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A Survey of the Homotopy Properties of Inclusion of Certain Types of Configuration Spaces into the Cartesian Product*

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Abstract Let X be a topological space. In this survey the authors consider several types of configuration spaces, namely, the classical (usual) configuration spaces $F_n(X)$ and $D_n(X)$, the orbit configuration spaces $F_n^G(X)$ and $F_n^G(X)/S_n$ with respect to a free action of a group G on X, and the graph configuration spaces $F_n^\Gamma(X)$ and $F_n^\Gamma(X)/H$, where Γ is a graph and H is a suitable subgroup of the symmetric group S_n. The ordered configuration spaces $F_n(X)$, $F_n^G(X)$, $F_n^\Gamma(X)$ are all subsets of the n-fold Cartesian product $\prod\limits_1^n X$ of X with itself, and satisfy $F_n^G(X) \subset F_n(X) \subset F_n^\Gamma(X) \subset \prod\limits_1^n X$. If A denotes one of these configuration spaces, the authors analyse the difference between A and $\prod\limits_1^n X$ from a topological and homotopical point of view. The principal results known in the literature concern the usual configuration spaces. The authors are particularly interested in the homomorphism on the level of the homotopy groups of the spaces induced by the inclusion $\iota\colon A \longrightarrow \prod\limits_1^n X$, the homotopy type of the homotopy fibre I_ι of the map ι via certain constructions on various spaces that depend on X, and the long exact sequence in homotopy of the fibration involving I_ι and arising from the inclusion ι. In this respect, if X is either a surface without boundary, in particular if X is the 2-sphere or the real projective plane, or a space whose universal covering is contractible, or an orbit space $\mathbb{S}^k/G$ of the k-dimensional sphere by a free action of a Lie group G, the authors present recent results obtained by themselves for the first case, and in collaboration with Golasiński for the second and third cases. The authors also briefly indicate some older results relative to the homotopy of these spaces that are related to the problems of interest. In order to motivate various questions, for the remaining types of configuration spaces, a few of their basic properties are described and proved. A list of open questions and problems is given at the end of the paper.

Keywords Configuration space, Equivariant configuration space, Fibration, Homotopy fibre, $K(\pi, 1)$ space, Braid groups
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1 Introduction

Let A and Y be topological spaces, let $f\colon A \longrightarrow Y$ be a continuous map, and let I denote the unit interval. If $y_0 \in Y$ is a basepoint, the homotopy fibre I_f of f is defined by $I_f = \{\, (a,\lambda) \in A \times Y^I \mid \lambda(0) = f(a) \text{ and } \lambda(1) = y_0\}$ (see Section 4 for more information about I_f). The knowledge of I_f is a relevant ingredient for many applications, such as the possible factorisation of a map $g\colon X \longrightarrow Y$ through A, leading to results in fixed point theory for suitable choices of A, Y and f (see [9]), as well as the study of the induced homomorphisms $\pi_i(A) \longrightarrow \pi_i(Y)$, where $i \geq 0$, that form part of a long exact sequence in homotopy involving $\pi_i(I_f)$. Intuitively, we are trying to measure how A differs from Y in some sense. At this level of generality, we cannot expect to obtain deep results, and we must restrict our attention to certain families of spaces and maps in order to make any meaningful progress.

In this paper, A will be a subset of Y, and f will be the inclusion map. Our aim is to study the case where $Y = \prod_1^n X$ is the n-fold Cartesian product of a topological space X, where $n \in \mathbb{N}$. The subspace $A \subset \prod_1^n X$ will be one of the following:

(a) A is the n-th ordered configuration space of X, defined by [6, 10–11]:

$$F_n(X) = \Big\{(x_1, \cdots, x_n) \in \prod_1^n X \,\Big|\, x_i \neq x_j \text{ for all } i \neq j\Big\}. \tag{1.1}$$

We will often refer to $F_n(X)$ as the "usual" configuration space of X. If X is a surface, it is well known that $\pi_1(F_n(X))$ is isomorphic to the pure braid group $P_n(X)$ of X on n strings (see [4, 11–12]). If additionally X is the two-dimensional disc, then $P_n(X)$ is the Artin pure braid group on n strings, denoted by P_n.

(b) A is the n-th ordered orbit configuration space with respect to a free action of a group G on X, defined by [7, 30]:

$$F_n^G(X) = \Big\{(x_1, \cdots, x_n) \in \prod_1^n X \,\Big|\, Gx_i \cap Gx_j = \emptyset \text{ for all } i \neq j\Big\}. \tag{1.2}$$

See Section 2 for more details.

(c) A is the n-th ordered graph configuration space $F_n^\Gamma(X)$ associated to a graph Γ whose vertices are labelled by $\{1, \cdots, n\}$, that has no loops, and that possesses at most one edge between two vertices, where we define

$$F_n^\Gamma(X) = \{(x_1, \cdots, x_n) \mid x_i \neq x_j \text{ if there is an edge in } \Gamma \text{ between vertices } i \text{ and } j\}. \tag{1.3}$$

This notion was defined in [2], using the notation X_n^Γ. See Section 6.1 for more information about $F_n^\Gamma(X)$.

One may see that $F_n^G(X) \subset F_n(X) \subset F_n^\Gamma(X) \subset \prod_1^n X$. In what follows, if A is one of the above spaces, then we will let $\iota_n\colon A \longrightarrow \prod_1^n X$ denote inclusion, and if $m \geq 0$, then $(\iota_n)_{\#m}$: $\pi_m(A) \longrightarrow \pi_m\big(\prod_1^n X\big)$ will denote the induced homomorphism on the level of π_m (relative to some choice of basepoint). If $m = 1$ then we will often just write $\iota_{n\#}$ for this homomorphism of fundamental groups if no confusion is possible. Two broad and important questions involving the pair of spaces (A, Y) are:

(I) describe the induced homomorphisms $(\iota_n)_{\#m}$, where $m \geq 0$. An example of this may be found in [3, 19], where it is proved that for surfaces other than $\mathbb{S}^2$ and $\mathbb{R}P^2$, $(\iota_n)_{\#m}$ is injective if $m \geq 2$, and if $m = 1$, the image of the induced homomorphism is the normal closure of the image of P_n by the homomorphism induced by embedding a topological disc in the surface.

(II) determine the homotopy type of the homotopy fibre of the map $\iota_n \colon A \hookrightarrow \prod_1^n X$.

The following classical result of Ganea illustrates nicely the type of answer that we would like to obtain to question (II). Let (X, x_0) and (Y, y_0) be pointed spaces, let $X \vee Y$ denote the wedge product of X and Y that we regard as a subspace of the Cartesian product $X \times Y$, and let ΩX denote the loop space of X equipped with the basepoint x_0. Ganea's theorem describes the homotopy type of the homotopy fibre of the inclusion $\iota \colon X \vee Y \longrightarrow X \times Y$.

Theorem 1.1 (see [15, p. 302], [16, Equation (6), p. 448] and [26]) *If X and Y have the homotopy type of a CW-complex, then the homotopy fibre of the inclusion $\iota \colon X \vee Y \longrightarrow X \times Y$ has the homotopy type of $\Omega X * \Omega Y$, where $*$ denotes the join operation.*

One may define an unordered version of each of the three types of ordered configuration spaces mentioned above. For the usual configuration space $F_n(X)$, the symmetric group S_n on n letters acts freely on $F_n(X)$ by permutation of coordinates, and the n-th unordered configuration space of X is defined to be the quotient space $D_n(X) = F_n(X)/S_n$. In a similar manner, the n-th unordered orbit configuration space of X is defined by $D_n^G(X) = F_n^G(X)/S_n$. Thirdly, for a graph Γ as given in (c) above, let H be a maximal subgroup of S_n that acts freely on $F_n^\Gamma(X)$ by permutation of coordinates. Then the n-th unordered graph configuration space of X (with respect to H) may be defined to be the quotient space $D_n^\Gamma(X) = F_n^\Gamma(X)/H$. Each such subgroup H gives rise to an unordered graph configuration space. Actually, the quotient space may be interesting even if H is not maximal, two examples being the usual configuration spaces, and the quotients that give rise to "mixed" braid groups (see [20, 22]). Some other information and remarks about the graph configuration spaces and their relationship with the automorphisms of the graph may be found in Section 6.1. In contrast with the other types of configuration space, the determination of the subgroups of S_n that act freely on $F_n^\Gamma(X)$ by permutation of coordinates, in particular the maximal subgroups, is not completely clear, and some work needs to be done to find them. Even for question (I), very little is known about the configuration spaces. In the case where X is a surface, the usual unordered configuration spaces have been widely studied (their fundamental groups are the full braid groups of X), but this is not the case so far for the unordered orbit and graph configuration spaces. Question (II) remains largely open if X is arbitrary and A is an ordered configuration space. Nevertheless, some progress has been made very recently in the case where A is the usual ordered configuration space $F_n(X)$, and X is either

(i) a surface without boundary, or

(ii) the orbit space of a sphere by a tame, free action of a compact Lie group G, or

(iii) a space that admits a contractible universal covering.

The aim of this paper is to give a survey of the state of the art of these types of questions, and to describe these recent results. It is worth mentioning that the ordered orbit configuration spaces appear in a natural way in the study of the problems for the ordinary configuration spaces. This provides extra motivation to study such spaces. The paper is comprised of five sections besides the introduction. In Section 2, we discuss the orbit configuration spaces $F_n^G(X)$. In Proposition 2.1, we show that if G acts freely and properly discontinuously on a surface X different from the 2-sphere $\mathbb{S}^2$ and the real projective plane $\mathbb{R}P^2$, then $F_n^G(X)$ is a $K(\pi, 1)$. In the rest of Section 2, we study the free action of $\mathbb{Z}_2$ induced by the antipodal map $\tau\colon\ \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ on the open cylinder C, which we interpret as the complement of the north and south poles in $\mathbb{S}^2$. In Propositions 2.2 and 2.3 respectively, we give a presentation of $F_n^{\langle\tau\rangle}(C)$ and we describe the homotopy fibre of the inclusion map $\iota_n^{\langle\tau\rangle}\colon\ F_n^{\langle\tau\rangle}(C) \hookrightarrow \prod_1^n C$. In Section 3, we describe the homomorphism $\iota_{n\#} : \pi_1(F_n(S)) \longrightarrow \pi_1\big(\prod_1^n S\big)$ where S is either $\mathbb{S}^2$ or $\mathbb{R}P^2$. The case $S = \mathbb{S}^2$ is much simpler, but its analysis aids us in the study of the case of $S = \mathbb{R}P^2$. If $S = \mathbb{R}P^2$, the kernel of $\iota_{n\#}$ may be written as the direct product of the cyclic subgroup of order 2 generated by the full twist braid of $P_n(\mathbb{R}P^2)$ and a subgroup L_n that may be described as an iterated semi-direct product of free groups (see Proposition 3.1 and Theorem 3.2). In Proposition 3.2, we determine the Abelianisation of L_n. This understanding of $\operatorname{Ker}(\iota_{n\#})$ enables us to prove the outstanding cases of a conjecture of Birman [3] in Theorem 3.1, which mirrors a similar well-known result for $\mathbb{S}^2$. In Section 4, we extend our analysis of the inclusion map $\iota_n\colon\ F_n(S) \hookrightarrow \prod_1^n S$, where S is either $\mathbb{S}^2$ or $\mathbb{R}P^2$, and determine the homotopy type of the homotopy fibre of ι_n in Theorem 4.2. If $S = \mathbb{R}P^2$, the description of the homotopy fibre makes use of the orbit configuration spaces $F_n^{\langle\tau\rangle}(C)$. This allows us to analyse the long exact sequence in homotopy associated to the fibration involving the homotopy fibre of ι_n (see Corollary 4.1), including the boundary homomorphism, using an analysis of the homomorphisms induced by ι_n (see Proposition 4.1). One interesting point about this long exact sequence is that the homotopy groups of $\mathbb{S}^2$ occur not only as the homotopy groups of S, but also as those of $F_n(S)$ since by Theorem 4.1, the universal covering of $F_n(S)$ has the same homotopy type as $\mathbb{S}^2$ or $\mathbb{S}^3$ (depending on the value of n). In Section 5, we determine the homotopy type of the homotopy fibre of the inclusion $\iota_n\colon\ F_n(X) \hookrightarrow \prod_1^n X$, where X satisfies one of the above conditions (ii) or (iii). The results are given in Propositions 5.1 and 5.2 respectively. In the latter case, if G is finite, we describe the long exact sequence in homotopy associated to the fibre sequence obtained by turning ι_n into a fibration in Proposition 5.3, which extends the results of Corollary 4.1. These two results make use of a more general analysis of the homotopy fibre of ι_n given by Theorem 5.1 and Corollary 5.1. Finally, in Section 6, we make some remarks about the graph configuration spaces defined in Equation (1.3), and we finish the paper with a list of open problems and questions.

2 Orbit Configuration Spaces

The notion of orbit configuration space was introduced in [30] and further studied in [7, 31]. It generalises the standard concept of configuration space, and has been applied recently in

[18, 24] to understand certain homotopy fibres and long exact sequences (see Sections 4 and 5 respectively). In this section, we will discuss these spaces, as well their unordered version, and develop some basic properties that are closely related to our questions.

Let X be a topological space. Given a free action $G\times X \longrightarrow X$ of a group G on a topological space X, the n-th orbit configuration space $F_n^G(X)$ of X (with respect to G) is defined as in Equation (1.2). Note that if G is the trivial group then $F_n^G(X)$ coincides with $F_n(X)$. The fundamental group $\pi_1(F_n^G(X))$ will be called the n-th orbit pure braid group of X (with respect to G).

Let $1 \leq i_1 < \cdots < i_{n-1} \leq n$ be integers. If X is a manifold that admits a properly discontinuous action of a group G such that the orbit space X/G is again a manifold, then by [7, Lemma 4], the projection $F_n^G(X) \longrightarrow F_{n-1}^G(X)$ onto the $n-1$ coordinates $x_{i_1},\cdots,x_{i_{n-1}}$ is a locally-trivial fibre bundle whose fibre may be identified with $F_1(X\setminus G(\{x_{i_1},\cdots,x_{i_{n-1}}\}))$. As in the case of the usual configuration space, we may prove the following result.

Proposition 2.1 *If X is a surface without boundary different from $\mathbb{S}^2$ and $\mathbb{R}P^2$, and G is a Lie group that acts freely and properly discontinuously on X, then the space $F_n^G(X)$ is a $K(\pi,1)$.*

Proof If G is discrete, the result follows from the fact that $F_1(X\setminus G(\{x_{i_1},\cdots,x_{i_{n-1}}\}))$ and X are $K(\pi,1)$'s. If G is not discrete, then it is a Lie group of dimension 1, and so contains $\mathbb{S}^1$ as a subgroup. But this implies that either the surface X is the torus, in which case X is compact, or X is the open cylinder, i.e., homeomorphic to $\mathbb{S}^2\times(0,1)$. This follows from the fact that the quotient is a one-dimensional connected manifold, so is either $\mathbb{S}^1$ or the open interval.

Let $x_0\in\mathbb{S}^2$, let C denote the open cylinder $\mathbb{S}^2\setminus\{x_0,-x_0\}$, and let $\tau\colon \mathbb{S}^2\longrightarrow\mathbb{S}^2$ denote the antipodal map defined by $\tau(x)=-x$, as well as its restriction to C. Then the cyclic group $\langle\tau\rangle$ of order 2 acts freely on $\mathbb{S}^2$ and on C. As a special case of Proposition 2.1, if $G=\langle\tau\rangle\cong\mathbb{Z}_2$ then $F_n^{\langle\tau\rangle}(C)$ is a $K(\pi,1)$ (see also [24, Lemma 16]). A presentation of the fundamental group of $F_n^{\langle\tau\rangle}(C)$, which we denote by G_n, was computed in [24, Proposition 19], and may be described as follows. We identify C with the open annulus $(0,1)\times\mathbb{S}^1$ in the complex plane, and if $z=re^{i\theta}\in C$, we take its antipodal point z' to be $(1-r)e^{i(\theta+\pi)}$. For $n\in\mathbb{N}$, we choose basepoints $x_1,\cdots,x_n$ that lie on the negative part of the x-axis, and we let $x_1',\cdots,x_n'$ be the corresponding antipodal points (see Figure 1). For $j\in\mathbb{N}$, the free group $\pi_1(C\setminus\{x_1,x_1',\cdots,x_{j-1},x_{j-1}'\},x_j)$ of rank $2j-1$ admits a basis $\{\,\rho_{j,i}\mid 0\leq i\leq 2j-2\}$, where for different values of i, the element $\rho_{j,i}$ is represented by a pair of antipodal loops. In Figure 1, for clarity, we depict one loop of every representative pair for each generator $\rho_{j,i}$. In Figures 2–4, for each $\rho_{j,i}$, we indicate the corresponding pair of antipodal loops. Using these geometric representatives and Fadell-Neuwirth type fibrations involving the spaces $F_n^{\langle\tau\rangle}(C)$, we obtain the following presentation of G_n.

Proposition 2.2 *Let $n\in\mathbb{N}$. The following constitutes a presentation of the group G_n:*

generating set: $\displaystyle\bigcup_{j=1}^{n}\{\,\rho_{j,i}\mid 0\leq i\leq 2j-2\}$,

relations: let $1\leq j<k\leq n$.

Then

(I)

$$\rho_{j,0}\rho_{k,l}\rho_{j,0}^{-1} = \begin{cases} \rho_{k,l}, & \text{if } l=0 \text{ or } j<l<k,\\ C_{k,j}\rho_{k,l}C_{k,j}^{-1}, & \text{if } 1\le l<j,\\ & \text{or if } k\le l\le k+j-2,\\ C_{k,j}, & \text{if } l=j,\\ C_{k,j}E_{k,k,k+j-2}^{-1}\rho_{k,0}^{-1}D_{1,k}^{-1}\rho_{k,l}D_{1,k}\rho_{k,0}E_{k,k,k+j-2}C_{k,j}^{-1}, & \text{if } l=k+j-1,\\ C_{k,j}\rho_{k,k+j-1}\rho_{k,l}\rho_{k,k+j-1}^{-1}C_{k,j}^{-1}, & \text{if } k+j\le l\le 2k-2, \end{cases}$$

where

(a) *for all* $1\le j<k\le n$, $D_{j,k}=\rho_{k,j}\rho_{k,j+1}\cdots\rho_{k,k-1}$;

(b) *for all* $1\le j<k\le n$, $C_{k,j}=\rho_{k,0}^{-1}D_{j+1,k}^{-1}\rho_{k,j}D_{j+1,k}\rho_{k,0}$;

(c) *for all* $1\le k\le n$ *and* $k\le m<q\le 2k-2$, $E_{k,m,q}=\rho_{k,m}\rho_{k,m+1}\cdots\rho_{k,q}$;

(II) *for all* $1\le i<j$, $\rho_{j,i}\rho_{k,l}\rho_{j,i}^{-1}=\rho_{k,l}$ *if* $0\le l\le i-1$, $j+1\le l\le k+i-2$, $k+i\le l\le k+j-2$ *or* $k+j\le l\le 2k-2$, *and*

$$\rho_{j,i}\rho_{k,l}\rho_{j,i}^{-1} = \begin{cases} \rho_{k,j}^{-1}\rho_{k,i}\rho_{k,j}, & \text{if } l=i,\\ [\rho_{k,j}^{-1},\rho_{k,i}^{-1}]\rho_{k,l}[\rho_{k,i}^{-1},\rho_{k,j}^{-1}], & \text{if } i<l<j,\\ \rho_{k,j}^{-1}\rho_{k,i}^{-1}\rho_{k,j}\rho_{k,i}\rho_{k,j}, & \text{if } l=j,\\ E_{k,k+i-1,k+j-1}E_{k,k+i,k+j-2}^{-1}\rho_{k,k+i-1}E_{k,k+i,k+j-2}E_{k,k+i-1,k+j-1}^{-1}, & \text{if } l=k+i-1,\\ E_{k,k+i,k+j-2}^{-1}E_{k,k+i-1,k+j-2}\rho_{k,k+j-1}E_{k,k+i-1,k+j-2}^{-1}E_{k,k+i,k+j-2}, & \text{if } l=k+j-1 \end{cases}$$

in the remaining cases;

(III) *for all* $j\le i\le 2j-2$,

$$\rho_{j,i}\rho_{k,l}\rho_{j,i}^{-1} = \begin{cases} \rho_{k,l}, & \text{if } l=0 \text{ or } j+1\le l\le k+i-j-1,\\ [\rho_{k,j}^{-1},\rho_{k,k+i-j}^{-1}]\rho_{k,l}[\rho_{k,k+i-j}^{-1},\rho_{k,j}^{-1}], & \text{if } 1\le l\le i-j,\ i-j+2\le l\le j-1,\\ & k+i-j+1\le l\le k+j-2 \text{ or}\\ & k+j\le l\le 2k-2,\\ \rho_{k,j}^{-1}\rho_{k,k+i-j}^{-1}\rho_{k,j}\rho_{k,k+i-j}\rho_{k,j}, & \text{if } l=j,\\ \rho_{k,j}^{-1}\rho_{k,k+i-j}\rho_{k,j}, & \text{if } l=k+i-j, \end{cases}$$

$$\begin{aligned}\rho_{j,i}\rho_{k,i-j+1}\rho_{j,i}^{-1} = {} & [\rho_{k,j}^{-1},\rho_{k,k+i-j}^{-1}]D_{i-j+1,k}\rho_{k,0}E_{k,k,k+j-1}E_{k,k,k+j-2}^{-1}C_{k,i-j+1}E_{k,k,k+j-2}\\ & \cdot E_{k,k,k+j-1}^{-1}\rho_{k,0}^{-1}D_{i-j+1,k}^{-1}[\rho_{k,k+i-j}^{-1},\rho_{k,j}^{-1}],\end{aligned}$$

which is the case $l=i-j+1$, *and*

$$\begin{aligned}\rho_{j,i}\rho_{k,k+j-1}\rho_{j,i}^{-1} = {} & [\rho_{k,j}^{-1},\rho_{k,i-j+1}^{-1}]E_{k,k,k+j-2}^{-1}C_{k,i-j+1}E_{k,k,k+j-2}\rho_{k,k+j-1}E_{k,k,k+j-2}^{-1}C_{k,i-j+1}^{-1}\\ & \cdot E_{k,k,k+j-2}[\rho_{k,i-j+1}^{-1},\rho_{k,j}^{-1}],\end{aligned}$$

which is the case $l=k+j-1$.

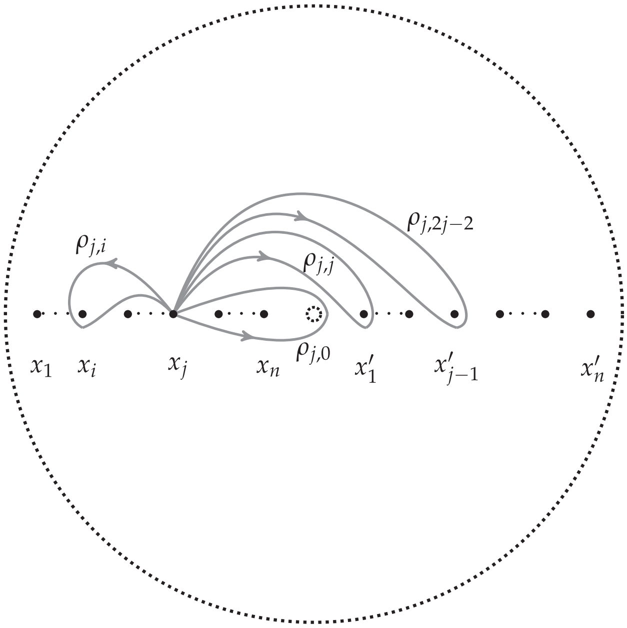

Figure 1 Generators of $\pi_1(C \setminus \{x_1, x'_1 \cdots, x_{j-1}, x'_{j-1}\}, x_j)$.

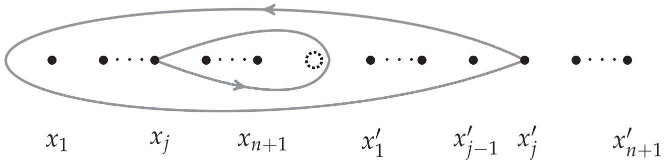

Figure 2 The generator $\rho_{j,0}$.

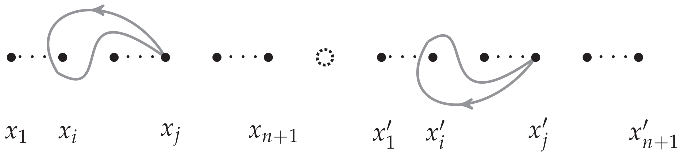

Figure 3 The generator $\rho_{j,i}$, $1 \leq i < j$.

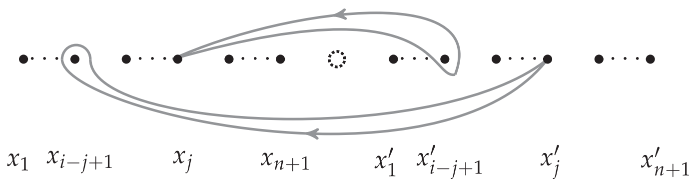

Figure 4 The generator $\rho_{j,i}$, $j \leq i \leq 2j-2$.

Some other properties of G_n will be highlighted in Section 4. As far as we know, this is the only case in which the fundamental group of an orbit configuration space (that is not a standard configuration space) has been computed. This allows us to understand better the homotopy fibre of the inclusion map $\iota_n^{\langle\tau\rangle}\colon F_n^{\langle\tau\rangle}(C) \hookrightarrow \prod_1^n C$. Using the presentation of G_n given by Proposition 2.2, we may prove the following result.

Proposition 2.3 *Let H_n denote the kernel of the homomorphism*

$$(\iota_n^{\langle\tau\rangle})_\# : \pi_1(F_n^{\langle\tau\rangle}(C)) \longrightarrow \pi_1\Bigl(\prod_1^n C\Bigr).$$

Then the homotopy fibre of the inclusion $\iota_n^{\langle\tau\rangle}\colon F_n^{\langle\tau\rangle}(C) \hookrightarrow \prod_1^n C$ is a $K(H_n,1)$, and the group H_n is isomorphic to the normal closure in G_n of $\{\rho_{j,i}\}_{i\neq 0}$.

Proof Since $\pi_k(F_n^{\langle\tau\rangle}(C)) = \pi_k\bigl(\prod_1^n C\bigr) = 0$ for all $k \neq 1$ and the homomorphism $(\iota_n^{\langle\tau\rangle})_\#$ is surjective, the same is true for the homotopy groups of the homotopy fibre of $\iota_n^{\langle\tau\rangle}$. Therefore this homotopy fibre is a $K(H_n,1)$. Note that the group $\pi_1\bigl(\prod_1^n C\bigr)$ is isomorphic to $\mathbb{Z}^n$. From the definition of the generators $\rho_{j,i}$, $i \neq 0$ (see Figures 3–4), it is clear that they belong to $\operatorname{Ker}\bigl(\iota_n^{\langle\tau\rangle}\bigr)_\#$. Conversely, given an element $w \in \operatorname{Ker}\bigl(\iota_n^{\langle\tau\rangle}\bigr)_\#$, for each $1 \leq j \leq n$, the exponent sum of $\rho_{j,0}$ in w is zero. Therefore w may be written as a product of conjugates of the $\rho_{j,i}$, where $i \neq 0$, multiplied by a product of commutators of elements of $\{\rho_{j,0}\}_{1\leq j\leq n}$. The result follows using the fact that the $\rho_{j,0}$ commute pairwise by relation (I) of Proposition 2.2.

As we mentioned in Section 1, the symmetric group S_n acts freely on $F_n^G(X)$, and we may define the n-th unordered orbit configuration space of X (with respect to G) to be the quotient space $D_n^G(X) = F_n^G(X)/S_n$. So there is an $n!$-fold regular covering map $F_n^G(X) \longrightarrow D_n^G(X)$. As in the case of the usual configuration spaces, many properties of $D_n^G(X)$ may be deduced from those of $F_n^G(X)$. For example, if X is a surface different from $\mathbb{S}^2$ and $\mathbb{R}P^2$, it follows from Proposition 2.1 that $D_n^G(X)$ is a $K(\pi,1)$. We also define the n-th orbit braid group of X (with respect to G) to be the fundamental group $\pi_1(D_n^G(X))$.

3 The Map ι and the Induced Homomorphism $\iota_\#$ for Surfaces

Let S be a connected surface, perhaps with boundary, and either compact, or with a finite number of points removed from the interior of the surface, and let $n \in \mathbb{N}$. In this section, we discuss recent results obtained in [23] regarding the inclusion map $\iota_n \colon F_n(S) \longrightarrow \prod_1^n S$ and the induced homomorphism $\iota_{n\#} : \pi_1(F_n(S)) \longrightarrow \pi_1\big(\prod_1^n S\big)$ on the level of fundamental groups, and more particularly, the cases where $S = \mathbb{S}^2$ or $\mathbb{R}P^2$. To simplify the notation, we will often just write ι and $\iota_\#$ if n is given. As we mentioned in Section 1, $\pi_1(F_n(S)) \cong P_n(S)$, the pure braid group of S on n strings (see [4, 11, 14]), from which one deduces that $\iota_{n\#}$ is surjective in a straightforward manner, and if S is the 2-disc $\mathbb{D}^2$ then $P_n(\mathbb{D}^2)$ is the Artin pure braid group P_n. If $\mathbb{D}^2$ is a topological disc lying in the interior of S and that contains the basepoints of the braids then the inclusion $j_n \colon \mathbb{D}^2 \longrightarrow S$ induces a group homomorphism $j_{n\#} \colon P_n \longrightarrow P_n(S)$. This homomorphism is known to be injective if S is different from $\mathbb{S}^2$ and $\mathbb{R}P^2$ (see [3, 19]). If $\beta \in B_n$ then we will denote its image $j_\#(\beta)$ in $P_n(S)$ simply by β. It is well known that the centre of P_n is infinite cyclic, generated by the full twist braid that we denote by Δ_n^2. If we consider Δ_n^2 as an element of $P_n(\mathbb{S}^2)$ or of $P_n(\mathbb{R}P^2)$, it is the unique element of order 2 and generates the centre (see [17, 20–21, 23, 27, 29]). If G is a group then we denote its commutator subgroup by $\Gamma_2(G)$ and its Abelianisation by G^{Ab}, and if H is a subgroup of G then we denote its normal closure in G by $\langle\!\langle H \rangle\!\rangle_G$. If $m \in \mathbb{N}$, $\mathbb{F}_m$ denotes the free group of rank m.

The study of the homomorphism $\iota_{n\#} : \pi_1(F_n(S)) \longrightarrow \pi_1\big(\prod_1^n S\big)$ was initiated by Birman in 1969 (see [3]). In that paper, she said that she had conjectured that $\mathrm{Ker}(\iota_{n\#}) = \langle\!\langle \mathrm{Im}\,(j_{n\#}) \rangle\!\rangle_{P_n(S)}$ if S is a compact orientable surface, but then stated without proof that her conjecture is false if S is of genus greater than or equal to 1 (see [3, p. 45]). However, several years later, Goldberg proved that Birman's conjecture is in fact true in both the orientable and non-orientable cases for compact surfaces without boundary different from $\mathbb{S}^2$ and $\mathbb{R}P^2$ (see [19, Theorem 1]). In connection with the study of Vassiliev invariants of surface braid groups, González-Meneses and Paris showed that $\mathrm{Ker}(\iota_{n\#})$ is also normal in the n-string full braid group $B_n(S)$ of S, and that the resulting quotient is isomorphic to the semi-direct product $\pi_1\big(\prod_1^n S\big) \rtimes S_n$, where the action is given by permuting coordinates (their work was within the framework of compact, orientable surfaces without boundary, but their construction is valid for any surface S, see [25]). In the case of $\mathbb{R}P^2$, this result was reproved using geometric methods (see [28]).

If $S = \mathbb{S}^2$, $\mathrm{Ker}(\iota_{n\#})$ is clearly equal to $P_n(\mathbb{S}^2)$, and so by [20, Theorem 4], it may be decomposed as

$$\mathrm{Ker}(\iota_{n\#}) = P_n(\mathbb{S}^2) \cong P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}) \times \mathbb{Z}_2, \tag{3.1}$$

where the first factor of the direct product is torsion free, and the $\mathbb{Z}_2$-factor is generated by Δ_n^2. Using Artin combing, one may show that $P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ may be decomposed as a repeated semi-direct product of the form $\mathbb{F}_{n-2} \rtimes (\mathbb{F}_{n-3} \rtimes (\cdots \rtimes \mathbb{F}_2) \cdots)$. This may be compared with the usual Artin combing operation of P_n:

$$P_n \cong \mathbb{F}_{n-1} \rtimes (\mathbb{F}_{n-2} \rtimes (\cdots \rtimes \mathbb{F}_2) \cdots) \times \langle \Delta_n^2 \rangle. \tag{3.2}$$

One of the aims of [23] is to resolve Birman's conjecture for surfaces without boundary in the remaining cases, namely $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, and in the case $S = \mathbb{R}P^2$, to elucidate the structure of $\operatorname{Ker}(\iota_{n\#})$ and to obtain a decomposition analogous to that of Equation (3.1) for $\mathbb{R}P^2$. One may see easily that the isomorphism given in (3.1) may not be generalised directly to the case of $\mathbb{R}P^2$, since for all $n \geq 2$, $P_n(\mathbb{R}P^2)$ possesses elements of order 4 (see [27, 29]), whilst the group $P_{n-m}(S \setminus \{x_1, \cdots, x_m\})$ is torsion free for all $n > m \geq 1$ and any surface S (including $\mathbb{S}^2$ and $\mathbb{R}P^2$).

Proposition 3.1 (see [23, Proposition 1]) *Let* $n \in \mathbb{N}$.

(a) (i) *Up to isomorphism, the homomorphism* $\iota_{n\#}\colon\ \pi_1(F_n(\mathbb{R}P^2)) \longrightarrow \pi_1(\Pi_1^n \mathbb{R}P^2)$ *coincides with Abelianisation. In particular,* $\operatorname{Ker}(\iota_{n\#}) = \Gamma_2(P_n(\mathbb{R}P^2))$.

(ii) *If* $n \geq 2$, *then there exists a torsion-free subgroup* L_n *of* $\operatorname{Ker}(\iota_{n\#})$ *such that*

$$\operatorname{Ker}(\iota_{n\#}) = L_n \times \langle \Delta_n^2 \rangle, \tag{3.3}$$

where the subgroup $\langle \Delta_n^2 \rangle$ *generated by the full twist is isomorphic to* $\mathbb{Z}_2$.

(b) *If* $n \geq 2$, *then any subgroup of* $P_n(\mathbb{R}P^2)$ *that is normal in* $B_n(\mathbb{R}P^2)$ *and that properly contains* $\operatorname{Ker}(\iota_{n\#})$ *possesses an element of order* 4.

Note that if $n = 1$ then $B_1(\mathbb{R}P^2) = P_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ and Δ_1^2 is the trivial element, so parts (a) (ii) and (b) of Proposition 3.1 do not hold. Equation (3.3) is the analogue of Equation (3.1), and as we will indicate shortly, the subgroup L_n may also be written as a repeated semi-direct product of free groups.

Proposition 3.1(a) (i) is proved by taking a generating set of $P_n(\mathbb{R}P^2)$ of the form $\{A_{i,j},\ \tau_k \mid 1 \leq i < j \leq n \text{ and } 1 \leq k \leq n\}$, where $\{A_{i,j}\}_{1 \leq i < j \leq n}$ is a standard set of generators of P_n, and for each $1 \leq k \leq n$, τ_k is represented by a braid that corresponds to a generator of $\pi_1(\mathbb{R}P^2)$ based at the k-th basepoint of $P_n(\mathbb{R}P^2)$. It is easy to see geometrically that $\iota_{n\#}(A_{i,j})$ is trivial for all $1 \leq i < j \leq n$ and that $\iota_{n\#}(\tau_k)$ is sent to the corresponding generator of $\pi_1(\Pi_1^n \mathbb{R}P^2)$. Using a presentation of $P_n(\mathbb{R}P^2)$ (see [21, Theorem 4]), one sees that the Abelianisation homomorphism has the same effect on these generators, which proves part (a). If $n \geq 3$, Proposition 3.1(a) (ii) may be obtained by considering the Fadell-Neuwirth pure braid group short exact sequence (see [11, 14]):

$$1 \longrightarrow P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow P_n(\mathbb{R}P^2) \xrightarrow{p_{2\#}} P_2(\mathbb{R}P^2) \longrightarrow 1, \tag{3.4}$$

where $p_{2\#}\colon\ P_n(\mathbb{R}P^2) \longrightarrow P_2(\mathbb{R}P^2)$ is the homomorphism given geometrically by forgetting all but the first two strings, and taking the restriction of $p_{2\#}$ to $\operatorname{Ker}(\iota_{n\#})$. Using part (i) and the fact that $P_2(\mathbb{R}P^2)$ is isomorphic to the quaternion group of order 8 (see [12]), it is straightforward to check that $p_{2\#}(\operatorname{Ker}(\iota_{n\#})) = \langle \Delta_2^2 \rangle$, which is isomorphic to $\mathbb{Z}_2$. One may also check that the kernel of the restriction of $p_{2\#}$ to $\operatorname{Ker}(\iota_{n\#})$ admits a section given by sending Δ_2^2 to the central element Δ_n^2, and is equal to $\operatorname{Ker}(\iota_{n\#}) \cap P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, which as we mentioned above, is torsion free. This intersection may be taken to be the group L_n in the statement. One may show that there are precisely $2^{n(n-2)}$ subgroups that satisfy the conclusions of part (a) (ii) (see [23, Remark 14]). If $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, the decompositions (3.1) and (3.3) are used in the proof of [23, Theorem 5] to compute the virtual cohomological dimension of $B_n(S)$, $P_n(S)$ and the corresponding mapping class groups of S with n marked points.

Using Proposition 3.1 in the case of $\mathbb{R}P^2$, we may then prove Birman's conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$.

Theorem 3.1 (see [23, Theorem 2]) *Let $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, and let $n \geq 1$. Then*

$$\langle\!\langle \operatorname{Im}(j_{n\#}) \rangle\!\rangle_{P_n(S)} = \operatorname{Ker}(\iota_{n\#}).$$

The proof of Theorem 3.1 is obtained by using the fact that $\{A_{i,j}\}_{1 \leq i < j \leq n}$ is a standard set of generators of P_n. The case $S = \mathbb{S}^2$ is straightforward since $\mathbb{S}^2$ is simply connected. In the case $S = \mathbb{R}P^2$, using this set of generators, one sees easily that $\langle\!\langle \operatorname{Im}(j_{n\#}) \rangle\!\rangle_{P_n(S)} \subseteq \operatorname{Ker}(\iota_{n\#})$. From the presentation of $P_n(\mathbb{R}P^2)$ given by [21, Theorem 4], one may determine a set of normal generators of $\Gamma_2(P_n(\mathbb{R}P^2))$, and then use Proposition 3.1(a) (i) to obtain the converse inclusion.

More detailed information may be obtained about the subgroup $L_n = \operatorname{Ker}(\iota_{n\#}) \cap P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Using Fadell-Neuwirth short exact sequences, the group $P_m(S \setminus \{x_1, \cdots, x_q\})$ may be decomposed as a repeated semi-direct product of the form

$$\mathbb{F}_{m-1+k} \rtimes (\mathbb{F}_{m-2+k} \rtimes (\cdots \rtimes \mathbb{F}_k) \cdots),$$

where k is the rank of $P_m(S \setminus \{x_1, \cdots, x_q\}) = \pi_1(S \setminus \{x_1, \cdots, x_q\})$. Notice that this rank is also equal to $q - 1 + (2 - \chi(S))$, where $\chi(S)$ is the Euler characteristic of S. In particular, in (3.1), $P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}) \cong \mathbb{F}_{n-2} \rtimes (\mathbb{F}_{n-3} \rtimes (\cdots \rtimes \mathbb{F}_2) \cdots)$. As we see in the following theorem, a similar result may be obtained for L_n.

Theorem 3.2 (see [23, Theorem 3]) *Let $n \geq 3$, and consider the Fadell-Neuwirth short exact sequence* (3.4)*. Then L_n may be identified with the kernel of the composition*

$$P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \longrightarrow P_n(\mathbb{R}P^2) \xrightarrow{\iota_{n\#}} \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ copies}},$$

where the first homomorphism is that appearing in Equation (3.4)*. The image of this composition is the product of the last $n-2$ copies of $\mathbb{Z}_2$. In particular, L_n is of index 2^{n-2} in $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Further, L_n is isomorphic to an iterated semi-direct product of free groups of the form $\mathbb{F}_{2n-3} \rtimes (\mathbb{F}_{2n-5} \rtimes (\cdots \rtimes (\mathbb{F}_5 \rtimes \mathbb{F}_3) \cdots))$.*

The first part of Theorem 3.2 may be obtained by analysing the commutative diagram involving the short exact sequence (3.4) and its restriction to $\operatorname{Ker}(\iota_{n\#})$. In the repeated semi-direct product decomposition of L_n, every factor acts on each of the preceding factors, and not just on the group formed by these factors. In this sense, the intersection of $\operatorname{Ker}(\iota_{n\#})$ with the repeated semi-direct product structure $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ restricts to each of the factors in a nice way. This is also the case for $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, and implies an Artin combing-type result for these groups. Taking into account the actions, these repeated semi-direct products may be used to prove the following result about the Abelianisation of $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ and L_n.

Proposition 3.2 (see [23, Proposition 4]) *If* $n \geq 3$, *then*
(a) $(P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}))^{\mathrm{Ab}} \cong \mathbb{Z}^{2(n-2)}$.
(b) $(L_n)^{\mathrm{Ab}} \cong \mathbb{Z}^{n(n-2)}$.

As we will see at the end of Section 4, L_n is closely related to the fundamental group of an orbit configuration space of the open cylinder.

4 Recent Results about the Homotopy Fibre and the Long Exact Sequence where A is a Configuration Space and X is a Surface

We start this section by recalling the construction of the homotopy fibre of a continuous map $f\colon A \longrightarrow Y$ of topological spaces that allows us to turn f into a fibration. Let y_0 be a basepoint of Y, and let

$$E_f = \{(x, \gamma) \in A \times Y^I \mid \gamma(0) = f(x)\} \quad \text{and} \quad I_f = \{(x, \gamma) \in E_f \mid \gamma(1) = y_0\} \tag{4.1}$$

denote the mapping path and homotopy fibre of f respectively. It is well known that $I_f \hookrightarrow E_f \longrightarrow Y$ is a fibration, and that E_f and A have the same homotopy type, a homotopy equivalence between E_f and A being given by the projection $E_f \longrightarrow A$ onto the first coordinate (see [1, Proposition 3.5.8 and Remark 3.5.9]). Let $p_f\colon I_f \longrightarrow A$ denote the composition of the inclusion $I_f \hookrightarrow E_f$ by this projection. We will refer to the sequence of maps $I_f \xrightarrow{p_f} A \xrightarrow{f} Y$ as a homotopy fibration. By [1, Proposition 3.3.17], the homotopy fibres of two homotopic maps have the same homotopy type, and if f is null homotopic then I_f has the same homotopy type as $A \times \Omega Y$, where ΩY denotes the loop space of Y. These two properties will be useful in what follows. Notice that I_f is determined by the homotopy pull-back diagram

$$\begin{tikzcd} I_f \arrow[r, "p_f"] \arrow[d, "q"] & A \arrow[d, "f"] \\ * \arrow[r, "c_{y_0}"] & Y \end{tikzcd} \tag{4.2}$$

where $c_{y_0} : * \longrightarrow Y$ is the constant map determined by a point $y_0 \in Y$, and $q : I_f \longrightarrow *$ is the constant map (see [1, p. 205]).

The main aim of [24] is to determine the homotopy type of the homotopy fibre of f in the case where $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, $A = F_n(S)$, $Y = \prod_1^n S$, and f is the inclusion map $\iota_n\colon F_n(S) \longrightarrow \prod_1^n S$. As a consequence, this leads to a better understanding of the induced homomorphisms $(\iota_n)_{\#m}$: $\pi_m(F_n(S)) \longrightarrow \pi_m\bigl(\prod_1^n S\bigr)$ on the π_m-level of the long exact sequence in homotopy of the homotopy fibration $I_{\iota_n} \xrightarrow{p_{\iota_n}} F_n(S) \overset{\iota_n}{\hookrightarrow} \prod_1^n S$, as well as an alternative interpretation of $P_n(S)$. One interesting point about this long exact sequence is that the homotopy groups of $\mathbb{S}^2$ appear, not just as factors in the groups $\pi_m\bigl(\prod_1^n S\bigr)$, but also in the groups $\pi_m(F_n(S))$. This is a consequence of the following result and the fact that $\pi_m(\mathbb{S}^3)$ is isomorphic to $\pi_m(\mathbb{S}^2)$ if $m \geq 3$.

Theorem 4.1 (see [5, 13] and [21, Proposition 6]) *Let $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, and let $n \in \mathbb{N}$. Then the universal covering $\widetilde{F_n(S)}$ of $F_n(S)$ has the homotopy type of $\mathbb{S}^2$ if $S = \mathbb{S}^2$ and $n \le 2$ or if $S = \mathbb{R}P^2$ and $n = 1$, and has the homotopy type of the 3-sphere $\mathbb{S}^3$ otherwise.*

Let $p\colon \mathbb{S}^2 \longrightarrow \mathbb{R}P^2$ denote the universal covering map, and for $i \in \{1, \cdots, n\}$, let $p_i : F_n(S) \longrightarrow S$ and $\widetilde{p_i}\colon \prod_1^n S \longrightarrow S$ denote the respective projections onto the i-th coordinate. Then $p_i = \widetilde{p_i} \circ \iota_n$. The following result is proved in [24, Lemma 7 and Proposition 9].

Proposition 4.1 (a) *Suppose that $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, and assume that $n, m \ge 2$. Then the induced homomorphism $(\iota_n)_{\#m} : \pi_m(F_n(S)) \longrightarrow \pi_m\big(\prod_1^n S\big)$ is injective.*

(b) *If $n = m = 2$, then the homomorphism $(\iota_2)_{\#2} : \pi_2(F_2(\mathbb{S}^2)) \longrightarrow \pi_2\big(\prod_{i=1}^2 \mathbb{S}^2\big)$ is the anti-diagonal homomorphism that sends a generator of $\pi_2(F_2(\mathbb{S}^2))$ to $(1, -1)$ or to $(-1, 1)$ depending on the choice of orientation of $\mathbb{S}^2$.*

(c) *If $n \ge 2$ and $m \ge 3$, then $(\iota_n)_{\#m} : \pi_m(F_n(\mathbb{S}^2)) \longrightarrow \pi_m\big(\prod_{i=1}^n \mathbb{S}^2\big)$ is a diagonal homomorphism.*

(d) *Let $m, n \ge 2$, and suppose that $n = 2$ if $m = 2$. Then for all $i = 1, \cdots, n$, the homomorphism $(p_i)_{\#m}\colon \pi_m(F_n(\mathbb{S}^2)) \longrightarrow \pi_m(\mathbb{S}^2)$ is an isomorphism.*

(e) *Let $n, m \ge 2$. Then $(\iota_n)_{\#m} : \pi_m(F_n(\mathbb{R}P^2)) \longrightarrow \pi_m\big(\prod_{i=1}^n \mathbb{R}P^2\big)$ is a diagonal homomorphism, and if additionally $m \ge 3$, then $(p_i)_{\#m}\colon \pi_m(F_n(\mathbb{R}P^2)) \longrightarrow \pi_m(\mathbb{R}P^2)$ is an isomorphism for all $i = 1, \cdots, n$.*

If $S = \mathbb{S}^2$, in the cases not covered by parts (b)–(d), namely $m = 2$ and $n \ge 3$, the homomorphism $(\iota_n)_{\#2} : \pi_2(F_n(\mathbb{S}^2)) \longrightarrow \pi_2\big(\prod_{i=1}^n \mathbb{S}^2\big)$ is trivially diagonal, since $\pi_2(F_n(\mathbb{S}^2)) = 1$, but the homomorphism $(p_i)_{\#2}$ of part (d) is not an isomorphism. The main ingredients in the proof of Proposition 4.1 are the fact that p_i is a fibration whose fibre is an Eilenberg-Mac Lane space of type $K(\pi, 1)$, Theorem 4.1, an analysis of ι_2 and $(\iota_2)_{\#m}$, and if $n \ge 3$, the comparison of the different projections of $F_n(S)$ onto $F_2(S)$ given by forgetting all but two coordinates. Proposition 4.1 allows us to determine the homotopy type of the homotopy fibre of $\iota_n\colon F_n(S) \longrightarrow \prod_1^n S$. The description in the case of $\mathbb{R}P^2$ makes use of the notion of orbit configuration space as defined in Section 1 and discussed in Section 2. The following results are stated in [24, Theorem 1].

Theorem 4.2 *Let $n \ge 2$, and let $S = \mathbb{S}^2$ or $\mathbb{R}P^2$. The homotopy fibre I_{ι_n} of the inclusion map $\iota_n\colon F_n(S) \longrightarrow \prod_1^n S$ has the homotopy type of $F_{n-1}(\mathbb{D}^2) \times \Omega\big(\prod_1^{n-1} \mathbb{S}^2\big)$, or equivalently of $K(P_{n-1}, 1) \times \Omega\big(\prod_1^{n-1} \mathbb{S}^2\big)$ if $S = \mathbb{S}^2$, and has the homotopy type of $F_{n-1}^{\langle\tau\rangle}(C) \times \Omega\big(\prod_1^{n-1} \mathbb{S}^2\big)$, or equivalently of $K(G_{n-1}, 1) \times \Omega\big(\prod_1^{n-1} \mathbb{S}^2\big)$ if $S = \mathbb{R}P^2$.*

The idea of the proof of Theorem 4.2 is to replace the map ι_n by another map that is null homotopic and whose homotopy fibre has the same homotopy type as that of I_{ι_n}. This allows us to apply the general properties of homotopy fibres given at the beginning of this section.

The projections p_i and $\widetilde{p_i}$ onto S are fibrations, and the restriction of ι_n to their topological fibres gives rise to a map $\iota_n\big|_{F_{n-1}(S\setminus\{x_0\})}\colon F_{n-1}(S\setminus\{x_0\}) \longrightarrow F_{n-1}(S)$. Further, the homotopy fibres of ι_n and $\iota_n\big|_{F_{n-1}(S\setminus\{x_0\})}$ have the same homotopy type. This is a consequence of the following result that seems to be well known to the experts in the field, but for which we were not able to find a proof in the literature.

Proposition 4.2 *Let $\alpha\colon E \longrightarrow B$ be a fibration, and let E_0 be a subspace of E such that the restriction $\alpha_0 = \alpha|_{E_0}\colon E_0 \longrightarrow B$ is also a fibration. Let F (resp. F_0) denote the fibre of α (resp. α_0) over a base point $b_0 \in B$, and let $\iota\colon E_0 \longrightarrow E$ and $\iota_0\colon F_0 \longrightarrow F$ denote the respective inclusions, where $\iota_0 = \iota|_{F_0}$. Then the homotopy fibres of ι and ι_0 are homotopy equivalent.*

A proof of Proposition 4.2 proposed by Michael Crabb may be found in [24, Proposition 36]. In the case where $S = \mathbb{S}^2$, by composing by the projections p_i and $\widetilde{p_i}$, we see that $\iota_n|_{F_{n-1}(\mathbb{S}^2\setminus\{x_0\})}$ is null homotopic, which yields the result. If $S = \mathbb{R}P^2$, the restriction map $\iota_n|_{F_{n-1}(\mathbb{R}P^2\setminus\{p(x_0)\})}$ is not null homotopic, but it lifts to the inclusion map $\iota'_n\colon F_{n-1}^{\langle\tau\rangle}(C) \longrightarrow \prod_1^{n-1} \mathbb{S}^2$ that is null homotopic (this may be seen once more by considering the projections onto each factor), and whose homotopy fibre has the same homotopy type as that of $\iota_n|_{F_{n-1}(\mathbb{R}P^2\setminus\{p(x_0)\})}$. One then concludes as in the case of $\mathbb{S}^2$.

Recall from Section 2 that $G_n = \pi_1(F_n^{\langle\tau\rangle}(C))$ and that $F_n^{\langle\tau\rangle}(C)$ is a $K(G_n, 1)$. From the presentation of G_n given by Proposition 2.2, one may show that the centre of G_n is infinite cyclic, generated by the element $\Theta_n = \rho_{1,0}\cdots\rho_{n,0}$ that is similar in nature to Δ_n^2, and that [24, Remark 21]:

$$G_n = H_n \times \langle\Theta_n\rangle, \quad \text{where } H_n \cong \mathbb{F}_{2n-1} \rtimes (\mathbb{F}_{2n-3} \rtimes (\cdots \rtimes \mathbb{F}_3)\cdots). \tag{4.3}$$

Taking the long exact sequence in homotopy of the homotopy fibration

$$I_{\iota_n} \xrightarrow{p_{\iota_n}} F_n(S) \xrightarrow{\iota_n} \prod_1^n S, \tag{4.4}$$

and using Proposition 4.1 and Theorem 4.2, we obtain the following corollary that may be found in [24, Section 1].

Corollary 4.1 *Let $n \geq 2$.*

(a) *Let $k \geq 3$ (resp. $k = n = 2$). Then we have the following split short exact sequence of Abelian groups:*

$$1 \longrightarrow \pi_k(F_n(\mathbb{S}^2)) \xrightarrow{(\iota_n)_{\#k}} \pi_k\Big(\prod_1^n \mathbb{S}^2\Big) \longrightarrow \pi_{k-1}\Big(\Omega\Big(\prod_1^{n-1} \mathbb{S}^2\Big)\Big) \longrightarrow 1, \tag{4.5}$$

where $(\iota_n)_{\#k}$ is diagonal (resp. anti diagonal). Up to isomorphism, this short exact sequence may also be written as $1 \longrightarrow \pi_k(\mathbb{S}^2) \longrightarrow \prod_1^n \pi_k(\mathbb{S}^2) \longrightarrow \prod_1^{n-1} \pi_k(\mathbb{S}^2) \longrightarrow 1$.

(b) *If $k = 2$ and $n \geq 3$, then we have the following non-split short exact sequence:*

$$1 \longrightarrow \pi_2\Big(\prod_1^n \mathbb{S}^2\Big) \longrightarrow \pi_1(K(P_{n-1}, 1)) \times \pi_1\Big(\Omega\Big(\prod_1^{n-1} \mathbb{S}^2\Big)\Big) \longrightarrow P_n(\mathbb{S}^2) \longrightarrow 1, \tag{4.6}$$

which may be also written as $1 \longrightarrow \mathbb{Z}^n \longrightarrow P_{n-1} \oplus \mathbb{Z}^{n-1} \longrightarrow P_n(\mathbb{S}^2) \longrightarrow 1$.

(c) *If* $k \geq 3$, *then we have the following split short exact sequence of Abelian groups:*

$$1 \longrightarrow \pi_k(F_n(\mathbb{R}P^2)) \xrightarrow{(\iota_n)_{\#k}} \pi_k\Big(\prod_1^n \mathbb{R}P^2\Big) \longrightarrow \pi_{k-1}\Big(\Omega\Big(\prod_1^{n-1} \mathbb{S}^2\Big)\Big) \longrightarrow 1, \tag{4.7}$$

where the homomorphism $(\iota_n)_{\#k}$ *is diagonal. Up to isomorphism, this short exact sequence may also be written as*

$$1 \longrightarrow \pi_k(\mathbb{R}P^2) \longrightarrow \prod_1^n \pi_k(\mathbb{R}P^2) \longrightarrow \prod_1^{n-1} \pi_k(\mathbb{R}P^2) \longrightarrow 1.$$

(d) *If* $k = 2$, *then we have the following exact sequence:*

$$1 \longrightarrow \pi_2\Big(\prod_1^n \mathbb{R}P^2\Big) \longrightarrow G_{n-1} \times \pi_1\Big(\Omega\Big(\prod_1^{n-1} \mathbb{S}^2\Big)\Big) \longrightarrow P_n(\mathbb{R}P^2) \xrightarrow{\iota_{n\#}} \mathbb{Z}_2^n \longrightarrow 1, \tag{4.8}$$

which up to isomorphism, may also be written as

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow G_{n-1} \times \mathbb{Z}^{n-1} \longrightarrow P_n(\mathbb{R}P^2) \longrightarrow \mathbb{Z}_2^n \longrightarrow 1.$$

In parts (a) and (c), the nature of the exact sequences is completely determined since we know that the homomorphism $(\iota_n)_{\#k}$ is diagonal by Proposition 4.1. In parts (b) and (d), the situation is less clear, and to understand the corresponding exact sequences, one must describe the boundary homomorphism $\partial_n \colon \pi_2\big(\prod_1^n S\big) \longrightarrow \pi_1(I_{\iota_n})$ of the homotopy fibration (4.4). This is not so straightforward, since to do so, it is necessary to determine geometric representatives of a generating set of $\pi_1(I_{\iota_n})$ via the identifications made in the proof of Theorem 4.2 between the different homotopy fibres, and then relate them to the images under ∂_n of geometric representatives of a basis of $\pi_2\big(\prod_1^n S\big)$. This is quite a long process, and the details are given in [24, Section 5]. The basic idea is first to understand the boundary homomorphism in the case $n = 3$ (resp. $n = 2$) if $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$), and then to deduce the result in the general case by considering the projections onto three (resp. two) coordinates. The exact sequences (4.6) and (4.8) may be written in the form:

$$1 \longrightarrow \pi_2\Big(\prod_{i=1}^n S\Big) \xrightarrow{\partial_n} \pi_1(I_{\iota_n}) \xrightarrow{(p_{\iota_n})_\#} P_n(S) \xrightarrow{\iota_{n\#}} \pi_1\Big(\prod_{i=1}^n S\Big) \longrightarrow 1. \tag{4.9}$$

Note that the term $\pi_1\big(\prod_{i=1}^n S\big)$ is trivial if $S = \mathbb{S}^2$. Fix a basepoint $(x_1, \cdots, x_n) \in F_n(\mathbb{S}^2)$. Identifying $\pi_2(S)$ with $\pi_1(\Omega S)$ in the usual way, for each $1 \leq i \leq n$, we choose a generator $\widetilde{\lambda}_{x_i}$ of $\pi_2(\mathbb{S}^2, x_i)$ that is represented by a family of loops based at x_i. This gives rise to a basis $(\widetilde{\lambda}_{x_i})_{1 \leq i \leq n}$ of $\pi_2\big(\prod_1^n \mathbb{S}^2\big)$ and to a basis $(\lambda_{p(x_i)})_{1 \leq i \leq n}$ of $\pi_2\big(\prod_1^n \mathbb{R}P^2\big)$. Using the above-mentioned identifications between the various homotopy fibres, if $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$), we construct an explicit homotopy equivalence $h \colon F_{n-1}(\mathbb{D}^2) \times \Omega\big(\prod_1^{n-1} \mathbb{S}^2\big) \longrightarrow I_{\iota_n}$ (resp.

$h\colon F_{n-1}^{\langle\tau\rangle}(C)\times\Omega\big(\prod_1^{n-1}\mathbb{S}^2\big)\longrightarrow I_{\iota_n})$ given by Theorem 4.2. Writing I_{ι_n} in the form of Equation (4.1), for $i=2,\cdots,n-1$, the induced isomorphism $h_\#$ on the level of fundamental groups sends the generator $\widetilde{\lambda}_{x_i}$ of the $(i-1)$-th factor of $\Omega\big(\prod_1^{n-1}\mathbb{S}^2\big)$ to an element $\widetilde{\delta}_{x_i}$ (resp. $\delta_{p(x_i)}$) of $\pi_1(I_{\iota_n})$ whose restriction to the i-th factor of $\pi_1(I_{\iota_n})$ is represented by $\widetilde{\lambda}_{x_i}$, and is constant elsewhere. Further, $h_\#$ sends Δ_{n-1}^2 (resp. Θ_{n-1}) to an element of $\pi_1(I_{\iota_n})$ that we denote by τ_n, and using the geometry, one may see that $(p_{\iota_n})_\#(\tau_n)=\Delta_n^2$ in $P_n(S)$. With our choices of the x_i and the corresponding elements $\widetilde{\lambda}_{x_i}$ for $i=2,\cdots,n$, we prove that (see [24, Theorem 3])

$$\begin{aligned}&\partial_n(\widetilde{\lambda}_{x_i})=\widetilde{\delta}_{x_i},\quad \partial_n(\widetilde{\lambda}_{x_1})=\tau_n^2-(\widetilde{\delta}_{x_2}+\ldots+\widetilde{\delta}_{x_{n-1}}-\widetilde{\delta}_{x_n})\quad \text{if } S=\mathbb{S}^2,\\ &\partial_n(\lambda_{p(x_i)})=\delta_{p(x_i)},\quad \partial_n(\lambda_{p(x_1)})=\tau_n^2-(\delta_{p(x_2)}+\cdots+\delta_{p(x_n)})\quad \text{if } S=\mathbb{R}P^2.\end{aligned}$$

Using these relations and exactness of (4.9), and up to the identification of $\pi_1\big(F_{n-1}(\mathbb{D}^2)\times\Omega\big(\prod_1^{n-1}\mathbb{S}^2\big)\big)$ $\big($resp. $\pi_1\big(F_{n-1}^{\langle\tau\rangle}(C)\times\Omega\big(\prod_1^{n-1}\mathbb{S}^2\big)\big)\big)$ with $\pi_1(I_{\iota_n})$, it follows that (see [24, Proposition 4])

$$\operatorname{Ker}(\iota_{n\#})=\operatorname{Im}\left((p_{\iota_n})_\#\right)\cong\pi_1(I_{\iota_n})/\operatorname{Im}(\partial_n)\cong\begin{cases}P_{n-1}/\left\langle\Delta_{n-1}^4\right\rangle, & \text{if } S=\mathbb{S}^2,\\ G_{n-1}/\left\langle\Theta_{n-1}^2\right\rangle, & \text{if } S=\mathbb{R}P^2.\end{cases}$$

By Equations (3.1)–(3.3) and (4.3), it follows that

$$\operatorname{Ker}(\iota_{n\#})\cong\begin{cases}(\mathbb{F}_{n-2}\rtimes(\cdots\rtimes\mathbb{F}_2)\cdots)\times\mathbb{Z}_2, & \text{if } S=\mathbb{S}^2,\\ (\mathbb{F}_{2n-3}\rtimes(\mathbb{F}_{2n-5}\rtimes(\cdots\rtimes\mathbb{F}_3)\cdots))\times\mathbb{Z}_2, & \text{if } S=\mathbb{R}P^2.\end{cases}\tag{4.10}$$

On the other hand, $\operatorname{Ker}(\iota_{n\#})$ is equal to $P_n(\mathbb{S}^2)$ if $S=\mathbb{S}^2$, and is equal to $\Gamma_2(P_n(\mathbb{R}P^2))$ if $S=\mathbb{R}P^2$ by Proposition 3.1(a) (i), so by exactness of (4.9) and Equation (4.10), we obtain

$$P_n(\mathbb{S}^2)\cong(\mathbb{F}_{n-2}\rtimes(\cdots\rtimes\mathbb{F}_2)\cdots)\times\mathbb{Z}_2,\tag{4.11}$$

$$\Gamma_2(P_n(\mathbb{R}P^2))\cong(\mathbb{F}_{2n-3}\rtimes(\mathbb{F}_{2n-5}\rtimes(\cdots\rtimes\mathbb{F}_3)\cdots))\times\mathbb{Z}_2.\tag{4.12}$$

Equation (4.11) is in agreement with the decomposition given by Equation (3.1). In the case of $\mathbb{R}P^2$, we recover the repeated semi-direct product decomposition of L_n given by Theorem 3.2 by comparing Equation (4.12) with Equation (3.3).

5 Recent Results about the Homotopy Type of the Homotopy Fibre and the Long Exact Sequence, where A is a Configuration Space and either the Universal Covering of X is Contractible or X is the Orbit Space $\mathbb{S}^n/G$, where G is a Lie Group

Let X be a topological manifold without boundary such that $\dim(X)\geq 3$. In this section, we describe recent results of [18] about the homotopy type of the homotopy fibre of the inclusion map $\iota_n\colon F_n(X)\hookrightarrow\prod_1^n X$, and the associated long exact sequence in homotopy. They include the cases where either the universal covering $\widetilde{X}$ of X is contractible, or X is an orbit space of the form $\mathbb{S}^k/G$ of a tame (i.e., an action for which the orbit space $\mathbb{S}^k/G$ is a manifold, see [8]) free action of a Lie group G on the k-dimensional sphere $\mathbb{S}^k$. In the former case, if $\widetilde{X}$ is contractible, the long exact sequence in homotopy of the homotopy fibration corresponding to ι_n is clear. In the latter case, if the group G is finite and k is odd, we will give a full description of the long exact sequence in homotopy of the homotopy fibration of ι_n.

To begin with, let X, X', Y and Y' be topological pointed spaces. The connectivity of X, denoted by $\mathrm{conn}(X)$, is defined to be the largest non-negative integer n (or infinity) for which $\pi_i(X) = 0$ for all $i \leq n$. Recall that a map $f\colon X \longrightarrow Y$ is said to be n-connected if the induced homomorphism $f_{\#j}\colon \pi_j(X) \longrightarrow \pi_j(Y)$ on the level of π_j is surjective if $j = n$ and is an isomorphism for all $j \leq n-1$. Note that the map f is n-connected if and only if its homotopy fibre I_f is $(n-1)$-connected, in other words, $\pi_i(I_f) = 0$ for all $0 \leq i \leq n-1$. Let $f\colon X \longrightarrow Y$, $f'\colon X' \longrightarrow Y'$, $g\colon X \longrightarrow X'$ and $g'\colon Y \longrightarrow Y'$ be maps such that the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \tag{5.1}$$

is homotopy-commutative. Let $H\colon X \times I \longrightarrow Y'$ be a homotopy between $g' \circ f$ and $f' \circ g$, and let $\varphi\colon X \longrightarrow Y'^I$ be the map given by $\varphi(x)(t) = H(x,t)$ for all $(x,t) \in X \times I$. Then the map $\mathcal{I}(g,g')\colon I_f \longrightarrow I_{f'}$ between the homotopy fibres of f and f' defined by

$$\mathcal{I}(g,g')(x,\gamma) = (g(x), \varphi(x)^{-1} * (g' \circ \gamma)) \quad \text{for all } (x,\gamma) \in I_f,$$

makes the following diagram

$$\begin{tikzcd} I_f \arrow[r, "p_f"] \arrow[d, dotted, "\mathcal{I}(g,g')"] & X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "g'"] \\ I_{f'} \arrow[r, "p_{f'}"] & X' \arrow[r, "f'"] & Y' \end{tikzcd} \tag{5.2}$$

homotopy-commutative. The two rows are homotopy fibrations, and $p_f : I_f \longrightarrow X$ and $p_{f'} : I_{f'} \longrightarrow X'$ are the projections onto the first factor described in Section 4.

Now let $G \times X \longrightarrow X$ be a tame, free action of a group G on a topological manifold X without boundary. Let $q(X)\colon X \longrightarrow X/G$ denote the quotient map, let $q_1 \in X$, let $Q = \{q_1\}$, and let $\overline{Q} = q(X)(Q)$. The map $\prod_1^n q(X)\colon \prod_1^n X \longrightarrow \prod_1^n X/G$ induces a map $\psi_n(X)\colon F_n^G(X) \longrightarrow F_n(X/G)$, which in turn restricts to a map

$$\psi_n(X \setminus GQ)\colon F_n^G(X \setminus GQ) \longrightarrow F_n((X/G) \setminus \overline{Q}).$$

Let $j_n(X/G)\colon \prod_1^{n-1} X/G \longrightarrow \prod_1^n X/G$ and $j'_n(X/G)\colon F_{n-1}((X/G) \setminus \overline{Q}) \longrightarrow F_n(X/G)$ denote the maps that insert the point $q(X)(q_1)$ into the first position. These two maps are the inclusions of the fibres over the point $q(X)(q_1)$ of the fibrations $F_n(X/G) \longrightarrow X$ and $\prod_1^n X \longrightarrow X$ given by forgetting all but the first coordinate. By applying the construction of (5.2), we obtain the following diagram that is homotopy commutative:

$$\begin{tikzcd}
I_{\iota''_{n-1}(X)} \arrow[r, "p_{\iota''_{n-1}(X)}"] \arrow[d, "\mathcal{I}(\psi_{n-1}(X\setminus GQ), \prod_1^{n-1} q(X))"'] & F^G_{n-1}(X \setminus GQ) \arrow[r, hook, "\iota''_{n-1}(X)"] \arrow[d, "\psi_{n-1}(X\setminus GQ)"] & \prod_1^{n-1} X \arrow[d, "\prod_1^{n-1} q(X)"] \\
I_{\iota'_{n-1}(X/G)} \arrow[r, "p_{\iota'_{n-1}(X/G)}"] \arrow[d, "\mathcal{I}(j'_n(X/G), j_n(X/G))"'] & F_{n-1}((X/G) \setminus \overline{Q}) \arrow[r, hook, "\iota'_{n-1}(X/G)"] \arrow[d, hook, "j'_n(X/G)"] & \prod_1^{n-1} X/G \arrow[d, hook, "j_n(X/G)"] \\
I_{\iota_n(X/G)} \arrow[r, "p_{\iota_n(X/G)}"] & F_n(X/G) \arrow[r, hook, "\iota_n(X/G)"] & \prod_1^{n} X/G
\end{tikzcd} \tag{5.3}$$

where the maps $\iota''_{n-1}(X)$, $\iota'_{n-1}(X/G)$ and $\iota_n(X/G)$ denote inclusion of the given (orbit) configuration space into the associated Cartesian product, the rows are homotopy fibrations, and the maps $p_{\iota''_{n-1}(X)}$, $p_{\iota'_{n-1}(X/G)}$ and $p_{\iota_n(X/G)}$ are the projections of the homotopy fibres onto the first factor. Using Proposition 4.2, the map

$$\mathcal{I}(j'_n(X/G), j_n(X/G))\colon\ I_{\iota'_{n-1}(X/G)} \longrightarrow I_{\iota_n(X/G)}$$

may be seen to be a homotopy equivalence. Applying [18, Corollary 1.7], one may also show that the map

$$\mathcal{I}\Big(\psi_{n-1}(X \setminus GQ), \prod_1^{n-1} q(X)\Big)\colon\ I_{\iota''_{n-1}(X)} \longrightarrow I_{\iota'_{n-1}(X/G)}$$

gives rise to a homotopy equivalence

$$\Omega\mathcal{I}\Big(\psi_{n-1}(X \setminus GQ), \prod_1^{n-1} q(X)\Big)\colon\ \Omega I_{\iota''_{n-1}(X)} \longrightarrow \Omega I_{\iota'_{n-1}(X/G)}$$

on the level of loop spaces. Let $k_n(X) : I_{i''_{n-1}(X)} \longrightarrow I_{i_n(X/G)}$ be the map defined by $k_n(X) = \mathcal{I}(j'_n(X/G), j_n(X/G)) \circ \mathcal{I}(\psi_{n-1}(X\setminus GQ), \Pi_1^{n-1} q(X))$. We then obtain the following theorem.

Theorem 5.1 (see [18, Theorem 3.5]) *Let $G \times X \longrightarrow X$ be a tame, free action of a Lie group G on a connected topological manifold X without boundary, and let $Q \in X$. Suppose that the inclusion map $\iota''_{n-1}(X)\colon F^G_{n-1}(X \setminus GQ) \hookrightarrow \prod_1^{n-1} X$ is null homotopic (this is the case if for example the inclusion map $X \setminus GQ \hookrightarrow X$ is null homotopic).*

(a) *The maps*

$$\mathcal{I}(\mathrm{Id}_{F^G_{n-1}(X\setminus GQ)}, \mathrm{Id}_{\prod_1^{n-1} X}) : I_{\iota''_{n-1}(X)} \longrightarrow F^G_{n-1}(X \setminus GQ) \times \Omega\Big(\prod_1^{n-1} X\Big)$$

and

$$\mathcal{I}(j_{n'}(X/G), j_n(X/G))\colon\ I_{\iota'_{n-1}(X/G)} \longrightarrow I_{\iota_n(X/G)}$$

are homotopy equivalences, and the map

$$\mathcal{I}\Big(\psi_{n-1}(X \setminus GQ), \prod_1^{n-1} q(X)\Big)\colon I_{\iota''_{n-1}(X)} \longrightarrow I_{\iota'_{n-1}(X/G)}$$

is a weak homotopy equivalence. Further, if G is discrete, X is 1-connected and $\dim(X/G) \geq 3$, then the map $\psi_{n-1}(X \setminus GQ)\colon F^G_{n-1}(X \setminus GQ) \longrightarrow F_{n-1}((X/G) \setminus \overline{Q})$ is the universal covering.

(b) *If $j \leq \min(\mathrm{conn}(X), \dim(X/G) - 2)$, then $\pi_j(F^G_{n-1}(X \setminus GQ)) = 0$.*

(c) *Suppose that the homomorphism $\pi_j(X) \longrightarrow \pi_j(X/G)$ induced by the quotient map $X \longrightarrow X/G$ is injective for all $j \geq 1$ (this is the case if for example the inclusion map $GQ \hookrightarrow X$ is null homotopic). For all $j \geq 2$, up to the identification of the groups $\pi_{j-1}(I_{\iota_n(X/G)})$ and $\pi_{j-1}(F^G_{n-1}(X \setminus GQ)) \times \pi_{j-1}\big(\Omega\big(\prod_1^{n-1} X\big)\big)$ via the isomorphism*

$$(\mathcal{I}(\mathrm{Id}_{F^G_{n-1}(X\setminus GQ)}, \mathrm{Id}_{\prod_1^{n-1} X}))_{\#(j-1)} \circ ((k_n(X))_{\#(j-1)})^{-1},$$

the restriction to the subgroup $\pi_j\big(\prod_1^{n-1} X\big)$ of $\pi_j\big(\prod_1^{n} X/G\big)$ of the boundary homomorphism $\widehat{\partial}_j\colon \pi_{j-1}\big(\Omega\big(\prod_1^{n} X/G\big)\big) \longrightarrow \pi_{j-1}(I_{\iota_n(X/G)})$ given by the long exact sequence in homotopy of the homotopy fibration $I_{\iota_n(X/G)} \xrightarrow{p_{\iota_n(X/G)}} F_n(X/G) \xhookrightarrow{i_n(X/G)} \prod_1^n X/G$ of (5.3) coincides with the inclusion of $\pi_j\big(\prod_1^{n-1} X\big)$ in $\pi_{j-1}(F^G_{n-1}(X \setminus GQ)) \times \pi_{j-1}\big(\Omega\big(\prod_1^{n-1} X\big)\big)$ via the usual identification of $\pi_j(X)$ with $\pi_{j-1}(\Omega(X))$.

The proof of Theorem 5.1 makes use of an explicit model for the homotopy type of the homotopy fibre I_f of a map $f\colon X \longrightarrow Y$ that is homotopic to a constant map, properties of homotopy pullbacks and universal coverings, as well as a generalisation of [3, Theorem 1] to the topological category given in [18, Theorem 2.2], and routine diagram-chasing arguments. The following corollary is obtained by considering the usual action of $\pi_1(X)$ on the universal covering of X.

Corollary 5.1 (see [18, Corollary 3.6]) *Assume that X be a connected topological manifold without boundary such that $\dim(X) \geq 3$, let $\widetilde{X}$ be its universal covering, and let $Q \in \widetilde{X}$. Assume that the map*

$$\iota''_{n-1}(\widetilde{X})\colon F^{\pi_1(X)}_{n-1}(\widetilde{X} \setminus \pi_1(X)Q) \hookrightarrow \prod_1^{n-1} \widetilde{X}$$

is null homotopic.

(a) *The spaces $F^{\pi_1(X)}_{n-1}(\widetilde{X} \setminus \pi_1(X)Q)$ and $F_{n-1}\widetilde{(X \setminus \overline{Q})}$ are homeomorphic, and there exists a homotopy equivalence between $I_{\iota''_{n-1}(\widetilde{X})}$ and $F_{n-1}\widetilde{(X \setminus \overline{Q})} \times \Omega\big(\prod_1^{n-1} \widetilde{X}\big)$, and a weak homotopy equivalence between $I_{\iota_n(X)}$ and $F_{n-1}\widetilde{(X \setminus \overline{Q})} \times \Omega\big(\prod_1^{n-1} \widetilde{X}\big)$.*

(b) *Let* $j \geq 2$. *Up to the identification of the groups* $\pi_j\bigl(\prod_1^n \widetilde{X}\bigr)$ *with* $\pi_j\bigl(\prod_1^n X\bigr)$ *via the isomorphism* $(q(\widetilde{X}))_{\#j}\colon \pi_j(\widetilde{X}) \longrightarrow \pi_j(X)$ *and the identification of the groups* $\pi_{j-1}(I_{\iota_n(X)})$ *and* $\pi_{j-1}(F_{n-1}^{\pi_1(X)}(\widetilde{X} \setminus \pi_1(X)Q)) \times \pi_{j-1}\bigl(\Omega\bigl(\prod_1^{n-1} \widetilde{X}\bigr)\bigr)$ *via the isomorphism given in Theorem* 5.1(c), *the restriction of the boundary homomorphism* $\widehat{\partial}_j\colon \pi_j\bigl(\prod_1^n \widetilde{X}\bigr) \longrightarrow \pi_{j-1}(I_{\iota_n(X)})$ *of the long exact sequence in homotopy of the homotopy fibration*

$$I_{\iota_n(X)} \xrightarrow{p_{\iota_n(X)}} F_n(X) \overset{\iota_n(X)}{\hookrightarrow} \prod_1^n X$$

to the subgroup $\pi_j\bigl(\prod_1^{n-1} \widetilde{X}\bigr)$ *of* $\pi_j\bigl(\prod_1^n \widetilde{X}\bigr)$ *coincides with the inclusion of* $\pi_j\bigl(\prod_1^{n-1} \widetilde{X}\bigr)$ *in the product* $\pi_{j-1}(F_{n-1}^{\pi_1(X)}(\widetilde{X} \setminus \pi_1(X)Q)) \times \pi_{j-1}\bigl(\Omega\bigl(\prod_1^{n-1} \widetilde{X}\bigr)\bigr)$ *via the usual identification of* $\pi_j(\widetilde{X})$ *with* $\pi_{j-1}(\Omega(\widetilde{X}))$.

Corollary 5.1 may be applied to two types of spaces, those for which the universal covering is contractible, and those of the form $\mathbb{S}^k/G$, where G is a Lie group. In the first case, the following result brings together various descriptions of the weak homotopy type of the homotopy fibre $I_{\iota_n(X)}$, where X is a topological manifold without boundary whose universal covering is contractible. Recall that if $f\colon W \longrightarrow Z$ and $g\colon Y \longrightarrow Z$ are maps of spaces, the pullback $W \times_Z Y$ is defined by

$$W \times_Z Y = \{\, (w, y) \in W \times Y \mid f(w) = g(y)\}.$$

Proposition 5.1 (see [18, Proposition 3.7]) *Let* X *be a connected, topological manifold without boundary such that* $\dim(X) \geq 3$ *and whose universal covering* $\widetilde{X}$ *is contractible, and let* $Q \in \widetilde{X}$. *Then the homotopy fibre* $I_{\iota_n(X)}$ *of the map* $\iota_n(X)\colon F_n(X) \longrightarrow \prod_1^n X$ *is weakly homotopy equivalent to each of the following six spaces:* $\widetilde{F_{n-1}(X \setminus \overline{Q})}$, $F_n(X) \times_{\prod_1^n X} \prod_1^n \widetilde{X}$, $\widetilde{F_n(X)}$, $F_n^{\pi_1(X)}(\widetilde{X})$, $F_{n-1}(X \setminus \overline{Q}) \times_{\prod_1^{n-1} X} \prod_1^{n-1} \widetilde{X}$ *and* $F_{n-1}^{\pi_1(X)}(\widetilde{X} \setminus \pi_1(X)Q)$.

In the remainder of this section, we assume that X is the form $\mathbb{S}^k/G$, where G is a Lie group that acts freely and tamely on $\mathbb{S}^k$, the aim being to generalise the results of Section 4. The following proposition may be proved by applying Theorem 5.1 to the map $I_{\iota''_{n-1}(\mathbb{S}^k)}$.

Proposition 5.2 (see [18, Corollary 4.3]) *Let* $G \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$ *be a free, tame action of a compact Lie group* G *on* $\mathbb{S}^k$, *and let* $Q \in \mathbb{S}^k$.

(a) *There is a homotopy equivalence between* $I_{\iota''_{n-1}(\mathbb{S}^k)}$ *and* $F_{n-1}^G(\mathbb{S}^k \setminus GQ) \times \Omega\bigl(\prod_1^{n-1} \mathbb{S}^k\bigr)$. *If the group* G *is finite and* $\dim(\mathbb{S}^k/G) \geq 3$, *then there is a homotopy equivalence between* $\widetilde{F_{n-1}(\mathbb{S}^k \setminus Q)}$ *and* $F_{n-1}^G(\mathbb{S}^k \setminus GQ)$.

(b) *If* $j \leq k - \dim(G) - 2$, *then* $\pi_j(F_{n-1}^G(\mathbb{S}^k \setminus GQ)) = 0$.

By analysing the long exact sequence in homotopy of the following homotopy fibration

$$I_{\iota_n(\mathbb{S}^k/G)} \xrightarrow{p_{\iota_n(\mathbb{S}^k/G)}} F_n(\mathbb{S}^k/G) \xhookrightarrow{\iota_n(\mathbb{S}^k/G)} \prod_1^n \mathbb{S}^k/G$$

in the case that the group G is finite and k is odd, we obtain short exact sequences similar to those of Corollary 4.1.

Proposition 5.3 (see [18, Proposition 4.6]) *Let $k \geq 3$ be odd, let $j \geq 2$, and let $G \times \mathbb{S}^k \longrightarrow \mathbb{S}^k$ be a free action of a finite group G on $\mathbb{S}^k$. Let $\Delta_j^n(\mathbb{S}^k/G)$ denote the diagonal subgroup of $\prod_1^n \pi_j(\mathbb{S}^k/G)$. Then*

(a) *the image of the homomorphism $(\iota_n(\mathbb{S}^k/G))_{\#j}\colon \pi_j(F_n(\mathbb{S}^k/G)) \longrightarrow \prod_1^n \pi_j(\mathbb{S}^k/G)$ is equal to $\Delta_j^n(\mathbb{S}^k/G)$;*

(b) *there are split short exact sequences of the form:*

$$0 \longrightarrow \Delta_j^n(\mathbb{S}^k/G) \longrightarrow \pi_j\Big(\prod_1^n \mathbb{S}^k/G\Big) \xrightarrow{\widehat{\partial}_j} \pi_{j-1}\Big(\prod_1^{n-1} \Omega(\mathbb{S}^k/G)\Big) \longrightarrow 0, \tag{5.4}$$

$$0 \longrightarrow \pi_j(F_{n-1}^G(\mathbb{S}^k \setminus GQ)) \longrightarrow \pi_j(F_n(\mathbb{S}^k/G)) \xrightarrow{(\iota_n(\mathbb{S}^k/G))_{\#j}} \Delta_j^n(\mathbb{S}^k/G) \longrightarrow 0. \tag{5.5}$$

In particular, if G is the trivial group, then there is a split short exact sequence:

$$0 \longrightarrow \pi_j(F_{n-1}(\mathbb{R}^k)) \longrightarrow \pi_j(F_n(\mathbb{S}^k)) \longrightarrow \Delta_j^n(\mathbb{S}^k) \longrightarrow 0.$$

6 Other Types of Configuration Spaces, and some Open Questions

Let Y be a topological space, let $A \subset Y$ be a subspace, and let $\iota\colon A \longrightarrow Y$ denote the inclusion map. The following questions/problems are valid in this general context:

(I) Compute the homotopy groups $\pi_i(A)$ of A.

(II) If $j \geq 0$, determine the induced homomorphisms $\iota_{\#j}\colon \pi_j(A) \longrightarrow \pi_j(Y)$.

(III) Describe the homotopy type of the homotopy fibre I_ι of the inclusion map ι.

(IV) Determine the long exact sequence in homotopy of the fibration $I_\iota \longrightarrow E_\iota \longrightarrow Y$ associated to the map ι.

For specific choices of A and Y, it is often more natural to study variations or weaker formulations of these questions. Up until now, we have considered the usual configuration spaces $F_n(X)$ and $D_n(X)$ (ordered and unordered), as well as the orbit configuration spaces $F_n^G(X)$ and $D_n^G(X)$ (ordered and unordered) with respect to a free action of a group G on a space X. In Section 6.1, we make some comments about the graph configuration spaces $F_n^\Gamma(X)$ defined in Section 1, which we will refer to as ordered graph configuration spaces. We discuss some possible quotients of $F_n^\Gamma(X)$ by subgroups of the symmetric group S_n that act freely, and that give rise to some kind of unordered graph configuration spaces. Finally, in Section 6.2, we propose some questions for the three types of configuration space discussed in this paper.

6.1 Graph configuration space

Let Γ be a graph whose vertices are labelled by $\{1, \cdots, n\}$, that has no loops, and that possesses at most one edge between any two vertices. The n-th graph configuration space $F_n^\Gamma(X)$ is defined in Equation (1.3) in Section 1. If Γ is the complete graph on the set $\{1, \cdots, n\}$, i.e., for all $1 \le i < j \le n$, there is an edge in Γ between the vertices i and j, then $F_n^\Gamma(X)$ is the usual ordered configuration space $F_n(X)$. If on the other hand, the set of edges of Γ is empty, then X_n^Γ is the Cartesian product $\prod\limits_1^n X$. In general, we have $F_n(X) \subset X_n^\Gamma \subset \prod\limits_1^n X$. Little is known about these spaces, and the study of the type of questions that we have in mind is at a very early stage.

In order to define a notion of unordered graph configuration space similar to that for the other types of configuration space, one needs to adjust the definition given for $D_n(X)$. Recall that the symmetric group S_n acts on $\prod\limits_1^n X$, and by restriction, we obtain a free action of S_n on $F_n(X)$. Unfortunately, not all elements of S_n induce a map of $F_n^\Gamma(X)$. Indeed, one may check that an element $\alpha \in S_n$ induces a map of $F_n^\Gamma(X)$ if and only if it induces an automorphism of the graph. Let $\mathrm{Aut}(\Gamma)$ denote the group of automorphisms of Γ, that we interpret as a subgroup of S_n. An unordered graph configuration space could be defined to be the quotient space $D_n^H(X) = F_n^\Gamma(X)/H$, where H is a subgroup of $\mathrm{Aut}(\Gamma)$ that acts freely on $F_n^\Gamma(X)$ and is maximal with respect to this property. In principle, with this definition, we may have more than one unordered graph configuration space because H is not unique in general, and such a phenomenon indeed occurs. Since the quotient map $F_n^\Gamma(X) \longrightarrow F_n^\Gamma(X)/H$ is a covering map, the spaces $F_n^\Gamma(X)$ and $F_n^\Gamma(X)/H$ have the same homotopy groups with the exception of the fundamental group, which coincide rarely.

6.2 Questions and final comments

Among the three types of configuration spaces described in Section 1 and Section 6.1, the usual configuration spaces have been studied in much greater detail than the other two types. More is known about orbit configuration spaces than graph configuration spaces. For example, at the current time, no analogue of Proposition 2.1 is known for the latter. However, not much is known about the homotopy fibre of the inclusion of the ordered orbit configuration spaces into the Cartesian product, or about the homotopy groups of the ordered and unordered orbit configuration spaces, the fundamental group in particular. To end this paper, we state a number of questions about the different types of configuration spaces that are open (to our knowledge).

(I) For the usual configuration space $F_n(X)$: If X is a manifold of dimension $\dim(X) \ge 3$, determine

(a) the homotopy type of the homotopy fibre of the inclusion $\iota_n \colon F_n(X) \hookrightarrow \prod\limits_1^n X$;

(b) the long exact sequence in homotopy of the fibration $I_{\iota_n} \longrightarrow E_{\iota_n} \longrightarrow \prod\limits_1^n X$ associated with the map ι_n.

(II) For orbit configuration spaces: If X is a manifold and G is a group that acts freely on X, determine

(a) the homotopy groups of $F_n^G(X)$, and in particular the fundamental group in the case where X a surface;

(b) the same questions as in (a) for the unordered orbit configuration space $F_n^G(X)/S_n$;

(c) the homotopy type of the homotopy fibre of the inclusion $\iota_n^G\colon\, F_n^G(X) \hookrightarrow \prod_1^n X$;

(d) the long exact sequence in homotopy of the fibration $I_{\iota_n^G} \longrightarrow E_{\iota_n^G} \longrightarrow \prod_1^n X$ associated with the map ι_n^G.

(III) For graph configuration spaces: If X is a manifold and Γ is a graph as in Section 6.1, determine

(a) the surfaces X and graphs Γ for which the space $F_n^\Gamma(X)$ is a $K(\pi,1)$;

(b) the homotopy groups of $F_n^\Gamma(X)$, and in particular the fundamental group in the case where X is a surface;

(c) the same questions as in (b) for the unordered graph configuration spaces $F_n^\Gamma(X)/H$, where H is as defined in Section 6.1. as defined above;

(d) the homotopy type of the homotopy fibre of the inclusion $\iota_n^\Gamma\colon\, F_n^\Gamma(X) \hookrightarrow \prod_1^n X$;

(e) the long exact sequence in homotopy of the fibration $I_{\iota_n^\Gamma} \longrightarrow E_{\iota_n^\Gamma} \longrightarrow \prod_1^n X$ associated with the map ι_n^Γ.

We conclude by mentioning that the following two problems constitute work in progress by the authors:

(i) the description of the homotopy type of $F_n^{\langle\tau\rangle}(\mathbb{S}^2)$, and

(ii) with respect to free $\mathbb{Z}_2$-actions, the computation of the pure orbit braid groups $\pi_1(F_n^G(X))$, where X is $\mathbb{S}^2$, the torus or the Klein bottle, and of the full orbit braid groups $\pi_1(F_n^G(X)/S_n)$, where X is the cylinder C, $\mathbb{S}^2$, the torus and the Klein bottle.

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