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An Example Using Improved Lefschetz Duality*

Pascal LAMBRECHTS[1] Jeremy LANE[2] Donald STANLEY[3]

Abstract A theorem of Lambrechts and Stanley is used to find the rational cohomology of the complement of an embedding $S^{4n-1} \to S^{2n} \times S^m$ as a module and demonstrate that it is not necessarily determined by the map induced on cohomology by the embedding, nor is it a trivial extension. This demonstrates that the theorem is an improvement on the classical Lefschetz duality.

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1 Introduction

Let $f\colon N \to M$ be an embedding of closed oriented manifolds of codimension k. Classical Lefschetz duality implies that there is a long exact sequence of H^*M modules

$$\cdots H^{*-k}N \xrightarrow{f^!} H^*M \to H^*(M \setminus N) \to H^{*-k+1}N \cdots, \tag{1.1}$$

where $f^!$ is the unique map preserving orientations. This gives us a short exact sequence $0 \to \operatorname{coker} f^! \to H^*M \setminus N \to \ker f^! \to 0$. Over the rationals this determines $H^*M \setminus N$ as a vector space, but knowing its $H^*(M)$ modules structure requires solving an extension problem.

One of the ways to think of the improved Lefschetz duality of [4] is that the sequence (1.1) can be lifted to a triangle in the derived category of C^*M (really $A_{pl}M$) dgmodules

$$C^{*-k}N \xrightarrow{\widetilde{f}^!} C^*M \to C^*(M \setminus N) \to C^{*-k+1}N, \tag{1.2}$$

whose associated long exact sequence on homology is the sequence (1.1). Here $\widetilde{f}^!$ is the unique map in the derived category of C^*M modules that preserves orientation classes. Since any map in a triangle determines the triangle, the map $\widetilde{f}^!$ determines $C^*(M \setminus N)$ as a C^*M dg-module and hence determines $H^*(M \setminus N)$ as an H^*M module.

The point of the short note is to describe two embeddings

$$f, g\colon S^{4n-1} \to S^{2n} \times S^m,$$

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[1]Pascal Lambrechts, Institut Mathématique, Université de Louvain, Louvain-le-Neuve, Belgium.
E-mail: pascal.lambrechts@uclouvain.be
[2]Department of Mathemtics, University of Toronto, Toronto, Canada.
E-mail: lanejere@math.toronto.edu
[3]Department of Mathematics and Statistics, University of Regina, Canada.
E-mail: stanley@math.uregina.ca
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whose homological shriek maps are the same

$$f^! = g^! \colon H^{*-m+2n-1}S^{4n-1} \to H^*(S^{2n} \times S^m),$$

but whose complements $S^{2n} \times S^m \setminus f(S^{4n-1})$ and $S^{2n} \times S^m \setminus g(S^{4n-1})$, have different $H^*(S^{2n} \times S^m)$ module structures. This gives an example of how the improved Lefschetz duality sequence (1.2) gives more information than classical Lefschetz duality. The easy computation also provides another example of how keeping chain level information a little longer can be helpful.

We begin in Section 2 by recalling the definition of a Poincaré embedding and theorems due to Klein and Wall which allow us to apply improved Lefschetz duality to maps $S^{4n-1} \to S^{2n} \times S^m$. In Section 3, we give the necessary definitions for algebraic models from rational homotopy theory, and in Section 4 we provide the details of our example.

2 Poincaré Embeddings and Smooth Embeddings

The main example of this paper is concerned with applying improved Lefschetz duality to algebraic models of continuous maps

$$S^{4n-1} \to S^{2n} \times S^m.$$

The theorem requires that the algebraic map is a model of a Poincaré embedding. Conveniently, every such map is a Poincaré embedding for sufficiently large m due to a result of Klein. For further details, the reader is referred to the survey [3].

Morally, an embedding of manifolds $N \to M$ is a decomposition of the codomain $M = T \cup_{\partial T} C$, where T and C are codimension 0 sub-manifolds with boundary $\partial T = T \cap C$, and we think of T as a tubular neighborhood of N, or we can think of N as a manifold with boundary that is a codimension 0 submanifold of M. The topological generalization of a closed manifold (manifold with boundary) is a Poincaré duality space (pair), which is a finite CW-complex (pair) that satisfies Poincaré duality. This motivates the following definition (the map i is required to be $(w-p-1)$-connected for transversality reasons).

Definition 2.1 *Let W and P be finite CW-complexes. If W is a connected Poincaré duality space of dimension w, and P has dimension p, then a Poincaré embedding of codimension $w-p$ is a commutative diagram of topological spaces*

$$\begin{array}{ccc} \partial T & \xrightarrow{i} & P \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{l} & W \end{array} \tag{2.1}$$

such that (2.1) is a homotopy pushout; $(P, \partial T)$ and $(C, \partial T)$ are Poincare duality pairs in dimension w; and the map i is $(w-p-1)$-connected. If f fits into such a commutative diagram then we say that f Poincaré embeds.

Theorem 2.1 (see [2]) *Let P be a finite CW-complex of dimension $\leq p$ and W a w-dimensional connected Poincaré duality space. If a map $f : P \to W$ is r-connected then f Poincaré embeds provided that $w \geq p + 3$ and*

$$w \geq 2p - r + 2.$$

Instead of using the notion of Poincaré embeddings, we may replace "Poincaré embedding" in the statement of Theorem 4.1 with "smooth embedding" and apply the following theorem.

Theorem 2.2 (see [5]) *If P and W are smooth manifolds with $p \leq w-3$, a map $f : P \to W$ Poincaré embeds, and $2w \geq 3(k+1)$, then f is homotopic to a smooth embedding.*

3 Algebraic Models in Rational Homotopy Theory

In order to discuss models of embeddings, we must introduce some notation from rational homotopy theory. Details can be found in [1].

Definition 3.1 *A differential graded algebra (DGA for short) R is a cochain complex (V, d) over $\mathbb{Q}$ with an associative multiplication, which is a morphism of cochain complexes $V \otimes V \longrightarrow V$ and an identity element $1 \in V^0$ such that*

$$d(ab) = (da)b + (-1)^{|a|}a(db).$$

We say that R is commutative, or a CDGA, if $ab = (-1)^{|a||b|}ba$ for all $a, b \in R$.

Definition 3.2 *Given a DGA R, a left R-DG module is a cochain complex M with a cochain morphism $\cdot : R \otimes M \longrightarrow M$ commuting with the differential in M such that $(xy) \cdot m = x \cdot (y \cdot m)$ and $1 \cdot m = m$.*

Example 3.1 If $\phi : M \to N$ and $\rho : R \to M$ are morphisms of DGA's, then we can consider ϕ as a morphism of left R-DG modules by letting $r \cdot m = \rho(r)m$ and $r \cdot n = \phi \circ \rho(r)n$ for $m \in M$ and $n \in N$. Similarly, we can consider the dual $M^* = \mathrm{Hom}(M, \mathbb{Q})$ of a DGA as right R-DG modules by letting $f \cdot r = f(r \cdot -)$ for $f \in M^*$. The differential in the dual is defined by

$$d(f) = -(-1)^{\deg f} fd.$$

Recall that if M is an R-DG module, then for an integer k, the differential on the suspension $s^k M$ is given by

$$d(s^k x) = (-1)^k s^k d(x)$$

and in general the symbol s^k commutes with other symbols following the Kozul sign convention as if it were an element or function of degree k.

Definition 3.3 *The cone of a morphism $\phi : M \longrightarrow N$ of left R-DG modules is defined as the left R-DG module $C(\phi) = N \oplus_\phi sM$ with differential $d(n, sm) = (dn + \phi m, -sdm)$ and R-DG module structure*

$$r \cdot (n, sm) = (r \cdot n, (-1)^{|r|} sr \cdot m).$$

4 Complements of Poincaré Embeddings $S^{4n-1} \to S^{2n} \times S^m$

In this section we use improved Lefschetz duality (see Theorem 4.1) to construct an example (Example 4.1) of two maps that have the same shriek map, but give different extensions in the long exact sequence (1.1).

Theorem 4.1 (Improved Lefschetz Duality, see [4]) *Consider a Poincaré embedding* (2.1) *with W connected and of dimension k. Suppose that a quasi-isomorphism of differential graded algebras $\rho : A \longrightarrow A_{PL}(W)$ has been given and let $\phi : R \longrightarrow Q$ be an A-DG module model of $f : P \to W$. Then there is an isomorphism*

$$H^*(C;\mathbb{Q}) \cong H^*(s^{-k}R^* \oplus_{s^{-k}\phi^*} s(s^{-k}Q^*))$$

of $H^*(W;\mathbb{Q})$-modules.

Theorem 4.2 *Fix $n \geq 1$ and $m \geq 2n+1$, $m \neq 4n-1$. Let $f, g\colon S^{4n-1} \to S^{2n} \times S^m$ be Poincaré embeddings such that f is homotopically trivial and g is rationally homotopy non-trivial. Then*

$$f^! = g^! \colon H^{*-m+2n-1}S^{4n-1} \to H^*(S^{2n} \times S^m)$$

as $H^*(S^{2n} \times S^m)$-module maps, but

$$H^*(S^{2n} \times S^m \setminus f(S^{4n-1})) \ncong H^*(S^{2n} \times S^m \setminus g(S^{4n-1}))$$

as $H^*(S^{2n} \times S^m)$-modules.

Proof For simplicity, we will write the dimension $k := 2n + m$ in the proof. Consider the Sullivan algebras $Q = (\Lambda(x), 0)$ and $R = (\Lambda(a, b, c), d)$ over $\mathbb{Q}$ with $|x| = 4n - 1$, $|a| = 2n$, $|c| = m$ and $db = a^2$. Then R is a model of $S^{2n} \times S^m$ and Q is a model of S^{4n-1}, so every map $S^{4n-1} \to S^{2n} \times S^m$ has a model of the form $R \to Q$. For degree reasons, a morphism $\phi_\alpha\colon R \to Q$ of CDGA's is uniquely determined by $\phi_\alpha(b) = \alpha x$ for $\alpha \in \mathbb{Q}$.

By improved Lefschetz duality, $H^*(s^{-k}R^* \oplus_{s^{-k}\phi^*} s(s^{-k}Q^*))$ is isomorphic to the cohomology of the complement of f as described in Definition 2.1 as an $H^*(S^{2n} \times S^m;\mathbb{Q})$-module. We denote the cone of the suspended dual map $s^{-k}\phi^*_\alpha : s^{-k}Q^* \longrightarrow s^{-k}R^*$ by

$$C_\alpha := s^{-k}R^* \oplus_{s^{-k}\phi^*_\alpha} s(s^{-k}Q^*), \quad k = 2n + m.$$

To compute C_α, let R and Q have bases

$$\{1, a, b, a^2, \cdots, c, ac, bc, a^2c, \cdots\} \quad \text{and} \quad \{1, x\}$$

respectively, and denote an element of the dual basis with an asterisk (for example, $a^*(a) = 1$). For any $\alpha \in \mathbb{Q}$, one can show that

$$H^i(C_\alpha) = \begin{cases} \mathbb{Q}, & \text{if } i = 0, 2n, m - 2n \text{ or } m, \\ 0, & \text{else.} \end{cases}$$

The module structure on $H^*(C_\alpha)$ is independent of α, except in degree $m-2n$, where the kernel of d depends on ϕ_α. Consider a homogeneous element

$$(\beta_1 s^{-k}(a^2)^*, \beta_2 s^{1-k}x^*)$$

in C_α of degree $m-2n$ with $\beta_i \in \mathbb{Q}$. Observe that

$$\begin{aligned} d(\beta_1 s^{-k}(a^2)^*, \beta_2 s^{1-k}x^*) &= (-\beta_1 s^{-k}b^* + \beta_2 s^{-k}\psi_\alpha^* x^*, \beta_2 s^{1-k}dx^*) \\ &= ((\alpha\beta_2 - \beta_1)s^{-k}b^*, 0), \end{aligned}$$

which is 0 if and only if $\beta_1 = \alpha\beta_2$. Hence, the cocycles in C_α^{m-2n} are of the form

$$(\alpha\beta s^{-k}(a^2)^*, \beta s^{1-k}x^*),$$

and are coboundaries if and only if $\beta = 0$. The R-DG module action on a representative of such a cohomology class is

$$a \cdot (\alpha\beta s^{-k}(a^2)^*, \beta s^{1-k}x^*) = (\alpha\beta s^{-k}a^*, 0).$$

Hence, in $H^*(C_\alpha)$

$$[a] \cdot [(\alpha\beta s^{-k}(a^2)^*, \beta s^{1-k}x^*)] = [(\alpha\beta s^{-k}a^*, 0)].$$

However, the induced map on H^* is trivial for every map $S^{4n-1} \to S^{2n} \times S^m$ for degree reasons, so it is clear that $f^! = g^!$. Furthermore, if $\alpha \neq 0$ then the following sequence of $H^*(A)$ modules does not split at $H^*(C_\alpha)$ (in other words, $H^*(C_\alpha)$ is not a trivial extension).

$$H^*(s^{-k}Q^*) \xrightarrow{H^*(s^{-k}\phi_\alpha^*)} H^*(s^{-k}R^*) \longrightarrow H^*(C_\alpha) \longrightarrow H^*(s^{1-k}Q^*) \longrightarrow \cdots .$$

Example 4.1 For all $n \geq 1$ and $m \geq 2n+1$, $m \neq 4n-1$, there are smooth embeddings

$$f, g \colon S^{4n-1} \to S^{2n} \times S^m,$$

such that

$$f^! = g^! \colon H^{*-m+2n-1}S^{4n-1} \to H^*(S^{2n} \times S^m)$$

as $H^*(S^{2n} \times S^m)$-module maps, but

$$H^*(S^{2n} \times S^m \setminus f(S^{4n-1})) \ncong H^*(S^{2n} \times S^m \setminus g(S^{4n-1}))$$

as $H^*(S^{2n} \times S^m)$-modules.

Proof Let

$$f' \colon S^{4n-1} \to S^{2n} \times S^m$$

be a trivial map and

$$g' \colon S^{4n-1} \to S^{2n} \to S^{2n} \times S^m$$

be the composition of the self-Whitehead product with the inclusion. By Theorem 2.2 there exist embeddings $f \simeq f'$ and $g \simeq g'$. Since g' and hence g are rationally nontrivial the example follows directly from Theorem 4.2.

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