Chin. Ann. Math. Ser. B
38(6), 2017, 1197–1212
DOI: 10.1007/s11401-017-1031-7

Chinese Annals of Mathematics, Series B

Embedded Surfaces for Symplectic Circle Actions*

Yunhyung CHO[1] Min Kyu KIM[2] Dong Youp SUH[3]

Abstract The purpose of this article is to characterize symplectic and Hamiltonian circle actions on symplectic manifolds in terms of symplectic embeddings of Riemann surfaces. More precisely, it is shown that (1) if (M,ω) admits a Hamiltonian S^1-action, then there exists a two-sphere S in M with positive symplectic area satisfying $\langle c_1(M,\omega),[S]\rangle > 0$, and (2) if the action is non-Hamiltonian, then there exists an S^1-invariant symplectic 2-torus T in (M,ω) such that $\langle c_1(M,\omega),[T]\rangle = 0$. As applications, the authors give a very simple proof of the following well-known theorem which was proved by Atiyah-Bott, Lupton-Oprea, and Ono: Suppose that (M,ω) is a smooth closed symplectic manifold satisfying $c_1(M,\omega) = \lambda\cdot[\omega]$ for some $\lambda\in\mathbb{R}$ and G is a compact connected Lie group acting effectively on M preserving ω. Then (1) if $\lambda < 0$, then G must be trivial, (2) if $\lambda = 0$, then the G-action is non-Hamiltonian, and (3) if $\lambda > 0$, then the G-action is Hamiltonian.

Keywords Symplectic geometry, Hamiltonian action
2000 MR Subject Classification 53D20, 53D05

1 Introduction

The purpose of this article is to characterize symplectic and Hamiltonian circle actions on symplectic manifolds in terms of symplectic embeddings of Riemann surfaces. We first consider the following simple situation which provides the motivation of our work.

Let Σ_g be a two-dimensional smooth closed oriented manifold with genus g, and let $\mathrm{Diff}(\Sigma_g)$ denote the diffeomorphism group of Σ_g. Suppose that G is a compact connected Lie group acting on Σ_g effectively (A G-action on a manifold M is called effective if the identity element $1\in G$ is the unique element which fixes whole M.), i.e., there is an injective Lie group homomorphism

$$\phi : G \hookrightarrow \mathrm{Diff}(\Sigma_g).$$

Since G is compact and connected, by averaging any given Riemannian metric over the Haar measure of G, we can get a G-invariant metric Ω on Σ_g so that we may regard G as a closed

Manuscript received November 24, 2015. Revised September 13, 2016.
[1]Department of Mathematics Education, Sungkyunkwan University, Seoul, Republic of Korea.
E-mail: yunhyung@skku.edu
[2]Department of Mathematics Education, Gyeongin National University of Education, San 59-12, Gyesan-dong, Gyeyang-gu, Incheon 407-753, Republic of Korea. E-mail: mkkim@kias.re.kr
[3]Department of Mathematical Sciences, KAIST, 335 Gwahangno, Yu-sung Gu, Daejeon 305-701, Republic of Korea. E-mail: dysuh@math.kaist.ac.kr
*The first author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP; Ministry of Science, ICT & Future Planning) (No. NRF-2017R1C1B5018168), the second author was partially supported by Gyeongin National University of Education Research Fund and the third author was partially supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) Funded by the Ministry of Science, ICT & Future Planning (No. 2016R1A2B4010823).

Lie subgroup of the identity component $\mathrm{Iso}(\Sigma_g, \Omega)_0$ of the full isometry group $\mathrm{Iso}(\Sigma_g, \Omega)$. In particular, we have the following inequalities

$$\dim G \leq \dim \mathrm{Iso}(\Sigma_g, \Omega)_0 = \dim \mathrm{Iso}(\Sigma_g, \Omega) \leq 3,$$

where the last inequality comes from the classical fact that

$$\dim \mathrm{Iso}(M^n, h) \leq \frac{n(n+1)}{2}$$

for any n-dimensional complete Riemannian manifold (M^n, h), see [6] for more details. Note that the isometry groups of a 2-sphere S^2 and a 2-torus T^2 are well-known such that

(1) $\mathrm{Iso}(S^2, h)_0$ is a subgroup of $\mathrm{SO}(3)$ for any metric h on S^2, and

(2) $\mathrm{Iso}(T^2, h)_0$ is a subgroup of $\mathrm{SO}(2) \times \mathrm{SO}(2)$ for any metric h on T^2.

In the case when the genus $g \geq 2$, then the isometry group of Σ_g is finite with respect to any metric on Σ_g, see [4, Chapter 7] for more details. In fact, the latter statement can be deduced from the following well-known formula

$$\chi(\Sigma_g) = \chi(\Sigma_g^{S^1}) = |\Sigma_g^{S^1}| \geq 0,$$

where $\Sigma_g^{S^1}$ denotes the set of fixed points of the S^1-action on Σ_g.

Thus the condition $g \geq 2$ is an obstruction to the existence of an action of a compact Lie group of positive dimension on compact Riemann surfaces. Together with the fact that $\chi(\Sigma_g) = 2 - 2g$, one might expect that the existence of a non-trivial compact connected Lie group G in $\mathrm{Diff}(M)$ might be obstructed by the Euler characteristic $\chi(M)$, i.e., the negativity of $\chi(M)$ would imply the non-existence of a G-action such as the case of compact Riemann surfaces. Unfortunately, it is inappropriate to use $\chi(M)$ since if we let $M = S^2 \times \Sigma_g$ for $g \geq 2$, then M admits a circle action on the first factor but M satisfies $\chi(M) < 0$. Therefore, instead of χ, we use some geometric structure as follows. Let us consider a G-invariant volume form ω on Σ_g. Then ω is a symplectic form[1] and thus (Σ_g, ω) is a symplectic manifold[2]. Hence we may think of G as a subgroup of the symplectomorphism group

$$\mathrm{Symp}(\Sigma_g, \omega) := \{g \in \mathrm{Diff}(\Sigma_g) \mid g^*\omega = \omega\}.$$

Let J be an ω-compatible almost complex structure on Σ_g, i.e., $\omega(\cdot,\cdot) = \omega(J\cdot, J\cdot)$ and $\omega(\cdot, J\cdot)$ is a Riemannian metric on M. Note that for any symplectic manifold (M, ω), ω-compatible almost complex structure J always exists. In fact, the space $\mathcal{J}(M, \omega)$ of ω-compatible almost complex structures is a contractible space (see [8]), which implies that (TM, J) and (TM, J') are isomorphic as a complex vector bundle for any J and J' in $\mathcal{J}(M, \omega)$. Hence the first Chern class of (TM, J) does not depend on the choice of a ω-compatible almost complex structure J, and we denote by

$$c_1(M, \omega) := c_1(TM, J)$$

for any $J \in \mathcal{J}(M, \omega)$.

Since ω represents a non-zero element $[\omega] \in H^2(\Sigma_g; \mathbb{R}) \cong \mathbb{R}$, there exists a constant $\lambda \in \mathbb{R}$ such that

$$c_1(\Sigma_g, \omega) = \lambda \cdot [\omega].$$

---

[1] A differential two-form ω on a manifold M is called symplectic if it is closed ($d\omega = 0$) and non-degenerate ($\omega_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate bilinear form for every $p \in M$).

[2] A symplectic manifold is a pair (M, ω) which consists of a smooth manifold M and a symplectic form ω on M.

Also, the Chern number $\langle c_1(\Sigma_g,\omega),[\Sigma_g]\rangle$ is the same as the Euler characteristic $\chi(\Sigma_g)=2-2g$ of Σ_g, where $[\Sigma_g]\in H_2(\Sigma_g,\mathbb{Z})$ is the fundamental homology class of Σ_g. Hence we may conclude as follows:

(1) If (Σ_g,ω) is a closed symplectic surface such that $\langle c_1(\Sigma_g,\omega),[\Sigma_g]\rangle<0$, then there is no compact connected Lie group action preserving ω.

In fact, we can say more about the G-action on (Σ_g,ω) in the symplectic setting. For a given symplectic manifold (M,ω) with the symplectomorphism group $\mathrm{Symp}(M,\omega)$, we say that $\phi\in\mathrm{Symp}(M,\omega)$ is a Hamiltonian diffeomorphism if there exists an isotopy ϕ_t for $0\le t\le 1$ such that

(1) $\phi_0=\mathrm{id}$,

(2) $\phi_1=\phi$, and

(3) $i_{X_t}\omega=\omega(X_t,\cdot)=\mathrm{d}H_t$ for some family of smooth functions $\{H_t:M\to\mathbb{R}\}_{0\le t\le 1}$, where X_t is a time-dependent vector field on M such that

$$X_t\circ\phi_t=\frac{\mathrm{d}}{\mathrm{d}t}\phi_t. \tag{1.1}$$

We denote by $\mathrm{Ham}(M,\omega)$ the set of all Hamiltonian diffeomorphisms on (M,ω) and it forms a group under the composition. In fact, the Hamiltonian diffeomorphism group $\mathrm{Ham}(M,\omega)$ is path-connected and it is a normal subgroup of $\mathrm{Symp}(M,\omega)$, see [8] for more details.

Now suppose that G is a compact connected Lie group acting on (M,ω) effectively, and $\mathfrak{g}$ is the Lie algebra of G. By definition of $\mathrm{Symp}(M,\omega)$ and by the surjectivity of the exponential map $\exp: T_eG\to G$, we can easily show that G is a subgroup of $\mathrm{Symp}(M,\omega)$ if and only if each one-parameter subgroup generated by each element $X\in\mathfrak{g}$ preserves ω, i.e.,

$$\mathcal{L}_{\underline{X}}\omega=(i_{\underline{X}}\circ d+d\circ i_{\underline{X}})\omega=d\circ i_{\underline{X}}\omega=0$$

for every $X\in T_eG$, where $\underline{X}$ is the vector field generated by X, i.e.,

$$\underline{X}_p:=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\exp tX)\cdot p$$

for each $p\in M$. In particular, if $i_{\underline{X}}\omega$ is exact for every $X\in\mathfrak{g}$, then $i_{\underline{X}}\omega=\mathrm{d}H_X$ for some smooth function $H_X:M\to\mathbb{R}$ for each $X\in\mathfrak{g}$. Then, the family $\{\exp tX\}_{0\le t\le 1}$ is an isotopy which connects $\mathrm{id}=\exp(0\cdot X)$ with $\exp(1\cdot X)$, and it satisfies Equation (1.1), which means that $\exp X\in\mathrm{Ham}(M,\omega)$. By surjectivity of $\exp:\mathfrak{g}\to G$ (since G is compact), we can deduce that G is a subgroup of $\mathrm{Ham}(M,\omega)$ if $i_{\underline{X}}\omega$ is exact for every $X\in T_eG$. Conversely, if G is a subgroup of $\mathrm{Ham}(M,\omega)$, then one can easily check that $i_{\underline{X}}\omega$ is exact for every $X\in\mathfrak{g}$.

Definition 1.1 *Let G be a compact connected Lie group acting on (M,ω). We say that a G-action is Hamiltonian if $i_{\underline{X}}\omega$ is exact for every $X\in\mathfrak{g}$. Equivalently, a G-action is Hamiltonian if G acts on M as a subgroup of* $\mathrm{Ham}(M,\omega)$.

Hence if the G-action is Hamiltonian, there exists a smooth map $H:M\to\mathfrak{g}^*$ (called a moment map) such that $\langle H,X\rangle=H_X$ with $i_{\underline{X}}\omega=\mathrm{d}H_X$ for every $X\in\mathfrak{g}$, where $\langle,\rangle:\mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$ is the usual pairing of the Lie algebra $\mathfrak{g}$ with its dual. Note that for each $X\in\mathfrak{g}$, the set of all critical points of H_X coincides with the set of all points fixed by the one-parameter subgroup generated by X by the non-degeneracy of ω. Note that if M is compact, then any smooth function on M has at least two critical points attaining its extrema. Hence if the G-action is Hamiltonian, the one-parameter subgroup $\{\exp(tX)\}_{t\in\mathbb{R}}$ has at least two fixed points for every $X\in\mathfrak{g}$. Therefore, we have the following proposition as follows.

Proposition 1.1 *Let (Σ_g, ω) be a closed two-dimensional symplectic manifold with genus g such that $c_1(\Sigma_g, \omega) = \lambda \cdot [\omega]$ and let G be a compact connected Lie group. Suppose that the G action is effective and it preserves ω. Then*
(1) *If $\lambda > 0$ $(g = 0)$, then the G-action is Hamiltonian.*
(2) *If $\lambda = 0$ $(g = 1)$, then the G-action is non-Hamiltonian, and*
(3) *If $\lambda < 0$ $(g > 1)$, then $G = \{1\}$.*

Proof If $g = 0$, then $\Sigma_0 \cong S^2$ is simply connected. In particular, we have $H^1(\Sigma_0; \mathbb{R}) = 0$ so that $\mathrm{d}i_{\underline{X}}\omega = 0$ if and only if $i_{\underline{X}}\omega$ is exact, which implies that any symplectic G-action on Σ_0 is automatically Hamiltonian. For the second statement, recall that $\mathrm{SO}(2) \times \mathrm{SO}(2)$ acts on Σ_1 freely and hence the G-action on Σ_1 is also free. In particular, every one-parameter subgroup of G has no fixed point so that the action is non-Hamiltonian. The last statement comes from the fact that the isometry group of Σ_g with $g \geq 2$ is finite (see p. 2).

In this point of view, we may think of the generalization of the closed Riemann surface case to the symplectic category. Note that if (M, ω) is a symplectic manifold such that $c_1(M, \omega) = \lambda \cdot [\omega]$, then we call (M, ω) a monotone symplectic manifold if $\lambda > 0$, a symplectic Calabi-Yau manifold if $\lambda = 0$, and a negatively monotone symplectic manifold if $\lambda < 0$. In this article, we will regard those three families of symplectic manifolds as a generalization of closed Riemann surfaces, and we prove the following theorem, which is a generalization of Proposition 1.1 to the higher dimensional cases.

Theorem 1.1 *Let (M, ω) be any smooth closed symplectic manifold such that $c_1(M, \omega) = \lambda \cdot [\omega]$ for some $\lambda \in \mathbb{R}$. Let G be a compact connected Lie group which acts on (M, ω) effectively and preserves ω. Then*
(1) *If $\lambda > 0$, then the G-action is Hamiltonian.*
(2) *If $\lambda = 0$, then the G-action is non-Hamiltonian.*
(3) *If $\lambda < 0$, then G is trivial.*

Note that Theorem 1.1 is not new. Theorem 1.1(1) was already proved independently by Atiyah-Bott [1] and Lupton-Oprea [5]. Also Theorem 1.1(2)–(3) was proved by Ono [10]. The proofs given by Atiyah-Bott [1] and Ono [10] are based on the equivariant cohomology theory, and the proof of Lupton-Oprea [5] is based on the homotopy theory, in particular the theory of Gottlieb groups. We do not refer to the details of their proofs, instead we give much more elementary and simple proof of Theorem 1.1 which can be obtained as a corollary of the following series of the propositions.

Proposition 1.2 *Let (M, ω) be a smooth closed symplectic manifold equipped with a smooth S^1-action preserving ω. Suppose that $[\omega]$ is a rational class in $H^2(M; \mathbb{Q})$. Then the action is non-Hamiltonian if and only if there exists an S^1-equivariant symplectic embedding of 2-torus $i\colon T^2 \hookrightarrow M$, where the circle acts freely on the left factor of $T^2 \cong S^1 \times S^1$. In particular, the normal bundle of $i(T^2)$ in M is trivial so that $\langle c_1(M, \omega), [i(T^2)] \rangle = 0$.*

Proposition 1.3 *Suppose that (M, ω) is a smooth closed symplectic manifold equipped with a Hamiltonian circle action. Then there exist a two-sphere S in M with positive symplectic area satisfying $\langle c_1(M, \omega), [S] \rangle > 0$.*

Proof of Theorem 1.1 First, suppose that the action is Hamiltonian. If G is not trivial, then there exists a maximal subtorus T with positive dimension in G. For any choice of a circle subgroup S^1 of T, there exists a two sphere S in M such that $\langle c_1(M, \omega), [S] \rangle = \lambda \cdot \langle [\omega], [S] \rangle > 0$

by Proposition 1.3. Since S has positive symplectic area, we have $\langle[\omega],[S]\rangle > 0$ so that λ must be positive. Secondly, if the action is non-Hamiltonian, we can easily show that it is non-Hamiltonian with respect to $k\omega$ for any positive real number $k \in \mathbb{R}$. Also, k can be chosen so that $[k\omega] \in H^2(M)$ is rational since $[\omega]$ is proportional to $c_1(M,\omega)$ by our assumption. By Proposition 1.2, there exists a symplectic two torus T in $(M, k\omega)$ such that

$$\langle c_1(M,k\omega),[T]\rangle = \langle c_1(M,\omega),[T]\rangle = \lambda \cdot \langle[\omega],[T]\rangle = 0,$$

where the first equality comes from the fact that $\mathcal{J}(M,\omega) = \mathcal{J}(M,k\omega)$ for $k > 0$. Since $\langle[k\omega],[T]\rangle = k\cdot\langle[\omega],[T]\rangle > 0$ and $k > 0$, we have $\lambda = 0$.

Note that if there is an effective symplectic S^1-action on (M,ω), then Propositions 1.2–1.3 imply that $\lambda \geq 0$. Thus if λ is negative, then there is no effective symplectic S^1-action on (M,ω). In other words, G must be trivial by the compactness and the connectivity of G.

We make the following two remarks. First, the reason why we only prove Proposition 1.2 for the rational case is that our proof relies on the existence of a generalized moment map [7] which can be defined only when $[\omega]$ is rational. If $[\omega]$ is not rational, since the non-degeneracy of ω is an open condition, we can always perturb a given ω slightly to another symplectic form ω' such that ω' is G-invariant and $[\omega']$ is rational. Hence if we apply Proposition 1.2 to (M,ω'), then there exists a symplectic embedding $i : T^2 \hookrightarrow M$ with respect to the new symplectic structure ω'. But there is no guarantee that the T^2-embedding i is symplectic with respect to ω. The authors could not fill the gap of the proof of Proposition 1.2 in the case when ω is not rational.

Secondly, unlike Proposition 1.2 which gives a necessary and sufficient condition for the existence of non-Hamiltonian G-action, Proposition 1.3 gives only the sufficient condition for the action to be Hamiltonian. The authors do not know whether the converse of Proposition 1.3 holds or not.

The organization of this paper is as follows. We give an introduction to the theory of Lie group actions on symplectic manifolds in Section 2, and we give the complete proof of Propositions 1.2–1.3 in Section 3.

2 Symplectic Circle Actions

In this section, we give a brief introduction to symplectic circle actions. Most of this section is contained in [2] or [8], but we give a complete proof for readers who are not familiar with symplectic geometry. Let M be a $2n$-dimensional smooth closed manifold. A differential 2-form ω is called a symplectic form if ω is closed and non-degenerate, i.e.,

(1) $d\omega = 0$, and

(2) ω^n is nowhere vanishing.

Let us assume that G is a compact connected Lie group acting effectively on M. The G-action on (M,ω) is called symplectic if $\mathcal{L}_{\underline{X}}\omega = 0$ for every $X \in T_eG$, where $\underline{X}$ is the fundamental vector field generated by X, i.e., G preserves a symplectic form ω. Equivalently, G-action is symplectic if and only if $i_{\underline{X}}\omega$ is closed. In particular, a G-action is called Hamiltonian if $i_{\underline{X}}\omega$ is exact for every $X \in T_eG$.

Now suppose that the unit circle group S^1 acts on (M,ω) symplectically, and let X be a fixed generator of $T_eS^1 \cong \mathbb{R}$. If the action is Hamiltonian, then there exists a smooth function $H : M \to \mathbb{R}$ such that

$$i_{\underline{X}}\omega = dH,$$

and we call H a moment map for the S^1-action. Note that $\mathcal{L}_X H = \omega(\underline{X}, \underline{X}) = 0$, i.e., a moment map H is S^1-invariant.

If the S^1-action is symplectic but non-Hamiltonian, then a moment map does not exist. Nevertheless, there exists an $\mathbb{R}/\mathbb{Z}$-valued function $\mu : M \to \mathbb{R}/\mathbb{Z}$ which locally looks like a moment map when $[\omega] \in H^2(M;\mathbb{R})$ is an integral class. We use the notation $\mathbb{R}/\mathbb{Z}$ instead of S^1 to avoid confusion with the acting group S^1.

Definition 2.1 (see [7]) *Let (M,ω) be a smooth closed symplectic manifold such that ω represents an integral cohomology class in $H^2(M;\mathbb{Z})$. Suppose that there is a symplectic non-Hamiltonian S^1-action on (M,ω). Fix a point $x_0 \in M$, and define an $\mathbb{R}/\mathbb{Z}$-valued map $\mu : M \to \mathbb{R}/\mathbb{Z}$ such that*

$$\mu(x) := \int_{\gamma_x} i_{\underline{X}}\omega \mod \mathbb{Z}$$

where γ_x is any path $\gamma_x : [0,1] \to M$ such that $\gamma_x(0) = x_0$ and $\gamma_x(1) = x$. We call μ an $\mathbb{R}/\mathbb{Z}$-valued moment map (or a generalized moment map) for the action.

By a direct computation, we can easily check that $[i_{\underline{X}}\omega]$ is an integral class in $H^1(M;\mathbb{Z})$ so that a generalized moment map given in Definition 2.1 is well-defined. Note that μ depends on the choice of a base point x_0 as in Definition 2.1. Consider two distinct points p and q on M, and let μ_p, respectively μ_q, be the $\mathbb{R}/\mathbb{Z}$-valued moment map with base point p, respectively q. For any point $x \in M$, let γ_q^p be a path from q to p and γ_p^x be a path from p to x respectively. Then

$$\mu_p(x) - \mu_q(x) = \int_{\gamma_p^x} i_{\underline{X}}\omega - \int_{\gamma_p^x \circ \gamma_q^p} i_{\underline{X}}\omega = -\int_{\gamma_q^p} i_{\underline{X}}\omega = -\mu_q(p) \mod \mathbb{Z}.$$

In other words, μ is unique up to a constant in $\mathbb{R}/\mathbb{Z} \cong S^1$. In particular, $\mathrm{d}\mu$ is independent of the choice of a base point. Since $\mathrm{d}\mu : TM \to TS^1 \cong S^1 \times \mathbb{R}$, we may regard $\mathrm{d}\mu$ as a differential 1-form on M.

Proposition 2.1 (see [7]) *Let $\mu : M \to \mathbb{R}/\mathbb{Z}$ be an $\mathbb{R}/\mathbb{Z}$-valued moment map for a symplectic non-Hamiltonian circle action on (M,ω). Then μ satisfies*

$$\mathrm{d}\mu = i_{\underline{X}}\omega.$$

Proof Let $x \in M$ be any point and let $\mathcal{U}$ be a contractible open neighborhood of x. Since $i_{\underline{X}}\omega$ is closed, it is locally exact by Poincaré lemma so that there exists a smooth function $f : \mathcal{U} \to \mathbb{R}$ such that $i_{\underline{X}}\omega = \mathrm{d}f$ on $\mathcal{U}$. Let μ be an $\mathbb{R}/\mathbb{Z}$-valued moment map with a base point $x_0 \in \mathcal{U}$. Let γ_x be a path from x_0 to x lying on $\mathcal{U}$. Then

$$\mu(x) = \int_{\gamma_x} i_{\underline{X}}\omega = \int_{\gamma_x} \mathrm{d}f = f(x) - f(x_0) \mod \mathbb{Z}$$

so that $\mathrm{d}\mu(x) = \mathrm{d}f(x) = i_{\underline{X}(x)}\omega_x$ for all $x \in \mathcal{U}$. Since x is chosen arbitrarily, we can conclude that $\mathrm{d}\mu = i_{\underline{X}}\omega$ on M.

It is an immediate consequence of Proposition 2.1 that μ is S^1-invariant, since $\mathcal{L}_{\underline{X}}\mu = i_{\underline{X}}\mathrm{d}\mu = \omega(\underline{X}, \underline{X}) = 0$.

Now, let us consider a critical point of a moment map H, i.e., $\mathrm{d}H(x) = i_{\underline{X}(x)}\omega_x = 0$. Since ω is non-degenerate on M, x is a critical point of H if and only if $\underline{X}(x) = 0$, i.e., x is a fixed point of given S^1-action. It is also true for a non-Hamiltonian case, i.e., x is a critical point of an $\mathbb{R}/\mathbb{Z}$-valued moment map μ if and only if x is a fixed point of the action by Proposition 2.1. Hence we have the following proposition.

Proposition 2.2 (see [2]) *Let (M,ω) be a smooth closed symplectic manifold equipped with a symplectic, respectively Hamiltonian, circle action. If μ, respectively H, is an $\mathbb{R}/\mathbb{Z}$-valued moment map, respectively moment map, of given action, then $x \in M$ is a critical point of μ, respectively H, if and only if x is a fixed point of the action.*

One of the most important property of symplectic geometry is that, for every point $p \in M$, there exists a neighborhood $\mathcal{U}_p$ of p with a local coordinate system $(\mathbb{R}^{2n}, x_1, y_1, \cdots, x_n, y_n)$ such that $\omega|_{\mathcal{U}_p} = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i$, i.e., a local symplectic structure for each point is isomorphic to the standard symplectic structure of $\mathbb{R}^{2n}$ so that symplectic geometry is locally the same as a linear symplectic geometry on $\mathbb{R}^{2n}$ with the standard symplectic structure $\sum \mathrm{d}x_i \wedge \mathrm{d}y_i$. This is known as the Darboux theorem. Similarly, there is an equivariant version of the Darboux theorem as follows.

Theorem 2.1 (Equivariant Darboux Theorem) *Let (M,ω) be a symplectic manifold and let G be a compact Lie group. Suppose that there is a symplectic G-action on (M,ω). For each fixed point p, there exists a neighborhood $\mathcal{U}_p$ together with a local coordinate system $(x_1, y_1, \cdots, x_n, y_n)$ such that*

(1) $\omega|_{\mathcal{U}_p} = \frac{1}{2\mathrm{i}} \sum \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ *with* $z_j = x_j + \mathrm{i}y_j$, *and*

(2) *G-action is linear with respect to $(z_1, \cdots, z_n)$. In particular if $G = S^1$, then there is a sequence of integers $\lambda_1, \cdots, \lambda_n$ such that the action is expressed as*

$$t \cdot (z_1, \cdots, z_n) = (t^{\lambda_1} z_1, \cdots, t^{\lambda_n} z_n)$$

for every $t \in S^1$.

Now, let p be a fixed point of symplectic S^1-action on $2n$-dimensional symplectic manifold (M,ω). By the equivariant Darboux theorem, there exists a local coordinate system $(\mathcal{U}_p, z_1, \cdots, z_n)$ centered at p and a sequence of integers $\lambda_1, \cdots, \lambda_n$ such that the S^1-action is expressed as

$$t \cdot (z_1, \cdots, z_n) = (t^{\lambda_1} z_1, \cdots, t^{\lambda_n} z_n)$$

for every $t \in S^1$. By solving $i_{\underline{X}}\omega = \mathrm{d}H$ on $\mathcal{U}_p$ with $\omega = \frac{1}{2\mathrm{i}} \sum \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$, we get

$$H(z_1, \cdots, z_n) = \text{constant} + \frac{1}{2} \sum \lambda_j |z_j|^2. \tag{2.1}$$

Therefore, we have the following corollary.

Corollary 2.1 (see [2]) *Let $H : M \to \mathbb{R}$ be a moment map on (M,ω). Then H is a Morse-Bott function. Similarly, if $\mu : M \to \mathbb{R}/\mathbb{Z}$ is an $\mathbb{R}/\mathbb{Z}$-valued moment map, then μ is an $\mathbb{R}/\mathbb{Z}$-valued Morse-Bott function. In either case, a Morse index of any critical submanifold of H or μ is even.*

Proof We need to show two things: (1) The critical point set is an embedded submanifold of (M,ω), and (2) The Hessian of H at p is non-degenerate along the normal direction of a critical submanifold containing p. The first claim is obvious since a sub-coordinate system

$$\{(z_1, \cdots, z_n) \in \mathcal{U}_p \mid z_j = 0 \text{ if } \lambda_j \neq 0\}$$

gives a coordinate system of a critical submanifold near p. The Hessian of H at p is a diagonal matrix is given by

$$\begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_n & 0 \\ 0 & 0 & \cdots & 0 & 0 & \lambda_n \end{pmatrix}$$

so that it finishes the proof.

Now, recall some basic Morse-Bott theory as follows. Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a closed manifold M, and let $M_t = \{p \in M \mid f(p) \leq t\}$ for every $t \in \mathbb{R}$. Suppose that a and b are regular values of f such that there exists a unique critical value c with $a < c < b$. Let $C_1, \cdots, C_r$ be connected components of the critical submanifold lying on $H^{-1}(c)$. According to classical Morse-Bott theory, M_b is homotopy equivalent to

$$M_a \cup_{\phi_1} D(\nu^-(C_1)) \cup_{\phi_2} D(\nu^-(C_2)) \cdots \cup_{\phi_r} D(\nu^-(C_r)),$$

where $\nu^-(C_j)$ is a negative normal bundle over C_j, $D(\nu^-(C_j))$ is a disk bundle of $\nu^-(C_j)$, and ϕ_j is an attaching map from a sphere bundle $S(\nu^-(C_j)) = \partial D(\nu^-(C_j))$ to $H^{-1}(a)$. Note that each $S(\nu^-(C_j))$ is an S^{k_j-1}-bundle over C_j, where $k_j = \mathrm{ind}(C_j)$ is a Morse index of C_j. In particular, $S(\nu^-(C_j))$ is connected if and only if $k_j \neq 1$.

Proposition 2.3 (see [2]) *Let H be a moment map on a (possibly non-compact) connected symplectic manifold (M, ω). Then every level set of H is empty or connected.*

Proof For the sake of simplicity, let $M(a, b) := H^{-1}((a, b))$ for $a, b \in \mathbb{R}$. Let us choose any S^1-invariant metric $\langle \cdot, \cdot \rangle$ on M so that the gradient flow $\bigtriangledown H$ of H is defined as

$$\mathrm{d}H(X) = \langle \bigtriangledown H, X \rangle$$

for every smooth vector field X on M. Suppose that there exists a regular value $r \in \mathbb{R}$ such that $H^{-1}(r)$ is non-empty and disconnected. By the connectivity of M, there exists a smallest $s \in \mathbb{R}^+$ such that $M(r - s, r + s)$ is connected. Note that $r + s$ or $r - s$ is a critical value of H, otherwise $M(r - s, r + s)$ is diffeomorphic to $M(r - (s - \epsilon), r + (s - \epsilon))$ along the gradient flow of H for a sufficiently small $\epsilon > 0$ so that it contradicts our assumption "smallest s".

Without loss of generality, we may assume that $c = r + s$ is a critical value of H. Let $C_1, \cdots, C_r$ be connected components of the critical submanifold lying on $H^{-1}(c)$. Then there exists some C_j such that $D(\nu^-(C_j))$ connects two disconnected components of $H^{-1}(c - \epsilon)$ via the attaching map $\phi_j : S(\nu^-(C_j)) \to H^{-1}(c - \epsilon)$, i.e., the index of C_j should equal to one. Since every critical submanifold of H has even index by Corollary 2.1, such C_j does not exist.

Similarly, if $c = r - s$ is a critical value of H, then there exists some C_j of co-index one, but there is no such C_j by Corollary 2.1. Hence it completes the proof.

Proposition 2.4 (see [2]) *Let (M, ω) be a $2n$-dimensional closed connected symplectic manifold equipped with a symplectic non-Hamiltonian circle action. Suppose $[\omega]$ is an integral class in $H^2(M; \mathbb{Z})$, and let $\mu : M \to \mathbb{R}/\mathbb{Z}$ be an $\mathbb{R}/\mathbb{Z}$-valued moment map defined in Definition* 2.1. *Then there is no critical submanifold of index zero nor co-index zero. In particular, every level set is non-empty and the number of connected components of $\mu^{-1}(t)$ is constant for all $t \in S^1$.*

Proof Let $r \in \mathbb{R}/\mathbb{Z}$ be a regular value of $\mu : M \to \mathbb{R}/\mathbb{Z}$ and let $N = \mu^{-1}(\mathbb{R}/\mathbb{Z} - \{r\})$ which is an open subset in M. With the induced S^1-action on N, we may regard N as a non-compact Hamiltonian S^1-manifold with a moment map $H = \mu|_N : N \to \mathbb{R}/\mathbb{Z} - \{r\} \cong (0,1)$. Let $N_1, N_2, \cdots, N_k$ be connected components of N and we denote by $H_j : N_j \to (0,1)$ the restriction of H onto N_j so that H_j is a moment map on $(N_j, \omega|_{N_j})$. By Proposition 2.3, every level set of H_j is empty or connected for every $j = 1, 2, \cdots, k$.

Firstly, we claim that each H_j is surjective. If not, $H(N_j)$ is either a half-closed interval of the form $[s,1)$ or $(0,s]$ for some $s \in (0,1)$ or a closed interval $[a,b] \subset (0,1)$. If $H(N_j) = [a,b]$, then N_j is also a connected component of $H^{-1}([a,b])$ so that N_j is both open and closed itself in M so that $N_j = M$ by the connectivity of M, which contradicts the assumption that the action is non-Hamiltonian. If $H(N_{j_1}) = [s_1, 1)$ for some $j_1 \in \{1, 2, \cdots, k\}$, then let us consider the closure $\overline{N}_{j_1}$ whose boundary $\partial\overline{N}_{j_1}$ is some connected component, namely B_1, of $\mu^{-1}(r)$. Since r is regular, there is a connected component N_{j_2} of N such that $H_{j_2}^{-1}(t)$ attains B_1 as $t \to 0$. If the image of N_{j_2} for H_{j_2} is a half-closed interval of the form $(0, s_2]$ for some $s_2 \in (0,1)$, then $\overline{N}_{j_1} \cup \overline{N}_{j_2}$ is connected and both open and closed in M so that we get $\overline{N}_{j_1} \cup \overline{N}_{j_2} = M$. Then the moment map μ factors such that

$$\mu : \overline{N}_{j_1} \cup \overline{N}_{j_2} = M \overset{H_{j_1,j_2}}{\longrightarrow} [s_1, 1+s_2] \overset{/\mathbb{Z}}{\longrightarrow} S^1,$$

where H_{j_1,j_2} maps $x \in N_{j_1}$ to $H_{j_1}(x)$, $y \in \mu^{-1}(r)$ to 1, and $z \in N_{j_2}$ to $1 + H_{j_2}(z)$. Then $i_{\underline{X}}\omega = \mathrm{d}(/\mathbb{Z}) \circ \mathrm{d}H_{j_1,j_2}$, but $\mathrm{d}(/\mathbb{Z})$ is the identity map so that $\mathrm{d}H_{j_1,j_2} = i_{\underline{X}}\omega$, i.e., the action is Hamiltonian which contradicts to our assumption. Hence $H(N_{j_2}) = (0,1)$, i.e., H_{j_2} is surjective so that $H_{j_2}^{-1}(t)$ attains some connected component $B_2 \neq B_1$ of $\mu^{-1}(r)$ as $t \to 1$. Take N_{j_3} such that $H_{j_3}^{-1}(t)$ attains B_2 as $t \to 0$. Then we can easily show that H_{j_3} is surjective by a similar reason. Hence we get a infinite sequence of pairwise distinct connected components $B_1, B_2, \cdots$, but it contradicts that the number of connected components of $\mu^{-1}(r)$ is finite by the compactness of M. Therefore, H_j is surjective for every j. In particular, there is no critical submanifold of index 0 nor co-index 0.

To complete the proof, recall that when a level set of μ passes through some critical value $c \in S^1$ such that $\mu^{-1}(c)$ does not contain a critical submanifold of index 0, 1, co-index 0, nor co-index 1, then the number of connected components does not change, i.e., $\mu^{-1}(c+\epsilon)$ and $\mu^{-1}(c-\epsilon)$ have the same number of connected components, see the proof of Proposition 2.3. Since an index and co-index of any critical component is even and there is no critical component of index 0 nor co-index 0, every level set has the same number of connected components.

3 Proof of the Main Theorem

Let G be a compact Lie group, and suppose (M,ω) is a closed symplectic manifold equipped with an effective symplectic G-action. In this section, we prove Theorem 1.1. To determine whether a given G-action is Hamiltonian or not, it is enough to check it for every circle subgroup of G since every element $g \in G$ is contained in some maximal torus of G. The following proposition characterizes a non-Hamiltonian circle action in terms of equivariant symplectic embedding of two-torus.

Proof of Proposition 1.2 Let $T^2 = S^1 \times S^1$ be a two-torus and assume that an S^1-action on T^2 is given by

$$t \cdot (t_1, t_2) = (t \cdot t_1, t_2)$$

for any $t \in S^1$ and $(t_1, t_2) \in T^2$. We denote the vector field on T^2 generated by the S^1-action by X. Suppose that there exists an S^1-equivariant symplectic embedding $i\colon T^2 \hookrightarrow M$. Then the vector field generated by the given S^1-action on M is $i_*(X)$. If the given S^1-action on (M, ω) is Hamiltonian with a moment map $H : M \to \mathbb{R}$, then for any smooth vector field Y on T^2, we have

$$i^*\omega(X, Y) = \omega(i_*X, i_*Y) = \mathrm{d}H(i_*Y) = \mathrm{d}(H \circ i)(Y)$$

which follows from the fact that $\mathrm{d}i(Y) = i_*Y$. Thus the S^1-action on $(T^2, i^*\omega)$ is Hamiltonian with a moment map $i^*H = H \circ i : T^2 \to \mathbb{R}$. But it contradicts the assumption that S^1-action on T^2 is free, because there must be at least two fixed points, namely the maximum and the minimum of i^*H. Hence the S^1-action on (M, ω) cannot be Hamiltonian.

Conversely, suppose that the given symplectic S^1-action on (M, ω) is non-Hamiltonian. By our assumption, there exists a natural number N big enough so that $N \cdot \omega$ is integral and we still denote by ω the new symplectic form $N \cdot \omega$. Obviously, the given S^1-action is symplectic with respect to the new symplectic form ω so that there exists an $\mathbb{R}/\mathbb{Z}$-valued moment map $\mu : M \to \mathbb{R}/\mathbb{Z}$ satisfying $i_{\underline{X}}\omega = \mathrm{d}\mu$ by Proposition 2.1, where $\underline{X}$ is the vector on M generated by the S^1-action. Let $M_{(1)}$ be the set of all points in M with the trivial isotropy subgroup. Now, suppose that there exists a smoothly embedded loop $\sigma : S^1 = [0, n]/_{0 \sim n} \longrightarrow M_{(1)}$ for some $n \in \mathbb{N}$ satisfying the following conditions:

(a) $\mu\big(\sigma(r)\big) = [r] \in \mathbb{R}/\mathbb{Z}$ for each $r \in [0, n]$,

(b) $\sigma(0) = \sigma(n)$, i.e., the image of σ is a loop, and

(c) for $r \neq r'$ with $(r, r') \neq (0, n)$, $t \cdot \sigma(r) \neq \sigma(r')$ for any $t \in S^1$.

If such σ exists, then we can define a smooth embedding of 2-torus as

$$i : S^1 \times [0, n]/_{0\sim n} \cong T^2 \to M, \quad (t_1, t_2) \longmapsto t_1 \cdot \sigma(t_2).$$

It is straightforward that i is S^1-equivariant for the S^1-action on T^2 by $t \cdot (t_1, t_2) = (t \cdot t_1, t_2)$. To show that $i(T^2)$ is a symplectic submanifold with respect to the induced symplectic structure, let us define a smooth vector field $\underline{Y}$ on $i(T^2)$ as

$$\underline{Y}(i(t_1, t_2)) := \frac{\mathrm{d}(t_1 \cdot \sigma(t))}{\mathrm{d}t}\Big|_{t=t_2}. \tag{3.1}$$

Since σ has no critical point, it is straightforward that $\underline{Y}$ has no zero and $\underline{X}(p)$ and $\underline{Y}(p)$ span the tangent space $T_p\, i(T^2)$ for every $p = i(t_1, t_2) \in i(T^2)$. Also,

$$\begin{aligned}
\omega\big(\underline{X}(p), \underline{Y}(p)\big) &= \mathrm{d}\mu\big(\underline{Y}(p)\big) && \text{by definition of } \mu \\
&= \mathrm{d}\mu\Big(\frac{\mathrm{d}(t_1 \cdot \sigma(t))}{\mathrm{d}t}\Big|_{t=t_2}\Big) && \text{by definition of } Y \\
&= \mathrm{d}\mu\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\Big|_{t=t_2}\Big) && \text{by } S^1\text{-invariance of } \mu \\
&= 1 && \text{by } (a).
\end{aligned}$$

So, ω is non-degenerate on $i(T^2)$, i.e., i is a symplectic embedding.

Now, we need to prove that such smoothly embedded loop σ in M actually exists.

Lemma 3.1 *$M_{(1)}$ is path-connected and open dense in M.*

Proof Let $\mathbb{Z}_n$ be the cyclic subgroup of S^1 of order n, and we denote by $M^{\mathbb{Z}_n}$ the set of all points in M fixed by $\mathbb{Z}_n$. Since M is compact, there are at most finitely many n's, say

$n_1, n_2, \cdots, n_k$ such that $M^{\mathbb{Z}_n} \neq \emptyset$ (see [3, Proposition IV.1.2]). Let M^{S^1} be the set of all points in M fixed by S^1. Since $M^{S^1} \subset M^{\mathbb{Z}_n}$ for every integer $n > 1$, we have

$$M_{(1)} = M - \bigcup_{n>1} M^{\mathbb{Z}_n} - M^{S^1} = M - \bigcup_{n>1} M^{\mathbb{Z}_n}. \tag{3.2}$$

Furthermore, for each $n > 1$, $M^{\mathbb{Z}_n}$ is closed symplectic submanifolds of M with the induced symplectic form by the equivariant Darboux theorem 2.1. Thus the set $\bigcup\limits_{n>1} M^{\mathbb{Z}_n}$ is the union of closed submanifolds with codimensions at least two, in particular $M_{(1)}$ is path-connected and open dense in M.

Corollary 3.1 *For any regular value $t_0 \in \mathbb{R}/\mathbb{Z}$ of μ, the subset $M_{(1)} \cap \mu^{-1}(t_0)$ is open dense in $\mu^{-1}(t_0)$.*

Proof The openness is obvious since $M_{(1)}$ is open. We will show that $M^{\mathbb{Z}_n} \cap \mu^{-1}(t_0)$ is of codimension at least two in $\mu^{-1}(t_0)$ for each $n > 1$, which implies that $\bigcup\limits_{n>1} (M^{\mathbb{Z}_n} \cap \mu^{-1}(t_0))$ is of codimension at least two in $\mu^{-1}(t_0)$ so that

$$M_{(1)} \cap \mu^{-1}(t_0) = \mu^{-1}(t_0) - \bigcup_{n>1} (M^{\mathbb{Z}_n} \cap \mu^{-1}(t_0))$$

is dense in $\mu^{-1}(t_0)$. We already know that $M^{\mathbb{Z}_n}$ is of codimension at least two in M for each $n > 1$, since $M^{\mathbb{Z}_n}$ is a (proper) symplectic submanifold of M. Also, we know that t_0 is a regular value of the restriction map $\mu|_{M^{\mathbb{Z}^n}} : M^{\mathbb{Z}_n} \to \mathbb{R}/\mathbb{Z}$, since there is no fixed point in $(\mu|_{M^{\mathbb{Z}^n}})^{-1}(t_0) = M^{\mathbb{Z}_n} \cap \mu^{-1}(t_0)$. Thus we have

$$\dim(M^{\mathbb{Z}_n} \cap \mu^{-1}(t_0)) = \dim M^{\mathbb{Z}_n} - 1.$$

Also, we have $\dim \mu^{-1}(t_0) = \dim M - 1$ which implies that the codimension of $M^{\mathbb{Z}_n} \cap \mu^{-1}(t_0)$ in $\mu^{-1}(t_0)$ is equal to $\dim M - \dim M^{\mathbb{Z}_n} \geq 2$. This finishes the proof.

Remark 3.1 The set $M_{(1)} \cap \mu^{-1}(t_0)$ is not necessarily path-connected in Corollary 3.1.

Without loss of generality, we may assume that $0 \in \mathbb{R}/\mathbb{Z}$ is a regular value of μ. For our convenience, we use the following terminology: For a fixed S^1-invariant metric h on M, we say that a smooth path $\gamma : [a, b] \to M_{(1)}$ winds along $\mathbb{R}/\mathbb{Z}$ if $h(\bigtriangledown\mu_{\gamma(t)}, \gamma'(t)) > 0$ for every $t \in [a, b]$, which means that the vector field generated by γ is a gradient-like vector field of μ with respect to h. We call such a path γ a winding path. Also, we say that two points $x, y \in M_{(1)}$ are winding path-connected, if there exists a path $\gamma : [a, b] \to M^{(1)}$ from $\gamma(a) = x$ to $\gamma(b) = y$ which winds along $\mathbb{R}/\mathbb{Z}$.

Lemma 3.2 *For $x \in M_{(1)} \cap \mu^{-1}(0)$, there exists some $y \in M_{(1)} \cap \mu^{-1}(0)$ such that x and y are winding path-connected.*

Proof Let J be an S^1-invariant almost complex structure compatible[1] with ω. We denote by $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$ the induced S^1-invariant metric on M so that the gradient vector field $\bigtriangledown\mu$ on M with respect to $\langle \cdot, \cdot \rangle$ is defined as $\mathrm{d}\mu = \langle \bigtriangledown\mu, \cdot \rangle$. Then for any vector field Y on M, we have

$$\langle \bigtriangledown\mu, Y \rangle = i_{\underline{X}}\omega(Y) = \omega(\underline{X}, Y) = \omega(J\underline{X}, JY) = \langle J\underline{X}, Y \rangle,$$

[1] We say that an almost complex structure J on (M, ω) is compatible with ω if (1) $\omega(\cdot, \cdot) = \omega(J\cdot, J\cdot)$ and (2) $\omega(\cdot, J\cdot)$ is a Riemannian metric. Such J always exists, see [8] for the details.

so that we obtain $\triangledown\mu = J\underline{X}$.

Note that $\triangledown\mu$ commutes with the S^1-action, since J, $\langle\cdot,\cdot\rangle$, and μ are chosen to be S^1-invariant. Thus the one-parameter group action generated by $\triangledown\mu$ preserves their isotropy subgroups, which means that if $x \in M$ is fixed by some subgroup $H \subset S^1$, then any point y in the orbit $\{(\exp t\triangledown\mu)\cdot x\}_{t\in\mathbb{R}}$ is fixed by H. In particular, the one-parameter group action generated by $\triangledown\mu$ acts on $M_{(1)}$.

Now, $x \in M_{(1)} \cap \mu^{-1}(0)$ and consider the integral curve along $\triangledown\mu$ passing through x

$$\begin{array}{rccc} \gamma_x & : \mathbb{R} & \longrightarrow & M, \\ & t & \longmapsto & \varphi^t_{\triangledown\mu}\cdot x, \end{array}$$

where $\{\varphi^t_{\triangledown\mu}\}_{t\in\mathbb{R}}$ denotes the one-parameter subgroup of $\mathrm{Diff}(M)$ generated by the vector field $\triangledown\mu$. If γ_x is a winding path from x to some point $y \in M_{(1)} \cap \mu^{-1}(0)$, then there is nothing to prove.

Since $\gamma_x'(t) = \triangledown\mu_{\gamma_x(t)}$ and $\triangledown\mu_p = 0$ if and only if $p \in M^{S^1}$, if γ_x is not a winding path from x to any point in $M_{(1)}$, then $\lim\limits_{t\to\infty}\mu(\varphi^t_{\triangledown\mu}\cdot x) = t_0$ for some $t_0 \in \mathbb{R}/\mathbb{Z}$, which is equivalent to saying that

$$\lim_{t\to\infty}\gamma_x(t) = \lim_{t\to\infty}\varphi^t_{\triangledown\mu}\cdot x = p$$

for some fixed point $p \in M^{S^1}$. By the equivariant Darboux Theorem 2.1, there exists a local coordinate system $(\mathcal{U}_p, z_1, \cdots, z_n)$ centered at p such that

$$\omega|_{\mathcal{U}_p} = \frac{1}{2\mathrm{i}}\sum \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$$

and

$$t\cdot(z_1,\cdots,z_n) = (t^{\lambda_1}z_1,\cdots,t^{\lambda_n}z_n)$$

for every $t \in S^1$, where $\lambda_1,\cdots,\lambda_n$ are weights of the tangential S^1-representation on T_pM. Let C_p be the fixed connected component containing p and let ν_+ and ν_- be subsets of $\mathcal{U}_p$ given by

(1) $\nu_+ = \{(z_1,\cdots,z_n) \in \mathcal{U}_p \mid z_j = 0 \text{ if } \lambda_j \le 0\}$, and

(2) $\nu_- = \{(z_1,\cdots,z_n) \in \mathcal{U}_p \mid z_j = 0 \text{ if } \lambda_j \ge 0\}$.

Then $\gamma_x(t)$ is lying on ν_- for any sufficiently large $t > 0$. Note that $\{(z_1,\cdots,z_n) \in \mathcal{U}_p \mid z_j \neq 0 \text{ if } \lambda_j \neq 0\} \subset M_{(1)}$ since the S^1-action is effective. Then we may perturb γ_x smoothly to a new gradient-like flow $\widetilde{\gamma}_x$ on $\mathcal{U}_p \cap M_{(1)}$ connecting the gradient flow γ_x to γ_q for some $q \in \mathcal{U}_p \cap M_{(1)}$ and $q \notin \nu_-$, where γ_q is the integral curve along $\triangledown\mu$ passing through q (see Figure 1).

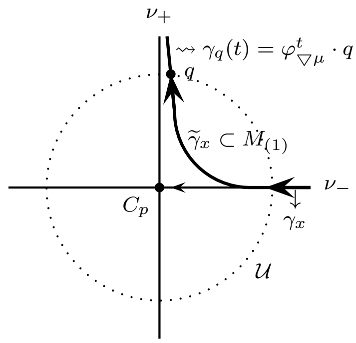

Figure 1 Perturbing γ_x to get $\widetilde{\gamma}_x$.

If $\widetilde{\gamma}_x$ is a winding path from x to some point $y \in M_{(1)} \cap \mu^{-1}(0)$, then it is done. If not, we apply the same procedure whenever our perturbed gradient-like flow converges to some fixed point. Then in finite steps, we can get a winding path connecting x to $M_{(1)} \cap \mu^{-1}(0)$ as is desired.

Lemma 3.3 *If a point x in $M_{(1)} \cap \mu^{-1}(0)$ is winding path-connected to some point y in a connected component C of $M_{(1)} \cap \mu^{-1}(0)$, then x is winding path-connected to every point in C.*

Proof For $x \in M_{(1)} \cap \mu^{-1}(0)$, let γ_x be a winding path connecting x to $y \in M_{(1)} \cap \mu^{-1}(0)$. Let C_y be the connected component of $M_{(1)} \cap \mu^{-1}(0)$ containing y. Fix $z \in C_y$ and let $\sigma : [0,1] \to M_{(1)} \cap \mu^{-1}(0)$ be a smooth path with $\sigma(0) = y$ and $\sigma(1) = z$. Since 0 is chosen to be a regular value of μ, the gradient vector field $\bigtriangledown\mu$ is non-zero on $\mu^{-1}(0)$. Also, we can find a sufficiently small ϵ such that any $r \in [-\epsilon, 0] \subset S^1$ is a regular value of μ.

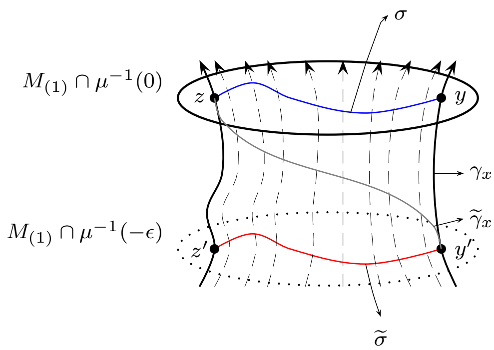

Figure 2 Winding path.

Let $\widetilde{\sigma}$ be the intersection of $M_{(1)} \cap \mu^{-1}(-\epsilon)$ and the trajectory of σ under the infinitesimal action generated by $\bigtriangledown\mu$. Let y' be the preimage of y for the infinitesimal action in $M_{(1)} \cap \mu^{-1}(-\epsilon)$. Then we can easily see that there exists a homotopy in $M_{(1)} \cap \mu^{-1}([-\epsilon, 0])$ from γ_x to a winding path $\widetilde{\gamma}_x$ connecting y' and z (see Figure 2). Then the path γ'_x defined by

$$\gamma'_x(t) = \begin{cases} \gamma_x(t), & \text{if } t \in (-\infty, t_0],\ \gamma_x(t_0) = y', \\ \widetilde{\gamma}_x(t), & \text{if } t \in (t_0, \infty) \end{cases}$$

is a winding path which connects x to z. It completes the proof.

To complete the proof of Proposition 1.2, pick a point x_1 in a connected component C_1 of $M_{(1)} \cap \mu^{-1}(0)$. By Lemma 3.2, x_1 is winding path-connected to some point x_2 in some connected component C_2 of $M_{(1)} \cap \mu^{-1}(0)$. If $C_1 = C_2$, then x_1 is winding path-connected to x_1 itself by Lemma 3.3 and it satisfies the conditions (a)–(c) automatically. If $C_1 \neq C_2$, then we can find a point x_3 in some connected component C_3 of $M_{(1)} \cap \mu^{-1}(0)$ which is winding path-connected to x_2. If $C_1 = C_3$, respectively $C_2 = C_3$, then we can take $x_3 = x_1$, respectively $x_3 = x_2$, so that we can obtain a smooth winding loop satisfying (a)–(c). Applying the above process inductively, we obtain a smooth winding loop satisfying (a)–(c). Then the following lemma finishes the proof.

Lemma 3.4 *Let E be an S^1-equivariant complex vector bundle of rank k over T^2 such that the induced S^1-action on the zero-section is free. Then E is trivial.*

Proof Let $Z \cong T^2$ be the zero section of E. Note that if the induced S^1-action on the zero section Z is free, then the given S^1-action on the total space E is free. Now, let us consider a following diagram:

$$\begin{array}{ccc} E & \xrightarrow{\;q\;} & E/S^1 \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ Z & \xrightarrow[\;q'\;]{} & Z/S^1 \end{array}$$

Since the action is free, $\pi' \colon E/S^1 \to Z/S^1 \cong S^1$ is a complex vector bundle of rank k over S^1 and the quotient map q is a bundle morphism. Note that any complex vector bundle over S^1 is trivial, since the structure group $U(n)$ is connected. Therefore $E/S^1 \cong S^1 \times \mathbb{C}^k$ and hence $E \cong q'^*(E/S^1) \cong Z \times \mathbb{C}^k$.

Now, let us consider the Hamiltonian case. The following proposition characterizes a Hamiltonian circle action in terms of equivariant symplectic embedding of two-spheres.

Proof of Proposition 1.3 Let $H : M \to \mathbb{R}$ be a moment map of the circle action. Let $Z_{\min}$, respectively $Z_{\max}$, be a critical submanifold which attains the minimum, respectively maximum, of H. Let g be an S^1-invariant Riemannian metric defined by $g(X, Y) = \omega(X, JY)$, where J is an ω-compatible S^1-invariant almost complex structure. Let $\bigtriangledown H$ be the gradient vector field with respect to g, i.e., $\mathrm{d}H = g(\bigtriangledown H, \cdot)$. Since H is a Morse-Bott function by Corollary 2.1, the unstable submanifold of $Z_{\min}$

$$W^u(Z_{\min}) = \left\{ p \in M \;\middle|\; \lim_{s \to -\infty} \varphi^s_{\bigtriangledown H} \cdot p \in Z_{\min} \right\}$$

is open and dense in M. Similarly, let $W^s(Z_{\max})$ be the stable submanifold of $Z_{\max}$. Since both $W^u(Z_{\min})$ and $W^s(Z_{\max})$ are open dense subsets, their intersection $W^u(Z_{\min}) \cap W^s(Z_{\max})$ is also open and dense in M, in particular it is non-empty. Now, pick a point $p \in W^u(Z_{\min}) \cap W^s(Z_{\max})$ and let

$$M(p) = \bigcup_{\substack{t \in S^1 \\ s \in \mathbb{R}}} t \cdot (\varphi^s_{\bigtriangledown H} \cdot p).$$

Then it is straightforward that the closure $\overline{M(p)}$ is homeomorphic to a two-sphere whose north pole, respectively south pole, is in $Z_{\min}$, respectively $Z_{\max}$.

Now, let $\underline{X}$ be the fundamental vector field generated by the S^1-action. Since the S^1-action and the gradient flow are smooth, it is obvious that $M(p)$ is a smoothly embedded two-sphere punctured at the two poles $\{z_N, z_S\}$. In particular, a tangent space $T_q M(p)$ for every $q \in M(p)$ is generated by $\underline{X}_q$ and $J\underline{X}_q$ since $\underline{X}_q$ never vanishes on $M(p)$. Since we have chosen J such that $\omega(\underline{X}_q, J\underline{X}_q) = g(\underline{X}_q, \underline{X}_q) > 0$, $M(p)$ is a smoothly embedded symplectic two sphere punctured at $\{z_N, z_S\}$. Then the closure $\overline{M(p)}$ is homeomorphic to S^2 and it defines a homology class $[\overline{M(p)}]$ in $H_2(M)$ even though the closure $\overline{M(p)}$ is not smooth at z_N and z_S in general. Obviously, the symplectic area $\langle [\omega], [\overline{M(p)}] \rangle$ is positive.

McDuff and Tolman proved that $\langle c_1(M, \omega), [\overline{M(p)}] \rangle > 0$ in [9, Lemma 2.2]. We sketch their idea as follows. Consider

$$\pi : E := S^3 \times_{S^1} M \to S^3/S^1 \cong S^2,$$

where S^1 acts on $S^3 \subset \mathbb{C}^2$ by $t \cdot (z_1, z_2) = (tz_1, tz_2)$ for $t \in S^1$. Then E is an M-bundle over S^2 with sections

$$\sigma_N = S^3 \times_{S^1} \{z_N\},$$
$$\sigma_S = S^3 \times_{S^1} \{z_S\}.$$

Let $c_{\text{vert}} \in H^2(E)$ be the first Chern class of the vertical subbundle[1] of TE. Then $c_{\text{vert}}(\sigma_N)$ is the first Chern number of the complex vector bundle

$$\pi_N : S^3 \times_{S^1} T_{z_N} M \to S^2,$$

over S^2. Note that $T_{z_N} M \cong \bigoplus_{i=1}^{n} \mathbb{C}_{\alpha_i}$ where $\{\alpha_1, \cdots, \alpha_n\}$ is the set of weights of the S^1-representation on $T_{z_N} M$ and $\mathbb{C}_{\alpha_i}$ is the one-dimensional S^1-representation with weight α_i. Therefore,

$$c_{\text{vert}}|_{\sigma_N} = \sum_{i=1}^{n} c_1(\pi_i),$$

where $\pi_i : S^3 \bigotimes_{S^1} \mathbb{C}_{\alpha_i} \to S^3 \times_{S^1} \{z_N\} = \sigma_N$ is the complex line bundle over σ_N whose first Chern number equals α_i. Thus $c_{\text{vert}}([\sigma_N]) = m_N$ where $m_N = \sum_{i=1}^{n} \alpha_i$ is the sum of weights of the S^1-representation on $T_{z_N} M$. Similarly, we have $c_{\text{vert}}([\sigma_S]) = m_S$, where m_S is the sum of weights of the S^1-representation on $T_{z_S} M$ so that

$$c_{\text{vert}}([\sigma_N] - [\sigma_S]) = m_N - m_S > 0,$$

since every nonzero weight of the S^1-representation at the maximum z_N (respectively at the minimum z_S) is positive (respectively negative). Thus it is enough to show that $[\overline{M(p)}] = [\sigma_N] - [\sigma_S]$.

Define $u_\omega \in \Omega^2(S^3 \times M)$ by

$$u_\omega = \omega + \mathrm{d}(H \cdot \theta),$$

where θ is a connection 1-form on S^3. Then we can easily check that $\mathcal{L}_{\underline{X}} u_\omega = i_{\underline{X}} u_\omega = 0$ so that u_ω induces a two-form, which we still denote by u_ω, on $S^3 \times_{S^1} M$. McDuff and Tolman's idea is that

$$\langle [\omega], [\overline{M(p)}] \rangle = \langle [u_\omega], [\sigma_{z_N}] - [\sigma_{z_S}] \rangle = H(z_N) - H(z_S)$$

holds. And if ω' is another symplectic form invariant under the S^1-action and $H_{\omega'}$ is a corresponding moment map, then

$$\langle [\omega'], [\overline{M(p)}] \rangle = \langle [u_{\omega'}], [\sigma_{z_N}] - [\sigma_{z_S}] \rangle = H_{\omega'}(z_N) - H_{\omega'}(z_S)$$

holds. Since the set of cohomology classes represented as S^1-invariant symplectic forms generates whole $H^2(M; \mathbb{R})$ as a vector space, we may conclude that

$$\langle \beta, [\overline{M(p)}] \rangle = \langle \beta, [\sigma_{z_N}] - [\sigma_{z_S}] \rangle$$

for every $\beta \in H^2(M; \mathbb{R})$, and hence we get $[\overline{M(p)}] = [\sigma_{z_N}] - [\sigma_{z_S}]$. It completes the proof.

[1] The vertical subbundle of TE is the kernel of the bundle map $\mathrm{d}\pi : TE \to TS^2$ induced by π.

Remark 3.2 In [9], they defined a Hamiltonian action with the following sign convention

$$i_{\underline{X}}\omega = -\mathrm{d}H,$$

while we use the equation $i_{\underline{X}}\omega = \mathrm{d}H$. In particular, the gradient vector field is $\bigtriangledown H = -J\underline{X}$ in [9] while $\bigtriangledown H = J\underline{X}$ in our paper, see [9, p. 12].

There are two effects of their sign convention on our proof of Proposition 1.3:

(1) In [9], $\omega(\underline{X}, J\underline{X})$ is negative so that our $[\overline{M(p)}]$ has an opposite orientation, and

(2) u_ω should be defined as $w - \mathrm{d}(H \cdot \theta)$ so that $\langle [u_\omega], [\sigma_z] \rangle = -H(z)$ for any fixed point $z \in M^{S^1}$.

After identifying our notation with the notation in [9] as above, we can easily check that our argument used in the proof of Proposition 1.3 coincides with Lemma 2.2 in [9].

Acknowledgements The authors would like to thank the referee for carefully reading this manuscript and pointing out that there was an error in the previous version of the manuscript.

References

[1] Atiyah, M. F. and Bott, R., The moment map and equivariant cohomology, *Topology*, **23**(1), 1984, 1–28.

[2] Audin, M., The Topology of Torus Actions on Symplectic Manifolds, Progress in Mathematics, **93**, Birkhäuser, Basel, 1991.

[3] Bredon, G. E., Introduction to Compact Transformation Groups, Academic Press, New York, London, 1972.

[4] Farb, B. and Margalit, D., A Primer on Mapping Class Groups, Princeton University Press, Princeton, NJ, 2012.

[5] Lupton, G. and Oprea, J., Cohomologically symplectic spaces: Toral actions and the Gottlieb group, *Trans. Amer. Math. Soc.*, **347**(1), 1995, 261–288.

[6] Mann, L. N., Gaps in the dimensions of isometry groups of Riemannian manifolds, *J. Differential. Geom.*, **11**, 1976, 293–298.

[7] McDuff, D., The moment map for circle actions on symplectic manifolds, *J. Geom. Phys.*, **5**(2), 1988, 149–160.

[8] McDuff, D. and Salamon, D., Introduction to Symplectic Topology, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995.

[9] McDuff, D. and Tolman, S., Topological properties of Hamiltonian circle actions, *Int. Math. Res. Pap.*, 72826, 2006, 1–77.

[10] Ono, K., Some remarks on group actions in symplectic geometry, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **35**, 1988, 431–437.

[11] Ono, K., Equivariant projective embedding theorem for symplectic manifolds, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **35**, 1988, 381–392.