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Virtual Braids, Virtual Temperley-Lieb Algebra and f-Polynomial*

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Abstract The authors study the properties of virtual Temperley-Lieb algebra and show how the f-polynomial of virtual knot can be derived from a representation of the virtual braid group into the virtual Temperley-Lieb algebra, which is an approach similar to Jones's original construction.

Keywords Virtual braids, Virtual Temperley-Lieb algebra, f-polynomial
2000 MR Subject Classification 57M25, 57M27

1 Introduction

In 1996, L. H. Kauffman introduced the notion of virtual knot, which is motivated by the study of knots in a thickened surface and abstract Gauss codes (see [10]). According to M. Goussarov, M. Polyak and O. Viro [2], two classical knot diagrams represent the same knot type if and only if they represent the same virtual knot type. Thus, the notion of a virtual knot is a generalization of a classical knot.

Let us recall some historical development of the knot theory since Jones's famous work [3–5]. A braid representation by the Temperley-Lieb (TL for short) algebra (see [15, 18]) leads to the Jones polynomial, while a braid representation through Hecke algebra derives the two variable HOMFLY polynomial which has Jones polynomial as a specialization.

L. H. Kauffman extends the bracket polynomial [7] to virtual knots and links by the usual formula for the state sum of the bracket in [11] and defined the f-polynomial which is an invariant of virtual links.

In the literature, the virtual TL algebra has been implicitly argued and proposed in [1, 12, 21]. In [20], Y. Zhang gave a virtual extension of TL algebra from a pure algebraic viewpoint, i.e., the mixed relations between TL algebra idempotents and virtual crossings are determined by a presumed requirement that virtual braids can be represented in the virtual TL algebra.

All of above information constitutes the motivation of this study. In the present paper, we give a geometric illustration of virtual TL algebra by diagrams algebra as in Kauffman's

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presentation (see [8]) and a representation of virtual braid group by the virtual TL algebra which leads to the f-polynomial.

We define a representation ρ_n from VB_n into virtual TL algebra $\mathcal{VT}_n$ by

$$\rho_n(\sigma_i) = A \cdot 1_n + A^{-1}E_i,$$
$$\rho_n(\sigma_i^{-1}) = A^{-1} \cdot 1_n + AE_i,$$
$$\rho_n(\zeta_i) = v_i.$$

Given a virtual braid $\beta = \sigma_{i_1}^{a_1}\zeta_{j_1}^{\epsilon_1}\sigma_{i_2}^{a_2}\zeta_{j_2}^{\epsilon_2}\cdots\sigma_{i_s}^{a_s}\zeta_{j_s}^{\epsilon_s} \in VB_n$, where $\epsilon_t = 0, 1,\ a_t \in \mathbb{Z},\ 1 \leq i_t, j_t \leq n-1,\ t = 1, 2, \cdots, s$. We plug it into ρ and get a product in $VT_n(\delta)$:

$$\rho_n(\beta) = (A + A^{-1}E_{i_1})^{a_1}v_{j_1}^{\epsilon_1}\cdots(A + A^{-1}E_{i_s})^{a_s}v_{j_s}^{\epsilon_s}.$$

We define $\langle\beta|t\rangle$ to be the coefficient of the U_t in the expanded expression of $\rho_n(\beta)$. Thus we have

$$\rho_n(\beta) = \sum_t \langle\beta|t\rangle U_t,$$

where t indexes all the terms in the sum, U_t is a product of some E_{i_t}'s and v_{j_t}'s.

We will discuss in Subsection 4.2 that $\rho_n(\beta)$ essentially resolves all the crossings of β. Furthermore, the U_t represent the resolved states of the braids. We define the bracket for the resolved states by

$$\langle U_t\rangle = \langle\overline{U_t}\rangle = \delta^{\|U_t\|}.$$

We can conclude that the bracket for the closed braid $\overline{\beta}$ is

$$\langle\overline{\beta}\rangle = \sum_t \langle\beta|t\rangle\delta^{\|U_t\|}.$$

But the bracket is not an invariant of virtual knot. To correct this, in a way similar to that in classical knot theory, we define the writhe number $w(\beta)$ of virtual braid β by the formula

$$w(\beta) = \sum_{t=1}^{s} a_t$$

for a virtual braid $\beta = \sigma_{i_1}^{a_1}\zeta_{j_1}^{\epsilon_1}\sigma_{i_2}^{a_2}\zeta_{j_2}^{\epsilon_2}\cdots\sigma_{i_s}^{a_s}\zeta_{j_s}^{\epsilon_s}$.

Then, we define the normalized bracket for a closed braid $\overline{\beta}$ as

$$\ll \overline{\beta} \gg = (-A^3)^{-w(\beta)}\sum_t \langle\beta|t\rangle\delta^{\|U_t\|}.$$

The main result of this research is as follows.

Theorem 1.1 *The normalized bracket $\ll K \gg$ is an invariant of virtual knots. That is to say, if $K \approx K'$, then $\ll K \gg = \ll K' \gg$.*

Finally, we give an example for calculation of the normalized bracket for virtual trefoil knot by using our representation into virtual TL algebra.

2 Virtual Braid

It is well known that classical braids have many different definitions: the algebraic definition from Artin presentation; the geometrical definition from diagrams on the plane, up to

Reidemeister moves; the topological definition from the fundamental group of the configuration space for a set of distinct points on the plane.

Virtual braid, a generalization of classical braid, was first mentioned by Kauffman in his first talk about virtual knot theory (see [10]). The first papers about virtual braids belong to S. Kamada [6] and V. Vershinin [19].

Firstly, let us recall some definition about virtual braids from [6].

Definition 2.1 *A virtual braid diagram on* n *strands is a union of* n *smooth curves in general position on the plane connecting points* $(i,1)$ *with points* $(a_i,0)$, *these curves are monotonic with respect to ordinate* (*here* $(a_1,\cdots,a_n)$ *is some permutation of the numbers* $(1,\cdots,n)$), *herewith some crossings are marked as virtual crossings, and at the other* (*classical*) *crossings the under/overcrossing structure is specified, i.e., it is indicated which branch forms an overcrossing and which one forms an undercrossing, see Figure* 1. *In this case it is said that the braid realizes the permutation* $\alpha: 1\to a_1,\cdots,n\to a_n$.

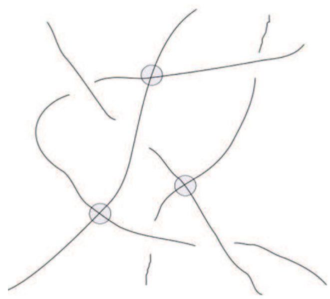

Figure 1 A virtual braid diagram.

Definition 2.2 *A virtual braid is an equivalence class of virtual braid diagrams by planar isotopies and all generalized Reidemeister moves except the first classical move and the first virtual move* (*see Figure* 5).

Like classical braids, virtual braids form a group (with respect to juxtaposition and rescaling the vertical coordinate). The unit element e of this group is the braid represented by all vertical parallel strands. The reverse element for a given braid is just its mirror image with respect to a horizontal line. The generators of the group of virtual braids with n strands are: $\sigma_1,\cdots,\sigma_{n-1}$ (classical crossings) and $\zeta_1,\cdots,\zeta_{n-1}$ (virtual crossings), see Figure 2.

It is evident that the equality $\zeta_i^2=e$ holds for each $i=1,\cdots,n-1$ (by virtue of the second virtual Reidemeister move). One can show that the following set of relations (see [7]) generates the group of virtual braids with n strands:

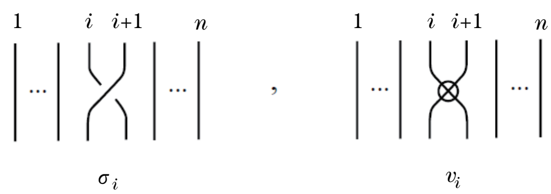

Figure 2 Generators of virtual braid group.

(1) relations of the Artin braid group:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j|>1,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1};$$

(2) relations of the permutation group:

$$\zeta_i\zeta_j = \zeta_j\zeta_i \quad \text{for } |i-j|>1,$$
$$\zeta_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\zeta_{i+1} \quad \text{and} \quad {\zeta_i}^2 = e;$$

(3) mixed relations:

$$\sigma_i\zeta_j = \zeta_j\sigma_i \quad \text{for } |i-j|>1,$$
$$\zeta_i\zeta_{i+1}\sigma_i = \sigma_{i+1}\zeta_i\zeta_{i+1}.$$

The group of virtual braids with n strands is denoted by VB_n. It is not difficult to show that VB_n is generated by one classical generator, for example σ_1 and all virtual generators. For more details concerning this reducing presentation see [13].

3 Generalized Alexander Theorem and Markov Theorem

It is a matter of observation that braid theory plays an important role in classical knot theory. The two theories are related by Alexander theorem and Markov theorem. Analogously, virtual braid theory is also of great importance to virtual knot theory, since we have the generalized Alexander theorem and Markov theorem for virtual knot theory as the following.

Theorem 3.1 ((Alexander Theorem, see [17]) *Every (oriented) virtual link can be represented by a virtual braid, whose closure is isotopic to the original link.*

In [6], S. Kamada proved an analogue of Markov's theorem for the case of virtual braids. Namely, he proved the following theorem.

Theorem 3.2 *Two virtual braids have equivalent closures as virtual links if and only if they are related to each other by a sequence of the following $VM1$, $VM2$ and $VM3$ moves:*

$VM1$: a conjugation in the virtual braid group,

$VM2$: a positive, negative, or virtual right stabilization, and their inverse operations,

$VM3$: a right or left virtual exchange move.

Here the moves $VM2, VM3$ see Figure 3 and Figure 4 respectively.

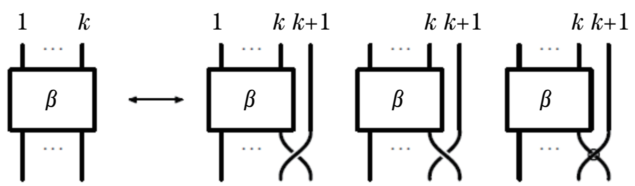

Figure 3 The right stabilization moves: Positive, negative, vitual types.

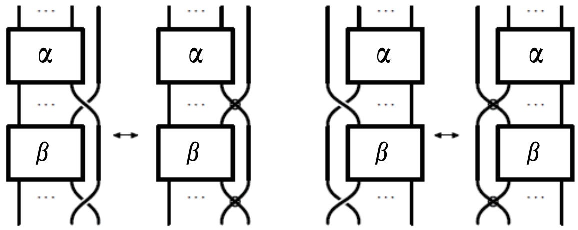

Figure 4 The right and left virtual exchange moves.

4 Representation of Virtual Braids and f-Polynomial

Theorems 3.1–3.2 give us a direct relationship between virtual links and virtual braids. With these theorems we can now use the virtual braid group to better understand virtual knots and links. In particular, we can now derive vitual knot invariants such as the f-polynomial using the virtual braid group. In order to derive the f-polynomial, we will construct a representation from the virtual braid group VB_n into the virtual TL algebra and show that this gives us a virtual version of the bracket polynomial for virtual braids, which we can extend to the original bracket polynomial for virtual knots. Theorem 3.2 will then guarantee that this is, in fact, an invariant for knots.

4.1 The virtual Temperley-Lieb algebra

Let Λ denote the ring $Z[A, A^{-1}]$ of Laurent polynomials in variable A. We denote by $\mathcal{VT}_n$ (see [16]) the free algebra over A generated by all virtual $2n$-tangle diagrams D modulo:

(i) $\times = A$)($+A^{-1}$ $\asymp$.

(ii) $D \cup \bigcirc = -(A^2 + A^{-2})D$.

(iii) Generalized Reidemeister move VRI, VRII, VRIIIa-b.

The figures in (i) stand for parts of larger ones that differ only as indicated by them. The left side of (ii) denotes the disjoint union of the diagram D with a jordan curve in the plane. For convenience, we denote $-(A^2 + A^{-2})$ by δ. For the generalized Reidemeister moves, see Figure 5.

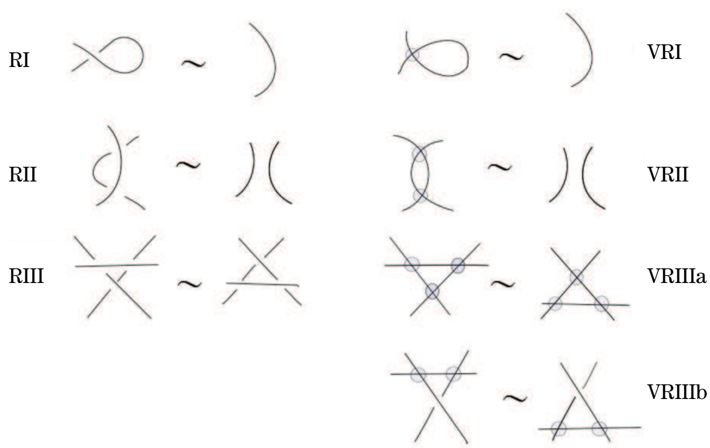

Figure 5 The generalized Reidemeister moves.

In order to elucidate the algebra $\mathcal{VT}_n$, we define the elementary tangles $E_1, E_2, \cdots, E_{n-1}$, virtual tangles $v_1, v_2, \cdots, v_{n-1} \in \mathcal{VT}_n$. That is, each U_i is a tangle with n input strands and n output strands. In E_i, the kth input is connected to the kth output for $k \neq i, i+1$, while the ith input is connected to the $(i+1)$st input and the ith output is connected to the $(i+1)$st output. In v_i, the kth input is connected to the kth output for $k \neq i, i+1$, while the ith input is connected to the $(i+1)$st output and the $(i+1)$st input is connected to the ith output, with the crossing encircled without over/under information. See Figure 6 for an illustration of $\mathcal{VT}_3$.

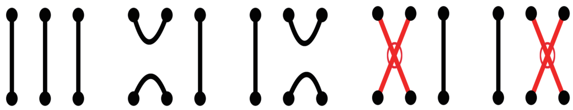

Figure 6 Generators for $\mathcal{VT}_3$.

Two tangles of the same number of strands are multiplied by attaching the output strands of the first tangle to the input strands of the second tangle. Two tangles are equivalent if they are virtual isotopic relative to their end points. This means that the virtual isotopy of a given tangle is restricted to a box; the input and output strands emanate from this box; the virtual isotopy leaves the intersections of the input and output strands with the boundary of the box fixed.

It is not hard to see that the equivalence class of a product of the E_i's, v_i's is determined entirely by the pattern of connections of inputs and outputs and that these products include all possible such connections that can be drawn in a planar box without self intersections. For example $E_1v_2 \in \mathcal{VT}_3$, see Figure 7.

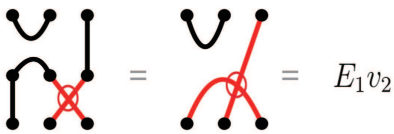

Figure 7 The product of E_1 and v_2.

The E_i's, v_i's satisfy the following basic relations (compare Figure 8):

$$
\begin{aligned}
&E_i^2 = \delta E_i,\\
&E_iE_{i\pm1}E_i = E_i,\\
&E_iE_j = E_jE_i, \quad |i-j| > 1,\\
&v_i^2 = 1, \quad v_iv_j = v_jv_i, \quad |i-j| > 1,\\
&v_iv_{i+1}v_i = v_{i+1}v_iv_{i+1},\\
&E_iv_j = v_jE_i, \quad |i-j| > 1,\\
&v_iE_{i+1}v_i = v_{i+1}E_iv_{i+1},\\
&E_iv_i = E_i = v_iE_i.
\end{aligned}
$$

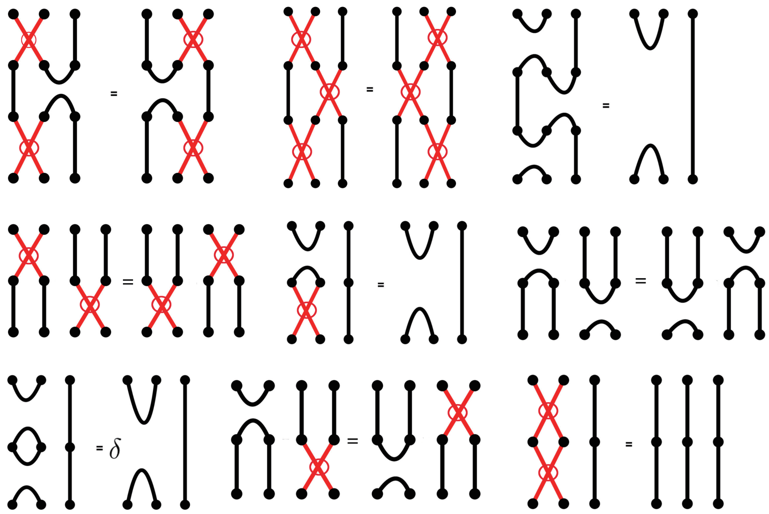

Figure 8 Some illustration diagrams for relations of VTL algebra.

We call $\mathcal{VT}_n$ the virtual Temperley-Lieb algebra (see [9, 16]). As a Λ-module, it is freely generated by $(2n-1)!!$ virtual $2n$-tangle diagrams corresponding to the various ways that the $2n$ points on the boundary of the 3-ball can be connected by n arcs without classical crossings.

To sum up, we can get the following proposition.

Proposition 4.1 (i) *The Temperley-Lieb algebra $\mathcal{VT}_n$ is a unital, associative R-linear algebra with generators: $1, E_1, \cdots, E_{n-1}$; $v_1, \cdots, v_{n-1}$ and relations:*

$$
\begin{aligned}
&E_i^2 = \delta E_i,\\
&E_iE_{i\pm 1}E_i = E_i,\\
&E_iE_j = E_jE_i, \quad |i-j|>1,\\
&v_i^2 = 1, \quad v_iv_j = v_jv_i, \quad |i-j|>1,\\
&v_iv_{i+1}v_i = v_{i+1}v_iv_{i+1},\\
&E_iv_j = v_jE_i, \quad |i-j|>1,\\
&v_iE_{i+1}v_i = v_{i+1}E_iv_{i+1},\\
&E_iv_i = E_i = v_iE_i.
\end{aligned}
$$

(ii) $\dim(\mathcal{VT}_n) = (2n-1)!!$.

For example, when $n = 3$, $\dim(\mathcal{VT}_n) = (2n-1)!! = 5!! = 15$. The basis of $\mathcal{VT}_3$ is $1_3, E_1, E_2, E_1E_2, E_2E_1, v_1, v_2, v_1E_2, v_2E_1, v_1E_2v_1, E_1v_2, E_2v_1, v_1v_2, v_2v_1, v_1v_2v_1$. See Figure 9.

Remark 4.1 Note that these relations used in defining the algebra are multiplicative ones. This suggests that we can obtain the algebra $VT_n(\delta)$ by presenting the multiplicative monoid $\mathscr{M}_n$, and then obtaining $VT_n(\delta)$ as the monoid algebra of formal R-linear combinations $\sum_i r_i \cdot a_i$ over $\mathscr{M}_n$, with the multiplication in $VT_n(\delta)$ defined as the bilinear extension of the monoid multiplication in $\mathscr{M}_n$. We can obtain $VT_n(\delta)$ as the monoid algebra over $\mathscr{M}_n$ subject to the identification $\tau = \delta \cdot 1$.

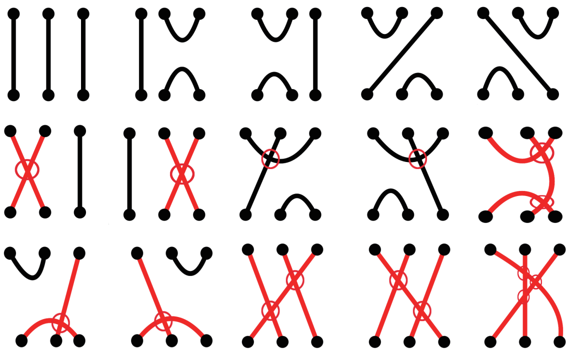

Figure 9 The basis of $\mathcal{VT}_3$.

Finally, we will be interested in the closure of virtual tangles. If we reduce our tangle to have no real crossings and no additional loops and take its closure (by a process similar to the closure of braids), then we will have a diagram of an unlink U up to virtual isotopy. We define $\|U\|$ to be the number of components in the unlink minus 1. For example, we can see that $\|\overline{E_1v_4}\| = 3 - 1 = 2$ from the following picture.

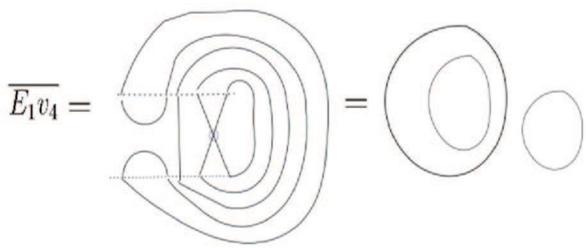

Figure 10 Closure of E_1v_4.

4.2 Representation of virtual braid into VTL algebra

In the mathematical field of representation theory, group representations describe abstract groups in terms of linear transformations of vector spaces; Formally, a linear representation of a group G is a homomorphism $\phi : G \to \mathrm{GL}_n(\mathbb{R})$, where $\mathrm{GL}_n(\mathbb{R})$ is the group of invertible $n \times n$ matrices over the real numbers. The study of group representation is a very big subject which we will not go into deeply here. More generally one can consider a representation of a group to be a homomorphism of the group into some other algebraic object such as symmetry group, a ring, a module or an algebra.

Here we give our representation of the virtual braid group VB_n in the virtual Temperley-Lieb algebra $\mathcal{VT}_n$. We define the following homomorphism ρ_n:

$$\rho_n : VB_n \to \mathcal{VT}_n$$

by the formulas

$$\rho_n(\sigma_i) = A \cdot 1_n + A^{-1}E_i,$$
$$\rho_n(\sigma_i^{-1}) = A^{-1} \cdot 1_n + AE_i,$$

$$\rho_n(\zeta_i) = v_i.$$

Note that we will generally omit the 1_n.

Proposition 4.2 *$\rho_n : VB_n \to \mathcal{VT}_n$ is a representation of the virtual braid group.*

Proof It can be checked directly.

The best way to understand ρ is to think of it as a function that resolves the crossings of the braid. Each σ_i in the virtual braid group represents a crossing of the strands in a braid. When we take ρ of it, we get the sum of the two possible resolutions of the crossing with the coefficients of A or A^{-1} depending on which type of crossing we are resolving. Understood this way, the representation ρ is clearly suggestive of the bracket polynomial for virtual knots (see [11]).

4.3 The normalized bracket polynomial

Recall that the bracket polynomial for virtual knots was defined as a sum over all states of a knot with the crossings resolved, for more details see [11]. Obviously, the bracket polynomial is closely related to the f-Polynomial. We will now see that we can derive the bracket polynomial from our representation ρ_n. Given a virtual braid $\beta = \sigma_{i_1}^{a_1}\zeta_{j_1}^{\epsilon_1}\sigma_{i_2}^{a_2}\zeta_{j_2}^{\epsilon_2}\cdots\sigma_{i_s}^{a_s}\zeta_{j_s}^{\epsilon_s} \in VB_n$, where $\epsilon_t = 0, 1,\ a_t \in \mathbb{Z},\ 1 \le i_t, j_t \le n-1,\ t = 1, 2, \cdots, s$, we plug it into ρ_n and get a product in $VT_n(\delta)$:

$$\rho_n(\beta) = (A + A^{-1}E_{i_1})^{a_1} v_{j_1}^{\epsilon_1} \cdots (A + A^{-1}E_{i_s})^{a_s} v_{j_s}^{\epsilon_s}.$$

We define $\langle\beta|t\rangle$ to be the coefficient of the U_t in the expanded expression of $\rho_n(\beta)$. Thus we have

$$\rho_n(\beta) = \sum_t \langle\beta|t\rangle U_t,$$

where t indexes all the terms in the sum, U_t is a product of some E_{i_t}'s and v_{j_t}'s. As we discussed above, $\rho_n(\beta)$ essentially resolves all the crossings of β. Furthermore, the U_t represent the resolved states of the braids. We define the bracket for the resolved states by

$$\langle U_t\rangle = \langle\overline{U_t}\rangle = \delta^{\|U_t\|}.$$

Now we can conclude that the bracket for the closed braid $\overline{\beta}$ is

$$\langle\overline{\beta}\rangle = \sum_t \langle\beta|t\rangle \delta^{\|U_t\|}.$$

We will see a few calculations of these techniques in Section 5. However, we must first fill in some gaps in our construction.

It has been shown that how we can use a representation of the virtual braid group to construct the bracket polynomial for virtual knots. However our construction is not quite complete. Recall that in order for our bracket polynomial to be an invariant for virtual knots, Kamada-Markovs theorem tells us that it must be invariant under the three moves VM1, VM2, VM3. It is easily checked that the above bracket polynomial fails to be invariant under these Markov moves. To correct this we introduce a new factor as in classical knot theory. We define the writhe of a braid, $w(\beta)$ to be the sum of exponents of the generators in the braid:

$$w(\beta) = \sum_{t=1}^{s} a_t$$

for a virtual braid $\beta = \sigma_{i_1}^{a_1}\zeta_{j_1}^{\epsilon_1}\sigma_{i_2}^{a_2}\zeta_{j_2}^{\epsilon_2}\cdots\sigma_{i_s}^{a_s}\zeta_{j_s}^{\epsilon_s}$.

And finally, we define the normalized bracket for a closed braid $\overline{\beta}$ as

$$\ll \overline{\beta} \gg = (-A^3)^{-w(\beta)}\sum_t \langle\beta|t\rangle\delta^{\|U_t\|}.$$

For convenience, we denote $\ll \overline{\beta} \gg$ by $\ll \beta \gg$.

Theorem 4.1 *The normalized bracket $\ll K \gg$ is an invariant of virtual knots. That is to say, if $K \approx K'$, then $\ll K \gg = \ll K' \gg$.*

Proof Suppose, by Alexander's theorem, that $K \approx \overline{\beta}$ and $K' \approx \overline{\beta'}$ for $\beta, \beta' \in VB_n$. Since K and K' are equivalent, Kamada-Markov's theorem tells us that β and β' are equivalent and thus β' can be obtained from β by a sequence of Markov moves of type VM1, VM2, and VM3 (see Theorem 3.2). So it suffices to show that the bracket is invariant under these three moves.

(1) Conjugation move. We must show that $\ll \beta \gg = \ll \alpha\beta\alpha^{-1} \gg$.

$$\begin{aligned}
\rho(\sigma_i\beta\sigma_i^{-1}) &= (A + A^{-1}E_i)\rho(\beta)(A^{-1} + AE_i)\\
&= (A\rho(\beta) + A^{-1}E_i\rho(\beta))(A^{-1} + AE_i)\\
&= \rho(\beta) + A^2\rho(\beta)E_i + A^{-2}E_i\rho(\beta) + E_i\rho(\beta)E_i,\\
\langle\sigma_i\beta\sigma_i^{-1}\rangle &= \langle\beta\rangle + A^2\langle\rho(\beta)E_i\rangle + A^{-2}\langle E_i\rho(\beta)\rangle + \langle E_i\rho(\beta)E_i\rangle\\
&= \langle\beta\rangle + (A^2 + A^{-2})\langle E_i\rho(\beta)\rangle + \delta\langle E_i\rho(\beta)\rangle\\
&= \langle\beta\rangle.
\end{aligned}$$

Similarly, we can get $\langle\zeta_i\beta\zeta_i\rangle = \langle\beta\rangle$.

(2) Right (positive, negative, virtual) stabilization.

Claim 1 $\ll \beta\sigma_n \gg = \ll \beta \gg$

$$\begin{aligned}
\ll \beta\sigma_n \gg &= (-A^3)^{-(w(\beta)+1)} \ll \rho(\beta)\rho(\sigma_n) \gg\\
&= (-A^3)^{-(w(\beta)+1)}\Big(\sum_t \langle\beta|t\rangle\delta^{\|U_t\|}\Big)(A\cdot\delta + A^{-1})\\
&= \ll \beta \gg (-A^{-3}(A\cdot\delta + A^{-1}))\\
&= \ll \beta \gg (-A^{-3}(A(-A^2 - A^{-2}) + A^{-1}))\\
&= \ll \beta \gg (-A^{-3})(-A^3 - A^{-1} + A^{-1})\\
&= \ll \beta \gg (-A^{-3})(-A^3)\\
&= \ll \beta \gg .
\end{aligned}$$

Claim 2 $\ll \beta\sigma_n^{-1} \gg = \ll \beta \gg$

$$\begin{aligned}
\ll \beta\sigma_n^{-1} \gg &= (-A^3)^{-(w(\beta)-1)}\langle\rho(\beta)\rho(\sigma_n^{-1})\rangle\\
&= (-A^3)^{-(w(\beta)-1)}\Big(\sum_t \langle\beta|t\rangle\delta^{\|U_t\|}\Big)(A^{-1}\cdot\delta + A)\\
&= \ll \beta \gg (-A^3(A^{-1}(-A^2 - A^{-2}) + A))\\
&= \ll \beta \gg (-A^3)(-A^{-3} - A + A)\\
&= \ll \beta \gg (-A^3)(-A^{-3})\\
&= \ll \beta \gg .
\end{aligned}$$

Claim 3 $\ll \beta\zeta_n \gg = \ll \beta \gg$

$$\begin{aligned} \ll \beta\zeta_n \gg &= (-A^3)^{-w(\beta)}\Big(\sum_t \langle\beta|t\rangle\delta^{\|U_t\|}\Big)\langle v_n\rangle \\ &= (-A^3)^{-w(\beta)}\Big(\sum_t \langle\beta|t\rangle\delta^{\|U_t\|}\Big) \\ &= \ll \beta \gg, \end{aligned}$$

where $\langle v_n\rangle = 1$ because Reidemeister VI-move.

(3) Right virtual exchange move.

Claim $\ll \alpha\sigma_n\beta\sigma_n^{-1} \gg = \ll \alpha\zeta_n\beta\zeta_n \gg$ $w(\alpha\sigma_n\beta\sigma_n^{-1}) = w(\alpha) + w(\beta) = w(\alpha\zeta_n\beta\zeta_n)$

$$\begin{aligned} &\ll \alpha\sigma_n\beta\sigma_n^{-1} \gg \\ &= (-A^3)^{-(w(\alpha)+w(\beta))}\Big(\sum_t \langle\alpha|t\rangle\delta^{\|U_t\|}\Big)(A\cdot\delta + A^{-1})\Big(\sum_t \langle\beta|t\rangle\delta^{\|V_t\|}\Big)(A^{-1}\cdot\delta + A) \\ &= (-A^3)^{-(w(\alpha)+w(\beta))}\Big(\sum_t \langle\alpha|t\rangle\delta^{\|U_t\|}\Big)\Big(\sum_t \langle\beta|t\rangle\delta^{\|V_t\|}\Big)(A\cdot\delta + A^{-1})(A^{-1}\cdot\delta + A) \\ &= (-A^3)^{-(w(\alpha)+w(\beta))}\Big(\sum_t \langle\alpha|t\rangle\delta^{\|U_t\|}\Big)\Big(\sum_t \langle\beta|t\rangle\delta^{\|V_t\|}\Big)(\delta^2 + (A^2 + A^{-2})\delta + 1) \\ &= (-A^3)^{-(w(\alpha)+w(\beta))}\Big(\sum_t \langle\alpha|t\rangle\delta^{\|U_t\|}\Big)\Big(\sum_t \langle\beta|t\rangle\delta^{\|V_t\|}\Big), \end{aligned}$$

$$\begin{aligned} &\ll \alpha\zeta_n\beta\zeta_n \gg \\ &= (-A^3)^{-(w(\alpha)+w(\beta))}\Big(\sum_t \langle\alpha|t\rangle\delta^{\|U_t\|}\Big)\langle\tau_n\rangle\Big(\sum_t \langle\beta|t\rangle\delta^{\|V_t\|}\Big)\langle\tau_n\rangle \\ &= (-A^3)^{-(w(\alpha)+w(\beta))}\Big(\sum_t \langle\alpha|t\rangle\delta^{\|U_t\|}\Big)\Big(\sum_t \langle\beta|t\rangle\delta^{\|V_t\|}\Big). \end{aligned}$$

Hence, $\ll \alpha\sigma_n\beta\sigma_n^{-1} \gg = \ll \alpha\zeta_n\beta\zeta_n \gg$.

(4) Left virtual exchange move.

We can do it in a way similar to the case (3).

5 Example

The virtual trefoil knot K_3 is a closure of the virtual braid $\beta = \zeta_1\sigma_1^2 \in VB_2$ (see Figure 11).

Figure 11 Virtual trefoil knot.

$$\begin{aligned} \rho(\sigma_1^2\zeta_1) &= (A + A^{-1}E_1)^2 v_1 \\ &= (A^2 + 2E_1 + A^{-2}E_1^2)v_1 \\ &= A^2 v_1 + 2E_1 + A^{-2}E_1^2. \end{aligned}$$

By the definition of normalized bracket, we have

$$\begin{aligned}\ll \sigma_1^2 v_1 \gg &= (-A^3)^{-2}[A^2\langle v_1\rangle + 2\langle E_1\rangle + A^{-2}\langle E_1^2\rangle]\\ &= A^{-6}[A^2 + 2 + A^{-2}\delta]\\ &= A^{-6}[A^2 + 2 + A^{-2}(-A^2 - A^{-2})]\\ &= A^{-6}[A^2 + 1 - A^{-4}]\\ &= A^{-4} + A^{-6} - A^{-10},\end{aligned}$$

which is exactly the f-polynomial of the virtual trefoil knot which is defined in Kauffman's famous paper [11].

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