the quasi-homogeneous directional blow-up in the positive x direction $x = u^p,\ y = u^q v$, then $x \geq 0$ and $\theta_k = \mathrm{ArcTn}(v_k^p) \geq 0$.

For the first subcase, suppose $u \geq 0,\ v_k \geq 0$ (resp. $u \geq 0,\ v_k \leq 0$), then $x \geq 0,\ y \geq 0$ (resp. $x \geq 0,\ y \leq 0$). There are exactly two orbits of (2.1) tending to the origin in the direction $\mathrm{sgn}(v_k)\theta_k$ (resp. $-\mathrm{sgn}(v_k)\theta_k$) and there is no orbit tending to the origin in the direction $-\mathrm{sgn}(v_k)\theta_k$ (resp. $\mathrm{sgn}(v_k)\theta_k$), so the statement (ii) is proved. The second subcase is similar.

If $X_m(-1, v_k) = 0,\ p\Psi^*_-(0, v_k) - qv_k\Phi^*_-(0, v_k) \neq 0$, employ the quasi-homogeneous directional blow-up in the negative x direction, the proof of statement (iii) is similar.

The statement (i) is obvious from the above analysis.

3.3 $G(\theta) = 0$ have finite real roots

Assume that θ_k is a root of multiplicity l of $G(\theta) = 0$, $\Delta\widehat{OA_kB_k} :|\ \theta - \theta_k\ | \leq \epsilon,\ 0 < r \leq r_1$, where ϵ, r_1 are sufficiently small.

In a similar way, we can obtain the following three propositions from the corresponding theorems of [14]. The idea of proof of these propositions is similar.

Proposition 3.1 *If l is odd and $G^{(l)}(\theta_k)H(\theta_k) > 0$, then $\Delta\widehat{OA_kB_k}$ is a QNS of type* I. *So there are infinitely number of orbits tending to the origin $O(0,0)$ in the direction θ_k.*

Proposition 3.2 *If l is odd and $G^{(l)}(\theta_k)H(\theta_k) < 0$, then $\Delta\widehat{OA_kB_k}$ is a QNS of type* II. *Hence, there exists either a unique orbit or infinitely number of orbits tending to the origin $O(0,0)$ in the direction θ_k.*

Moreover, assume that $\frac{\Phi(r,\theta)}{r^{m+p-1}}, \frac{\Psi(r,\theta)}{r^{n+q-1}}$ satisfy the Lipschitz conditions

$$\begin{cases} \dfrac{|\ \Phi(r,\theta_2) - \Phi(r,\theta_1)\ |}{r^{m+p-1}} \leq C(r)\ |\ \theta_2 - \theta_1\ |, \\ \dfrac{|\ \Psi(r,\theta_2) - \Psi(r,\theta_1)\ |}{r^{n+q-1}} \leq C(r)\ |\ \theta_2 - \theta_1\ |, \end{cases} \tag{3.5}$$

and that if $l = 1$,

$$\frac{\Phi(r,\theta)}{r^{m+p-1}}, \frac{\Psi(r,\theta)}{r^{n+q-1}}, C(r) \to \circ(1), \quad r \to 0; \tag{3.6}$$

if $l > 1$,

$$\Phi(r,\theta) = \circ(r^{m+p}), \quad \Psi(r,\theta) = \circ(r^{n+q}), \quad r \to 0, \tag{3.7}$$

then (2.1) *has a uniqueness orbit tending to the origin $O(0,0)$ in the direction θ_k.*

Proposition 3.3 *If l is even, then $\Delta\widehat{OA_kB_k}$ is a QNS of type* III. *Hence, there exists either no orbit or infinitely number of orbits tending to the origin $O(0,0)$.*

Moreover, let

$$\eta(r,\theta) = pr^p \mathrm{Cs}\,\theta \Psi(r^p\mathrm{Cs}\,\theta, r^q\mathrm{Sn}\,\theta) - qr^q\mathrm{Sn}\,\theta\Phi(r^p\mathrm{Cs}\,\theta, r^q\mathrm{Sn}\,\theta).$$

(i) *If in some QNS there exists C_1 such that*

$$\eta(r,\theta) \leq C_1 A(r), \quad 0 < C_1 < D, \tag{3.8}$$

then there exist infinitely many orbits tending to the origin $O(0,0)$ *in the direction* θ_k.

(ii) *If in a QNS there exists* C_2 *such that*

$$\eta(r,\theta) \geq C_2 A(r), \quad C_2 > D, \tag{3.9}$$

then there is no orbit tending to the origin $O(0,0)$ *in the direction* θ_k, *where*

$$A(r) = r^{m+p+q-2}\Big(\ln\Big(\frac{1}{r}\Big)\Big)^{\frac{-l}{l-1}},$$
$$D = \Big(\frac{H(\theta_k)}{l}\Big)^{\frac{l}{l-1}}(G^{(l)}(\theta_k)(l-1))^{\frac{-1}{l-1}}.$$

Remark 3.1 If the differential system (2.1) is an analytic system near the origin, then the condition of Proposition 3.2 and the condition (3.8) of Proposition 3.3 are naturally satisfied.

Furthermore, if l is odd, then the following statements hold.

Theorem 3.5 *Consider the* C^k *differential system* (2.1) *with* k *sufficiently large, and assume that* p *and* q *are odd.*

(i) *If* $G^{(l)}(\theta_k)H(\theta_k) > 0$, *then there exist infinitely many orbits tending to the origin* $O(0,0)$ *in the direction* θ_k *as well as* $\frac{T}{2}+\theta_k$.

(ii) *If* $G^{(l)}(\theta_k)H(\theta_k) < 0$ *and the condition of Proposition* 3.2 *holds, then there exists a unique orbit tending to the origin* $O(0,0)$ *in the direction* θ_k *as well as* $\frac{T}{2}+\theta_k$.

Proof Assume that p and q are odd. From the proof of Proposition 2.1 we have $G\big(\frac{T}{2}+\theta\big) = (-1)^{m+1}G(\theta)$, $H\big(\frac{T}{2}+\theta\big) = (-1)^{m+1}H(\theta)$, then

$$G^{(l)}\big(\frac{T}{2}+\theta_k\big)H\big(\frac{T}{2}+\theta_k\big) = (-1)^{2m+2}G^{(l)}(\theta_k)H(\theta_k).$$

(i) If $G^{(l)}(\theta_k)H(\theta_k) > 0$, then $G^{(l)}\big(\frac{T}{2}+\theta_k\big)H\big(\frac{T}{2}+\theta_k\big) > 0$. By Proposition 3.1, there are infinitely number of orbits tending to the origin $O(0,0)$ in the direction θ_k as well as $\frac{T}{2}+\theta_k$.

(ii) If $G^{(l)}(\theta_k)H(\theta_k) < 0$, then $G^{(l)}\big(\frac{T}{2}+\theta_k\big)H\big(\frac{T}{2}+\theta_k\big) < 0$. By Proposition 3.2, there exists a unique orbit tending to the origin $O(0,0)$ in the direction θ_k as well as $\frac{T}{2}+\theta_k$.

The proofs of following two theorems are similar to Theorem 3.5.

Theorem 3.6 *Consider the* C^k *differential system* (2.1) *with* k *sufficiently large, and assume that* p *is odd and* q *is even.*

(i) *If* $G^{(l)}(\theta_k)H(\theta_k) > 0$, *then there exist infinitely many orbits tending to the origin* $O(0,0)$ *in the direction* θ_k *as well as* $\frac{T}{2}-\theta_k$.

(ii) *If* $G^{(l)}(\theta_k)H(\theta_k) < 0$ *and the condition of Proposition* 3.2 *holds, then there exists a unique orbit tending to the origin* $O(0,0)$ *in the direction* θ_k *as well as* $\frac{T}{2}-\theta_k$.

Theorem 3.7 *Consider the* C^k *differential system* (2.1) *with* k *sufficiently large, and assume that* p *is even and* q *is odd.*

(i) *If* $G^{(l)}(\theta_k)H(\theta_k) > 0$, *then there exist infinitely many orbits tending to the origin* $O(0,0)$ *in the direction* θ_k *as well as* $-\theta_k$.

(ii) *If* $G^{(l)}(\theta_k)H(\theta_k) < 0$ *and the condition of Proposition* 3.2 *holds, then there exists a unique orbit tending to the origin* $O(0,0)$ *in the direction* θ_k *as well as* $-\theta_k$.

4 Applications

Example 4.1 Consider the differential system

$$\begin{cases} \dot{x} = x^4 + y^3, \\ \dot{y} = x^3 y. \end{cases} \tag{4.1}$$

(4.1) is a (3,4) quasi-homogeneous system of weight degree 10. Apply the generalized polar coordinate $x = r^3 \mathrm{Cs}\,\theta$, $y = r^4 \mathrm{Sn}\,\theta$, then we have

$$\begin{aligned} G(\theta) &= -\mathrm{Sn}\,\theta(\mathrm{Cs}^4\theta + 4\mathrm{Sn}^3\theta), \\ H(\theta) &= \mathrm{Cs}^3\theta(\mathrm{Cs}^8\theta + \mathrm{Cs}^4\theta\mathrm{Sn}^3\theta + \mathrm{Sn}^6\theta), \\ G'(\theta) &= \mathrm{Cs}^3\theta(4\mathrm{Sn}^6\theta - 16\mathrm{Sn}^3\theta\mathrm{Cs}^4\theta - \mathrm{Cs}^8\theta). \end{aligned}$$

The regular critical direction is determined by $\mathrm{Sn}\,\theta = 0$ or $\mathrm{Tn}\,\theta = -\frac{1}{4}$.

By direct a calculation, we have $G'(\theta)H(\theta)|_{\mathrm{Sn}\,\theta=0} < 0, H(\theta)|_{\mathrm{Sn}\,\theta=0} = \mathrm{Cs}^{11}\theta$. If $x = r^3\mathrm{Cs}\,\theta > 0$ (resp. $x < 0$), then $\frac{\mathrm{d}r}{\mathrm{d}t} > 0$ (resp. $\frac{\mathrm{d}r}{\mathrm{d}t} < 0$) and the orbit leaves (resp. enter into) the origin. By the statement (ii) of Theorem 3.6, there is a unique orbit leaves the origin in the direction $\theta = 0$ as well as a unique orbit enters into the origin in the direction $\theta = \frac{T}{2}$.

Since $G'(\theta)H(\theta)|_{\mathrm{Tn}\,\theta=-\frac{1}{4}} > 0, H(\theta)|_{\mathrm{Tn}\,\theta=-\frac{1}{4}} = \frac{13}{16}\mathrm{Cs}^{13}\theta$, by the statement (i) of Theorem 3.6, there are infinitely many orbits leaving the origin in the direction $y^3 = -\frac{1}{4}x^4$, $x > 0$ as well as infinitely many orbits entering into the origin in the direction $y^3 = -\frac{1}{4}x^4$, $x < 0$.

In conclusion, there are infinitely many orbits leaving the origin along the positive x-axis and infinitely many orbits entering into the origin along the negative x-axis (see Figure 8.1).

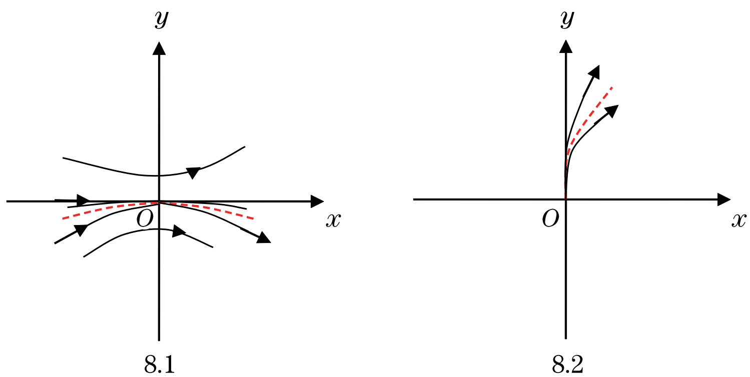


Figure 8 In Figure 8.1 the dashed curve is $y^3 = \frac{-1}{4}x^4$.
In Figure 8.2, the dashed curve is $y^{p+q-1} = \frac{b(q-1)}{a(p+q-1)}x^{q-1}$.

Note that if we use the normal sector method for the system (4.1), then $G(\theta) = -\sin^4(\theta)$, $H(\theta) = \cos(\theta)\sin^3(\theta)$. We can not determine the number of orbits tending to the origin in the direction $\theta = 0$ and $\theta = \pi$.

Example 4.2 Consider the differential system

$$\begin{cases} \dot{x} = bx^q + x^{q+1} + (a-b)xy^{p+q-1} - xy^{p+q}, \\ \dot{y} = x^q y + ay^{p+q} - y^{p+q+1}, \end{cases} \tag{4.2}$$

where both p and q are positive integers and $q \geq 2,\ a > 0,\ b > 0$. The above differential system is considered in [13]. We show that our method of QNS is more effective than the method of generalized normal sector in this case.

Since the system (4.2) has concrete background as stated in [13], we only consider the first quadrant ($x \geq 0,\ y \geq 0$).

For the system (4.2), $X_m(x,y) = bx^q + (a-b)xy^{p+q-1},\ Y_n(x,y) = ay^{p+q}$. Apply the generalized polar coordinate $x = r^{p+q-1}\mathrm{Cs}\,\theta,\ y = r^{q-1}\mathrm{Sn}\,\theta$, we have

$$\begin{aligned} G(\theta) &= \mathrm{Cs}\,\theta\mathrm{Sn}\,\theta\big(a(p+q-1)\mathrm{Sn}^{p+q-1}\theta - b(q-1)\mathrm{Cs}^{q-1}\theta\big), \\ H(\theta) &= a\mathrm{Sn}^{3(p+q-1)}\theta + b\mathrm{Cs}^{3(q-1)}\theta, \\ G'(\theta) &= -a(p+q-1)\mathrm{Sn}^{3(p+q-1)}\theta + bq(q-1)\mathrm{Sn}^{2(p+q-1)}\theta\mathrm{Cs}^{q-1}\theta \\ &\quad + a(p+q)(p+q-1)\mathrm{Sn}^{p+q-1}\theta\mathrm{Cs}^{2(q-1)} - b(q-1)\mathrm{Cs}^{3(q-1)}\theta. \end{aligned}$$

The regular critical direction is determined by $\mathrm{Sn}\,\theta = 0,\ \mathrm{Cs}\,\theta = 0$ or $\mathrm{Tn}\,\theta = \frac{b(q-1)}{a(p+q-1)}$.

Since $G'(\theta)H(\theta)|_{\mathrm{Sn}\,\theta=0} < 0, H(\theta)|_{\mathrm{Sn}\,\theta=0} = b\mathrm{Cs}^{3(q-1)}\theta > 0$, there is a unique orbit, which coincides with the positive x axis, and leaves the origin $O(0,0)$.

Since $G'(\theta)H(\theta)|_{\mathrm{Cs}\,\theta=0} < 0, H(\theta)|_{\mathrm{Cs}\,\theta=0} = a\mathrm{Sn}^{3(p+q-1)}\theta > 0$, there is unique orbit, which coincides with the positive y-axis, and leaves the origin $O(0,0)$.

Since $G'(\theta)H(\theta)|_{\mathrm{Tn}\,\theta=\frac{b(q-1)}{a(p+q-1)}} > 0, H(\theta)|_{\mathrm{Tn}\,\theta=\frac{b(q-1)}{a(p+q-1)}} = K\mathrm{Cs}^{3(q-1)}\theta > 0$, where

$$K = \frac{b^3(q-1)^3 + a^2b(p+q-1)^3}{a^2(p+q-1)}.$$

There are infinitely many orbits leaving the origin in the direction $y^{p+q-1} = \frac{b(q-1)}{a(p+q-1)}x^{q-1}$.

If we use the classic polar coordinate, then the curve $y^{p+q-1} = \frac{b(q-1)}{a(p+q-1)}x^{q-1}$ goes into the origin $O(0,0)$ in the direction $\frac{\pi}{2}$. (4.2) has two critical directions in the first quadrant ($x \geq 0,\ y \geq 0$), $\theta = 0$ and $\theta = \frac{\pi}{2}$ near $O(0,0)$. In $\theta = 0$ there is a unique orbit which coincides with the positive half x-axis and leaves $O(0,0)$. In $\theta = \frac{\pi}{2}$, infinitely many orbits leave $O(0,0)$ (see Figure 8.2).

5 Appendix: Method of Newton Polyhedron

In this appendix we show how to obtain the parameters p, q and d by Newton polyhedron, see for instance [1].

Consider the following system

$$\begin{cases} \dot{x} = X(x,y) = \sum\limits_{i+j\geq 1} a_{i,j}x^iy^j, \\ \dot{y} = Y(x,y) = \sum\limits_{i+j\geq 1} b_{i,j}x^iy^j. \end{cases} \tag{5.1}$$

We denote (5.1) by $\chi = (X, Y)$ conveniently.

We define the set

$$N = \{(i-1, j) : a_{i,j} \neq 0\} \cup \{(i, j-1) : b_{i,j} \neq 0\}. \tag{5.2}$$

The point $(-1, j)$ is associated to the monomial $a_{0,j}y^j$ and the point $(i, -1)$ is associated to the monomials $b_{i,0}x^i$. The point $(0,0)$ is associated to the monomials $a_{1,0}x$ and $b_{0,1}y$. If the origin of system (5.1) is degenerate, then $(0,0) \notin N$. We define the Newton polyhedron as the convex hull of $N + \mathbb{R}_+^2$ in the (i,j)-plane. We call γ_k the segments of this polyhedron. If one of these segments is completely in the half-plane $i \le 0$ (resp. $j \le 0$) we call it γ_0 (resp. γ_{n+1}). The rest of the segments are called $\gamma_1, \gamma_2, \cdots, \gamma_n$ from left to right, and they have at least one endpoint in the first quadrant of the (i,j)-plane. For $k = 1, \cdots, n$, the segment γ_k satisfies the equation of the straight line $p_k i + q_k j = d_k$ for some coprime α_k, β_k and δ_k. We choose the suitable (p,q,d) from the set $\{(p_k, q_k, d_k), k = 1, \cdots, n\}$ provided by $\gamma_1, \cdots, \gamma_n$.

If $(i,j) \in N$, then either $a_{i+1,j}x^{i+1}y^j$ is a monomial of $X(x,y)$, or $b_{i,j+1}x^i y^{j+1}$ is a monomial of $Y(x,y)$. We call $d_k = pi + qj$ the quasi-degree of type (p,q) of these monomials. The monomials $a_{i+1,j}x^{i+1}y^j$ (resp. $b_{i,j+1}x^i y^{j+1}$) of quasi-degree d_k of type (p,q) are grouped in a polynomial X_{d_k} (resp. Y_{d_k}). Hence the vector field $\chi = (X,Y)$ can be decomposed into its quasi-homogeneous components of type (p,q) : $\chi = \sum\limits_{d_k \ge d} \chi_{d_k}$, where $\chi_{d_k} = (X_{d_k}, Y_{d_k})$ and $d = \min\{d_k : \exists (i,j) \in N,\ pi + qj = d_k\}$. This is a different way of writing system (5.1).

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References

[1] Alvarez, M. J., Ferragut, A. and Jarque, X., A survey on the blow-up technique, *Internat. J. Bifur. Chaos*, **21**, 2011, 3103–3118.

[2] Andreev, A., Investigation of the behavior of the integral curves of a system of two differential equations in the neighborhood of a singular point, *Trans. Amer. Math. Soc.*, **8**, 1958, 187–207.

[3] Andronov, A. A., Leontovich, E. A., Gordon, I. I. and Maier, A. G., Qualitative Theory of 2nd Ordinary Dynamic Systems, J. Wiley Sons, New York, 1973.

[4] Broer, H. W., Dumortier, F., Strien, S. J. and Takens, F., Structures in Dynamics, Studies Math. Phys., North-Holland, Elsevier Science Publishers, B. V., 1981.

[5] Coll, B., Gasull, G. and Prohens, R., Differential equations defined by the sum of two quasi-homogeneous vector feilds, *Can. J. Math.*, **49**, 1997, 212–231.

[6] Dumortier, F., Singularities of vector fields on the plane, *J. Differential Equations*, **23**, 1977, 53–106.

[7] Frommer, M., Die intergralkurven einer gewohnlichen differentialgleichung erster ordnung in der umgebung rationaler unbestimmtheitsstellen, *Math. Appl.*, **99**, 1928, 222–272.

[8] Lefschetz, S., Differential Equations: Geometric Theory, Interscience, New York, 1957.

[9] Lyapunov, A. M., Stability of Motion, Mathematics in Science and Engineering, Academic Press, New York, London, 1966.

[10] Nemytskii, V. V. and Stepanov, V. V., Qualitative theory of differential equation, Princeton Math. Series, **22**, Princeton University Press, Princeton, N. J., 1960.

[11] Oliveira, R. D. S. and Zhao, Y., On the structural stability of planar quasi-homogeneous polynomial vector fields, *Qual. Theory Dyn. Syst.*, **13**, 2013, DOI: 10.1007/s12346-013-0105-5.

[12] Tang, Y., Li, W. and Zhang, Z., Focus-center problem of planar degenerate system, *J. Math. Anal. Appl.*, **345**, 2008, 934–940.

[13] Tang, Y. and Zhang, W., Generalized normal sectors and orbits in exceptional directions, *Nonlinearity*, **17**, 2004, 1407–1426.

[14] Zhang, Z., Ding. T., Huang, W. and Dong, Z., Qualitative Theory of Differential Equation, Transl. Math. Monogr., Providence, Rhode Island, 1992.