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Topology of Moment-Angle Manifolds Arising from Flag Nestohedra*

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Abstract The author constructs a family of manifolds, one for each $n \geq 2$, having a nontrivial Massey n-product in their cohomology for any given n. These manifolds turn out to be smooth closed 2-connected manifolds with a compact torus $\mathbb{T}^m$-action called moment-angle manifolds $\mathcal{Z}_P$, whose orbit spaces are simple n-dimensional polytopes P obtained from an n-cube by a sequence of truncations of faces of codimension 2 only (2-truncated cubes). Moreover, the polytopes P are flag nestohedra but not graph-associahedra. The author also describes the numbers $\beta^{-i,2(i+1)}(Q)$ for an associahedron Q in terms of its graph structure and relates it to the structure of the loop homology (Pontryagin algebra) $H_*(\Omega\mathcal{Z}_Q)$, and then studies higher Massey products in $H^*(\mathcal{Z}_Q)$ for a graph-associahedron Q.

Keywords Moment-angle manifold, Flag nestohedra, Stanley-Reisner ring, Massey products, Graph-associahedron
2000 MR Subject Classification 13F55, 55S30, 52B11

1 Introduction

The main aim of this work is to show that one of the key objects of study in toric topology—the moment-angle manifold $\mathcal{Z}_P$ of a simple convex n-dimensional polytope P—gives us an example of a smooth closed 2-connected manifold with a compact torus action such that its rational cohomology ring may contain a nontrivial higher Massey product of order n. These polytopes P are 2-truncated cubes and, moreover, flag nestohedra (see [22–23]). The class of 2-truncated polytopes was studied in toric topology by Buchstaber and Volodin, who proved that flag nestohedra can be realized as 2-truncated cubes and that Gal conjecture on γ-vectors of simple polytopes holds for 2-truncated cubes and, therefore, for all flag nestohedra (see [7]). We generalize in the polytopal sphere case the result of Baskakov [2] who constructed a family of triangulated spheres K whose moment-angle complexes $\mathcal{Z}_K$ have nontrivial triple Massey products of 3-dimensional classes in $H^*(\mathcal{Z}_K)$. In the lowest dimension Baskakov's construction gives a 2-sphere with 8 vertices K—the only K with a nontrivial triple Massey product in $H^*(\mathcal{Z}_K)$ among all the fourteen 2-spheres on 8 vertices. Denham and Suciu [9] generalized the result of Baskakov by proving a combinatorial criterion for K to give a $\mathcal{Z}_K$ with a nontrivial triple Massey product of 3-dimensional classes in $H^*(\mathcal{Z}_K)$.

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Denote by K a simplicial complex of dimension $n-1$ on the vertex set $[m]=\{1,\cdots,m\}$ and by $\Bbbk$ the base field or the ring of integers. Let $\Bbbk[v_1,\cdots,v_m]$ be a graded polynomial algebra on m variables, $\deg(v_i)=2$. The Stanley-Reisner ring (or the face ring) of K over $\Bbbk$ is the quotient ring

$$\Bbbk[K]=\Bbbk[v_1,\cdots,v_m]/\mathcal{I}_K,$$

where $\mathcal{I}_K$ is the ideal generated by square free monomials $v_{i_1}\cdots v_{i_k}$ such that $\{i_1,\cdots,i_k\}$ is not a simplex in K. The monomial ideal $\mathcal{I}_K$ is called the Stanley-Reisner ideal of K. Then $\Bbbk[K]$ has a structure of a $\Bbbk$-algebra and a module over $\Bbbk[v_1,\cdots,v_m]$ via the quotient projection.

In what follows we denote by P a simple n-dimensional convex polytope with m facets (i.e., faces of codimension 1) $F_1,\cdots,F_m$. Such a polytope P can be defined as a bounded intersection of m halfspaces:

$$P=\big\{\boldsymbol{x}\in\mathbb{R}^n\colon \langle\boldsymbol{a}_i,\boldsymbol{x}\rangle+b_i\geqslant 0\quad\text{for } i=1,\cdots,m\big\},\tag{1.1}$$

where $\boldsymbol{a}_i\in\mathbb{R}^n$, $b_i\in\mathbb{R}$. We assume that the hyperplanes defined by the equations $\langle\boldsymbol{a}_i,\boldsymbol{x}\rangle+b_i=0$ are in general position, that is, at most n of them meet at a single point. We also assume that there are no redundant inequalities in (1.1), that is, no inequality can be removed from (1.1) without changing P. Then the facets of P are given by

$$F_i=\big\{\boldsymbol{x}\in P\colon \langle\boldsymbol{a}_i,\boldsymbol{x}\rangle+b_i=0\big\}\quad\text{for } i=1,\cdots,m.$$

Let A_P be the $m\times n$ matrix of row vectors $\boldsymbol{a}_i$, and denote by $\boldsymbol{b}_P$ the column vector of scalars $b_i\in\mathbb{R}$. Then we can rewrite (1.1) as

$$P=\{\boldsymbol{x}\in\mathbb{R}^n\colon A_P\boldsymbol{x}+\boldsymbol{b}_P\geqslant\boldsymbol{0}\}.$$

Consider the affine map

$$i_P\colon\mathbb{R}^n\to\mathbb{R}^m,\quad i_P(\boldsymbol{x})=A_P\boldsymbol{x}+\boldsymbol{b}_P$$

which embeds P into

$$\mathbb{R}^m_{\geqslant}=\{\boldsymbol{y}\in\mathbb{R}^m\colon y_i\geqslant 0\quad\text{for } i=1,\cdots,m\}.$$

Definition 1.1 *We define the space $\mathcal{Z}_P$ as a pullback in the following commutative diagram (see [5, Lemma 3.1.6, Construction 3.1.8]):*

$$\begin{array}{ccc}\mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m\\ \big\downarrow & & \big\downarrow{\scriptstyle\mu}\\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant}\end{array}$$

where $\mu(z_1,\cdots,z_m)=(|z_1|^2,\cdots,|z_m|^2)$. The latter map may be thought of as the quotient map for the coordinatewise action of the standard torus

$$\mathbb{T}^m=\{\boldsymbol{z}\in\mathbb{C}^m\colon |z_i|=1\quad\textit{for } i=1,\cdots,m\}$$

on $\mathbb{C}^m$. Therefore, $\mathbb{T}^m$ acts on $\mathcal{Z}_P$ with a quotient space P, i_Z is a $\mathbb{T}^m$-equivariant embedding with a trivial normal bundle, and $\mathcal{Z}_P$ is embedded into $\mathbb{C}^m$ as a nondegenerate intersection of Hermitian quadrics. One can easily see that $\mathcal{Z}_P$ has a structure of a smooth closed manifold of dimension $m+n$, called the moment-angle manifold of P.

Suppose $(\mathbf{X},\mathbf{A}) = \{(X_i, A_i)\}_{i=1}^m$ is a set of topological pairs. The following construction appeared firstly in the work of Buchstaber and Panov [5] and then was studied intensively and generalized in the works of Bahri, Bendersky, Cohen, Gitler [1], Grbić and Theriault [13], Iriye and Kishimoto [16], and others.

Definition 1.2 *A polyhedral product is a topological space:*

$$(\mathbf{X},\mathbf{A})^K = \bigcup_{I\in K} (\mathbf{X},\mathbf{A})^I,$$

where $(\mathbf{X},\mathbf{A})^I = \prod_{i=1}^m Y_i$ *for* $Y_i = X_i$, *if* $i \in I$, *and* $Y_i = A_i$, *if* $i \notin I$. *Particular cases of a polyhedral product* $(\mathbf{X},\mathbf{A})^K$ *include moment-angle-complexes* $\mathcal{Z}_K = (\mathbb{D}^2, \mathbb{S}^1)^K$ *and real moment-angle complexes* $\mathcal{R}_K = (\mathbb{D}^1, \mathbb{S}^0)^K$.

Denote by K_P the nerve complex of P, i.e., the boundary ∂P^* of the dual simplicial polytope. It can be viewed as an $(n-1)$-dimensional simplicial complex on the set $[m]$, whose simplices are subsets $\{i_1, \cdots, i_k\}$ such that $F_{i_1} \cap \cdots \cap F_{i_k} \neq \emptyset$ in P. By [6, Theorem 6.2.4], $\mathcal{Z}_P$ is $\mathbb{T}^m$-equivariantly homeomorphic to the moment-angle-complex $\mathcal{Z}_{K_P}$.

The Tor-groups of K acquire a topological interpretation by means of the following result due to Buchstaber and Panov.

Theorem 1.1 (see [6, Theorem 4.5.4] or [21, Theorem 4.7]) *The cohomology algebra of the moment-angle-complex* $\mathcal{Z}_K$ *is given by the isomorphisms*

$$\begin{aligned} H^{*,*}(\mathcal{Z}_K;\Bbbk) &\cong \operatorname{Tor}^{*,*}_{\Bbbk[v_1,\cdots,v_m]}(\Bbbk[K],\Bbbk) \\ &\cong H[\Lambda[u_1,\cdots,u_m]\otimes \Bbbk[K], d] \\ &\cong \bigoplus_{I\subset[m]} \widetilde{H}^*(K_I;\Bbbk), \end{aligned}$$

where bigrading and differential in the cohomology of the differential bigraded algebra are defined by

$$\operatorname{bideg} u_i = (-1,2), \quad \operatorname{bideg} v_i = (0,2), \quad \mathrm{d}u_i = v_i,\ \mathrm{d}v_i = 0.$$

In the third row, $\widetilde{H}^*(K_I)$ *denotes the reduced simplicial cohomology of the induced subcomplex* K_I *of* K *(the restriction of* K *to* $I \subset [m]$*). The last isomorphism is the sum of isomorphisms*

$$H^p(\mathcal{Z}_K) \cong \sum_{I\subset[m]} \widetilde{H}^{p-|I|-1}(K_I),$$

and the ring structure is given by the maps

$$\widetilde{H}^{p-|I|-1}(K_I)\otimes \widetilde{H}^{q-|J|-1}(K_J) \to \widetilde{H}^{p+q-|I|-|J|-1}(K_{I\cup J}), \tag{1.2}$$

which are induced by the canonical simplicial maps $K_{I\cup J} \hookrightarrow K_I * K_J$ *(join of simplicial complexes) for* $I \cap J = \emptyset$ *and zero otherwise.*

Additively the following theorem of Hochster holds.

Theorem 1.2 (see [15]) *For any simplicial complex* K *on* m *vertices, we have*

$$\operatorname{Tor}^{-i,2j}_{\Bbbk[v_1,\cdots,v_m]}(\Bbbk[K],\Bbbk) \cong \bigoplus_{\substack{J\subset[m]\\ |J|=j}} \widetilde{H}^{j-i-1}(K_J;\Bbbk).$$

The ranks of the bigraded components of the Tor-algebra

$$\beta^{-i,2j}(\Bbbk[K]) = \mathrm{rk}_{\Bbbk}\,\mathrm{Tor}^{-i,2j}_{\Bbbk[v_1,\cdots,v_m]}(\Bbbk[K],\Bbbk)$$

are called the bigraded Betti numbers of $\Bbbk[K]$ or K, when $\Bbbk$ is fixed. In what follows we need a particular case of the Hochster result for $j = i+1$. One has

$$\beta^{-i,2(i+1)}(P) = \sum_{\substack{J\subset[m]\\|J|=i+1}} (\mathrm{cc}(P_J) - 1),$$

where $P_J = \bigcup\limits_{j\in J} F_j$ and $\mathrm{cc}(P_J)$ equals the number of connected components of P_J.

Due to [6, Construction 3.2.8, Theorem 3.2.9], the Tor-algebra of K acqures a multigrading and the multigraded components can be calculated in terms of induced subcomplexes.

Theorem 1.3 *For any simplicial complex K on m vertices, we have*

$$\mathrm{Tor}^{-i,2J}_{\Bbbk[v_1,\cdots,v_m]}(\Bbbk[K],\Bbbk) \cong \widetilde{H}^{|J|-i-1}(K_J;\Bbbk),$$

where $J \subset [m]$ and $\mathrm{Tor}^{-i,2\mathbf{a}}_{\Bbbk[v_1,\cdots,v_m]}(\Bbbk[K],\Bbbk) = 0$, if $\mathbf{a}$ is not a $(0,1)$-vector.

Moreover, if we denote by $R(K) = \Lambda[u_1,\cdots,u_m] \otimes \Bbbk[K]/(v_i^2 = u_iv_i = 0, 1 \le i \le m)$ a graded algebra with the differential d as in Theorem 1.1, then $R(K)$ also acquires multigrading and the following isomorphism holds:

$$\mathrm{Tor}^{-i,2\mathbf{a}}_{\Bbbk[v_1,\cdots,v_m]}(\Bbbk[K],\Bbbk) \cong H^{-i,2\mathbf{a}}[R(K),d]$$

for any simplicial complex K.

2 Nestohedra and Graph-Associahedra

We begin with a definition of a family of simple polytopes called nestohedra and state the result of Buchstaber and Volodin on geometric realization of flag nestohedra.

Definition 2.1 *Let $[n+1] = \{1,2,\cdots,n+1\}$, $n \ge 2$. A building set on $[n+1]$ is a family of nonempty subsets $B = \{S \subseteq [n+1]\}$, such that*

(1) $\{i\} \in B$ for all $1 \le i \le n+1$,

(2) if $S_1 \cap S_2 \ne \emptyset$, then $S_1 \cup S_2 \in B$.

A building set is called connected if $[n+1] \in B$.

Then a nestohedron is a simple convex n-dimensional polytope $P_B = \sum\limits_{S\in B} \Delta_S$, where in the Minkowski sum, one has

$$\Delta_S = \mathrm{conv}\{e_j \mid j \in S\} \subset \mathbb{R}^{n+1}.$$

Note that facets of P_B are in $1-1$ correspondence with proper elements S in B (see [10] and [6, Proposition 1.5.11]).

Example 2.1 If P is a combinatorial n-simplex, then the subset of $2^{[n+1]}$ consisting of all the singletons $\{i\}, 1 \le i \le n+1$ and the whole set $[n+1]$ gives a connected building set B, such that $P = P_B$ for any $n \ge 2$.

If P is a combinatorial n-cube, then the following set B consisting of

$$\{1\}, \cdots, \{n+1\}, \{1,2\}, \{1,2,3\}, \cdots, [n+1]$$

will be a connected building set for P for any $n \geq 2$.

Any n-dimensional nestohedron P_B on a connected building set B can be obtained from an n-simplex by a sequence of its face truncations. In order to give the precise statement, suppose $B_0 \subset B_1$ being building sets on $[n+1]$, and $S \in B_1$. Then define a decomposition of S into elements of B_0 as $S = S_1 \sqcup \cdots \sqcup S_k$, where S_j are pairwise nonintersecting elements of B_0 and k is minimal among such disjoint representations of S. One can see easily that this decomposition exists and is unique.

Theorem 2.1 (see [6, Lemma 1.5.17, Theorem 1.5.18]) *Every nestohedron P_B corresponding to a connected building set B can be obtained from a simplex by a sequence of face truncations.*

More precisely, let $B_0 \subset B_1$ be connected building sets on $[n+1]$. Then P_{B_1} is combinatorially equivalent to the polytope obtained from P_{B_0} by a sequence of truncations at the faces $G_i = \bigcap_{j=1}^{k_i} F_{S^i_j}$ corresponding to the decompositions $S^i = S^i_1 \sqcup \cdots \sqcup S^i_{k_i}$ of elements $S^i \in B_1 \setminus B_0$, numbered in any order that is inverse to inclusion (i.e., $S^i \supset S^{i'} \Rightarrow i \leqslant i'$).

Buchstaber suggested to call a simple convex n-dimensional polytope P a 2-truncated cube if it can be obtained from an n-cube by a sequence of cut off some faces of codimension 2 only. It is allowed to cut off any codimension 2 face that we have on a previous step of the sequence of face truncations.

Example 2.2 Here is an example of a 3-dimensional 2-truncated cube $\mathcal{P}$ which we shall use later.

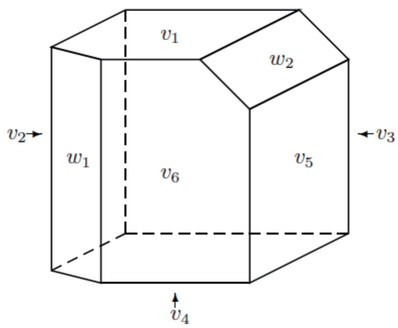

Figure 1 A 2-truncated cube $\mathcal{P}$.

Then any flag nestohedron can be realized as a 2-truncated cube. The following statement holds.

Theorem 2.2 (see [7, Proposition 6.1, Theorem 6.5]) *A nestohedron P_B is a flag polytope if and only if it is a 2-truncated cube.*

More precisely, if P_B is a flag polytope, then there exists a sequence of building sets $B_0 \subset B_1 \subset \cdots \subset B_N = B$, where P_{B_0} is a combinatorial cube, $B_i = B_{i-1} \cup \{S_i\}$, and P_{B_i} is obtained from $P_{B_{i-1}}$ by a 2-truncation at the face $F_{S_{j_1}} \cap F_{S_{j_2}} \subset P_{B_{i-1}}$ of codimension 2, where $S_i = S_{j_1} \sqcup S_{j_2}$, and $S_{j_1}, S_{j_2} \in B_{i-1}$.

The next family of polytopes introduced by Carr and Devadoss [8] are flag nestohedra and, therefore, by Theorem 2.2 can be realized as 2-truncated cubes.

Definition 2.2 *A graphical building set $B(\Gamma)$ for a (simple) graph Γ on the vertex set $[n+1]$ consists of such S that the induced subgraph Γ_S on the vertex set $S \subset [n+1]$ is a connected graph.*

Then $P_\Gamma = P_{B(\Gamma)}$ is called a graph-associahedron.

Example 2.3 The following families of graph-associahedra are of particular interest in convex geometry, combinatorics and representation theory.

(1) Γ is a complete graph on $[n+1]$.

Then $P_\Gamma = Pe^n$ is a permutohedron, see Figure 2.

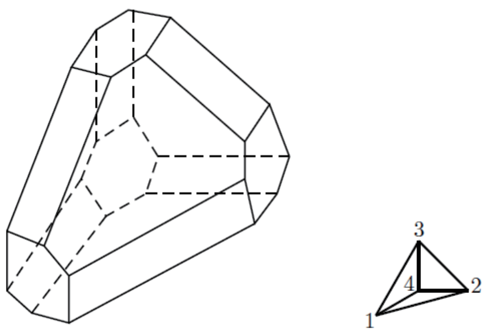

Figure 2 3-dimensional permutohedron and the corresponding graph.

(2) Γ is a stellar graph on $[n+1]$.

Then $P_\Gamma = St^n$ is a stellahedron, see Figure 3.

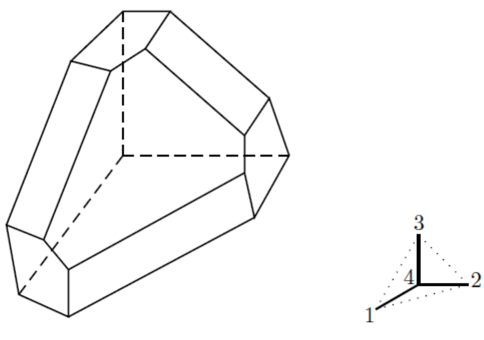

Figure 3 3-dimensional stellahedron and the corresponding graph.

(3) Γ is a cycle graph on $[n+1]$.
Then $P_\Gamma = Cy^n$ is a cyclohedron (or Bott-Taubes polytope, see [4]), see Figure 4.

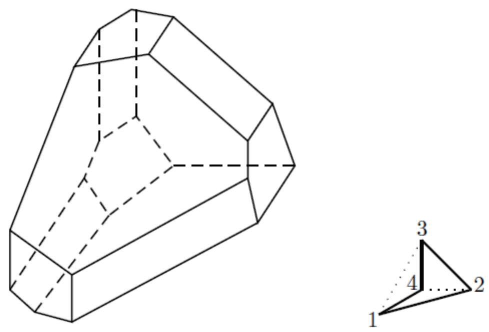

Figure 4 3-dimensional cyclohedron and the corresponding graph.

(4) Γ is a chain graph on $[n+1]$.
Then $P_\Gamma = As^n$ is an associahedron (or Stasheff polytope, see [24]), see Figure 5.

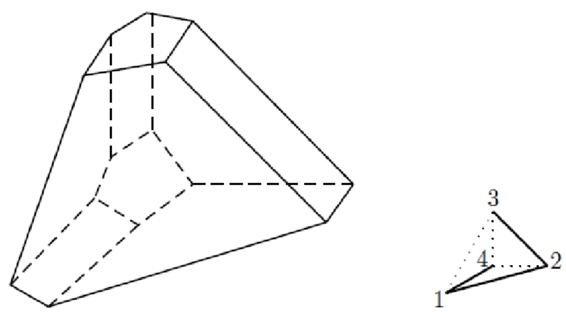

Figure 5 3-dimensional associahedron and the corresponding graph.

In order to determine the nerve complex K_P of a graph-associahedron $P = P_\Gamma$, we should describe the combinatorial structure of its face poset. The following is a reformulation of the general property stated in [6, Theorem 1.5.13].

Proposition 2.1 *Facets of* P_Γ *are in* 1-1 *correspondence with non-maximal connected subgraphs of* Γ.

Moreover, a set of facets corresponding to such subgraphs $\Gamma_{i_1}, \cdots, \Gamma_{i_s}$ *has a nonempty intersection if and only if*

(1) *For any two subgraphs* $\Gamma_{i_k}, \Gamma_{i_l}$*, either they do not have a common vertex or one is a subgraph of another;*

(2) *If any two of the subgraphs* $\Gamma_{i_{k_1}}, \cdots, \Gamma_{i_{k_l}}$*,* $l \geqslant 2$ *do not have common vertices, then their union graph is disconnected.*

Note that if P is a permutohedron, then its facets F_1 and F_2 have a nonempty intersection if and only if the corresponding subgraphs Γ_1 and Γ_2 are subgraphs of one another.

3 Bigraded Betti Numbers of Graph-Associahedra

In this section we describe certain bigraded Betti numbers of associahedra P in terms of combinatorics of their graphs Γ. This approach can be viewed as another argument to prove our previous result (see [18, Theorem 2.9]) and can be used to compute bigraded Betti numbers $\beta^{-i,2(i+1)}(P)$ of all graph-associahedra $P = P_\Gamma$. We begin with a following generalization of a result of Fenn (see [11, Theorem 4.6.4]).

Proposition 3.1 *Suppose $P = P_{B_1}$ and $Q = P_{B_2}$ are n-dimensional nestohedra on connected building sets $B_i, i = 1, 2$ and $J \subset B_1 \subset B_2$. Consider the following set*

$$\overline{J} = J \sqcup \{S \in B_2 \setminus B_1 \mid \exists S_1 \in J, S_1 \subset S\}.$$

Then P^n_J is homeomorphic to $Q^n_{\overline{J}}$.

Proof By Theorem 2.1, any nestohedron P_B on a connected building set $B \subset 2^{[n+1]}$ can be obtained as a result of a sequence of face truncations starting with a simplex Δ^n. Thus the nerve complex of our nestohedron $K_P = \partial P^*$ can be obtained from a boundary of a simplex as a result of a number of barycentric subdivisions in some of its simplices. Moreover, Theorem 2.1 states that the new vertices (barycenters of those simplices) correspond to the decompositions of the elements in $B_2 \setminus B_1$ in a disjoint unions of elements of B_1. Applying the descriprion of the face poset of Q in [6, Theorem 1.5.13] finishes the proof as any triangulation of a topological space is homeomorphic to the space itself.

Another way to prove this statement is similar to that of the proof in [11, Theorem 4.6.4]. Indeed, the centers of the geometric realizations of P and Q in $\mathbb{R}^{n+1}$ are Minkowski sums of the centers of their simplices from the definition of a nestohedron. Then we can translate P and Q so that their centers coincide and project the boundary of P onto the boundary of Q outwards from their common center. Obviously, the image of P_J is in $Q^n_{\overline{J}}$ and every facet in $Q^n_{\overline{J}}$ contains a point in the image of P_J. Finally, we make a continuous bijective transformation of the image (on each of the facet in $Q^n_{\overline{J}}$) onto the whole $Q^n_{\overline{J}}$.

In particular, when $B_2 = 2^{[n+1]}$ and Q is a permutohedron, we get the result of Fenn [11, Theorem 4.6.4]. In order to describe the bigraded Betti numbers of associahedra combinatorially, we introduce the following notion of a special subgraph γ in Γ.

Definition 3.1 *Suppose Γ is a graph. For any of its connected subgraphs γ, one can compute the number $i(\gamma)$ of such connected subraphs $\widetilde{\gamma}$ in Γ that either $\gamma \cap \widetilde{\gamma} \neq \emptyset, \gamma, \widetilde{\gamma}$ (in this case we say they have a nontrivial intersection) or $\gamma \cap \widetilde{\gamma} = \emptyset$, $\gamma \sqcup \widetilde{\gamma}$ is a connected subgraph in Γ. From now on we describe a subgraph in Γ as a vertex set meaning that the subgraph consists of its vertices and all edges in Γ connecting these vertices (induced subgraph). We denote by $i_{\max} = i_{\max}(\Gamma)$ the maximal value of $i(\gamma)$ over all connected subgraphs γ in Γ. A connected subgraph γ, on which $i_{\max}$ is achieved, will be called a special subgraph.*

Example 3.1 On Figure 5, we have 3 special subgraphs: $\{1, 2\}, \{1, 4\}$ and $\{2, 3\}$. The number $i_{\max}$ is equal to 4 and is achieved, for example, on $\gamma = \{1, 2\}$ with the graphs $\widetilde{\gamma}$ being

$\{3\}, \{4\}, \{1,4\}, \{2,3\}$ (the latter two intersect γ nontrivially).

The following statement for the bigraded Betti numbers of the type $\beta^{-i,2(i+1)}(P)$ for associahedra P holds.

Theorem 3.1 *Let $P = P_\Gamma$ be an associahedron of dimension $n \geq 3$. Then for $i > i_{\max}(\Gamma)$, one has*

$$\beta^{-i,2(i+1)}(P) = 0.$$

Denote the number of special subgraphs in Γ by s. Let $\omega = -i_{\max}, 2(i_{\max}+1)$. Then

$$\beta^{\omega}(P) = s.$$

Proof By Theorem 1.2 and Proposition 2.1 it is sufficient to prove the following three cases.

(a) We have $\mathrm{cc}(P_J) \leq 2$ if $|J| > i_{\max}$. In the latter case, if $P_J = P_{J_1} \sqcup P_{J_2}$ with $|J_{1,2}| \geq 2$, then there exists another $J' \subset B(\Gamma)$ such that $P_{J'} = P_{J_1'} \sqcup P_{J_2'}$ with $|J_1'| = 1$ and $|J'| > |J|$.

(b) Suppose $\mathrm{cc}(P_J) = 2,\ P_J = P_{J_1} \sqcup P_{J_2},\ |J| > i_{\max}$. Then either $|J_1| = 1$ or $|J_2| = 1$. Moreover, if $|J| = i_{\max}+1,\ |J_1| = 1$ then J_1 consists of a special subgraph of Γ and J_2 consists of all the $i_{\max}$ connected subgraphs in Γ determined in the definition of a special graph above.

(c) Suppose $|J| > i_{\max}+1$. Then $\mathrm{cc}(P_J) = 1$.

For an associahedron As^n the statement (a) follows from [18, Lemmas 2.13–2.14], the statement (b) follows from [18, Lemmas 2.15–2.16] and the statement (c) follows from [18, Lemma 2.17].

Remark 3.1 Using Propostion 2.1 one can see easily that Theorem 3.1 states that the last nonzero bigraded Betti number $\beta^{\omega}(P)$ in the sequence of $\beta^{-i,2(i+1)}(P),\ 1 \leq i \leq m-n$ is achieved precisely on P_J which is a union of a facet of P corresponding to a special subgraph in Γ and all the facets of P that do not intersect this facet. All the P_J with a greater cardinality $|J|$ of J are connected spaces in $\mathbb{R}^n$. An argument similar to that in the proof of Theorem 3.1 shows the same holds for a permutohedron $Pe^n,\ n \geq 3$ and applying Proposition 3.1 one can get the same result for any graph-associahedron on a connected graph Γ.

As an application of Theorem 3.1, the values of $i_{\max}(\Gamma)$ and s can be computed explicitly in terms of the combinatorics of the graph Γ. Using induction on the polytope dimension n for combinatorial enumerations in Γ it can be seen that a special graph γ is a path graph in Γ on either $\left[\frac{n+1}{2}\right]$ or $\left[\frac{n}{2}\right]+1$ vertices. This follows also from the proof of [18, Theorem 2.9], where the special graphs correspond to the longest diagonals in a regular $(n+3)$-gon G and the numbers of the vertices in such a graph are the numbers of vertices of G lying in one of the open halves of G divided by the diagonal. Thus, we get the following result (see [18, Theorem 2.9]).

Corollary 3.1 *For an associahedron P_Γ of dimension $n \geq 3$, one has the following values of $i_{\max} = q(n)$ and s:*

$$\beta^{-q,2(q+1)}(As^n) = \begin{cases} n+3, & \text{if } n \text{ is even,} \\ \dfrac{n+3}{2}, & \text{if } n \text{ is odd,} \end{cases}$$

$$\beta^{-i,2(i+1)}(As^n) = 0 \quad \text{for } i \geqslant q+1,$$

where $q = q(n)$ *is*

$$q = q(n) = \begin{cases} \dfrac{n(n+2)}{4}, & \text{if } n \text{ is even,} \\ \dfrac{(n+1)^2}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

As graph-associahedra are flag polytopes, we can apply the previous result to studying the loop homology algebra $H_*(\Omega\mathcal{Z}_P)$ for associahedra P. Namely, due to [12, Theorem 4.3] the minimal number of multiplicative generators of $H_*(\Omega\mathcal{Z}_P)$ is equal to $\sum_{i=1}^{m-n} \beta^{-i,2(i+1)}(P)$. Then Theorem 3.1 gives us lower bounds for the number of multiplicative generators in the Pontryagin algebra of $\mathcal{Z}_P$.

4 Massey Products

In this section we prove the main result of this article concerning Massey higher products in $H^*(\mathcal{Z}_P)$ (see Theorem 4.2) and a criterion when a nontrivial triple Massey product of 3-dimensional classes exists in $H^*(\mathcal{Z}_{P_\Gamma})$ (see Proposition 4.1). We first prove the statement on triple Massey products in the graph-associahedron P_Γ case, where Γ is an arbitrary (possibly disconnected) graph.

Let us state the following theorem due to Denham and Suciu which gives a combinatorial criterion for a simplicial complex K to produce a nontrivial triple Massey product of 3-dimensional classes in $H^*(\mathcal{Z}_K)$.

Theorem 4.1 (see [9, Theorem 6.1.1]) *The following are equivalent:*

(1) *There exist cohomology classes* $\alpha_i \in H^3(\mathcal{Z}_K)$, $i = 1, 2, 3$ *for which* $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ *is defined and nontrivial.*

(2) *The underlying graph* (1-*skeleton*) *of* K *contains an induced subgraph isomorphic to one of the five graphs in Figure* 6.

Moreover, all Massey products arising in this fashion are decomposable.

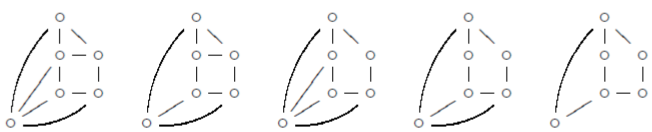

Figure 6 The five obstruction graphs.

Applying Theorem 4.1 to the graph-associahedra case gives us the following result.

Proposition 4.1 *There is a nontrivial triple Massey product* $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ *of* 3-*dimensional cohomology classes* $\alpha_i \in H^3(\mathcal{Z}_{P_\Gamma})$ *for* $i = 1, 2, 3$ *if and only if there is a connected component of* Γ *on* $m \geq 4$ *vertices which is different from a complete graph* K_4.

Proof We start with a connected graph Γ case. Suppose that the number of vertices in Γ is less than 4. Then P_Γ is either a point, a segment, a pentagon or a hexagon. The corresponding moment-angle manifold $\mathcal{Z}_P$ is either a disk D^2, a sphere S^3, or a connected sum of products of spheres respectively (see [3, 20]). These manifolds are formal spaces, therefore, there are no nontrivial higher Massey products in $H^*(\mathcal{Z}_P)$.

Suppose that there are 4 vertices in Γ. There are 6 combinatorially different connected graphs Γ on 4 vertices, thus, giving 6 combinatorially different 3-dimensional graph-associahedra P_Γ. If Γ is a complete graph K_4, then $P = P_\Gamma$ is a permutohedron and the boundary of its dual simplicial polytope $K = K_P$ is combinatorially equivalent to a barycentric subdivision of a boundary of a 3-simplex. As there are no induced 5-cycles in K and the first two graphs in Figure 6 can not also be induced graphs in K, by Theorem 4.1 there are no nontrivial triple Massey products in $H^*(\mathcal{Z}_P)$. On the other hand, using Figures 2–5 and Theorem 2.1 one can check easily that the third (middle) of the five graphs in the Figure 6 is an induced subgraph in the underlying graph (1-skeleton) of K_P for P being any of the other five 3-dimensional graph-associahedra on a connected graph with 4 vertices. The case of a connected graph on 4 vertices now holds from Theorem 4.1.

Suppose now, that Γ is a connected graph on more than 4 vertices. Using induction on the number of edges in Γ, we get an induced subgraph γ in Γ on 4 vertices. Using Proposition 2.1 the induced subcomplex in $K_P,\ P = P_\Gamma$ on the vertex set corresponding to all connected subgraphs in $\gamma \neq K_4$ will give us a nontrivial triple Massey product in $H^*(\mathcal{Z}_P)$ by the argument above. On the other hand, if any connected subgraph on 4 vertices in Γ is a complete graph K_4, then Γ is a complete graph K_{n+1}. Indeed, consider two different vertices α and β in Γ. Then there is a connected subgraph containing them in Γ. Such a graph γ with a minimal number of edges will obviously be a path between α and β. If it has more than 2 edges then it has more than 3 vertices and thus contains K_4 as an induced graph on some 4 of its vertices, thus γ being not minimal (any pair of vertices in K_4 is connected by one edge). Similarly, if γ has 2 edges then one of its 3 vertices is conected to another vertex of Γ (as Γ has more than 4 vertices and is connected) and we get K_4 as an induced subgraph. So, γ is not minimal again. Thus, γ has one edge, i.e., α and β are connected by an edge in Γ and Γ is a complete graph.

It remains to consider the case when Γ is a complete graph $K_{n+1},\ n \geq 4$ and $P = P_\Gamma$ is a permutohedron. Note that K_Q is an induced subcomplex in K_P for any such P when Q is a 4-dimensional permutohedron. Consider the graph $\Gamma = K_5$ for Q and an induced subgraph in K_Q on the following vertices:

$$\{1\},\ \{2\},\ \{1,3\},\ \{1,2,4\},\ \{1,2,3,4\},\ \{1,2,3,5\}.$$

One can see easily that this induced subgraph is the first (left) graph in Figure 6. By Theorem 4.1 and Theorem 1.1 (see (1.2)), any permutohedron P of dimension 4 and greater gives us a nontrivial triple Massey product in $H^*(\mathcal{Z}_P)$.

Finally, the case of a disconnected graph Γ follows from Proposition 2.1 and Theorem 1.1 and the connected graph case as if two graphs Γ_1 and Γ_2 are disjoint, then for their union graph Γ, one gets $P_\Gamma = P_{\Gamma_1} \times P_{\Gamma_2}$ and the moment-angle functor $\mathcal{Z}$ preserves products of polytopes (see [6, Chapter 4]). This finishes the proof.

Remark 4.1 Note that each of the six 3-dimensional graph-associahedra $P = P_\Gamma$ mentioned above is a 2-truncated cube and, moreover, Pe^3 can be obtained from Cy^3 by cut off its 2 non-adjacent edges, if realized as a simplex truncation (see Theorem 2.1 and Figures 2 and 4). As

$\mathcal{Z}_P$ for $P = I^n$ is a product of spheres and, therefore, is a formal manifold, it follows that a nontrivial higher Massey product in $H^*(\mathcal{Z}_P)$ can either appear or vanish after a (codimension 2) face truncation (or after a stellar subdivision in the dual simplicial sphere K_P).

Example 4.1 Consider $P = Pe^3$ (see Figure 2). It has $n = 3$ and $m = 14$. Letting us label its facets by the numbers $1, \cdots, 14$ such that the bottom and upper 6-gon facets are 1 and 14 respectively, the bottom facets are labeled by $2, \cdots, 7$ and the upper facets are labeled by $8, \cdots, 13$, both clockwisely.

Consider the following 3-dimensional cocycles:

$$a_1 = v_1 u_{14}, \quad a_2 = v_6 u_{10}, \quad a_3 = v_8 u_4, \quad a_4 = v_2 u_{12}.$$

They correspond to 4 pairs of parallel facets of P if realized as a result of face truncations from Δ^3. Suppose that they are representatives of the cohomology classes $\alpha_i \in H^3(\mathcal{Z}_P)$, that is, $\alpha_i = [a_i]$ for $i = 1, \cdots, 4$.

Then we get the following defining system A (see [17]) for the Massey 4-product $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ (up to signs):

$$a_{13} = v_6 u_1 u_{14} u_{10}, \quad a_{24} = v_6 u_{10} u_8 u_4, \quad a_{35} = v_2 u_8 u_4 u_{12},$$
$$a_{14} = v_6 u_1 u_8 u_4 u_{10} u_{14}, \quad a_{25} = 0,$$

so $0 \in \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$. Thus the two 3-products $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ and $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$ are defined and vanish simultaneously and the 4-product $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is defined and trivial.

Remark 4.2 Note that the same calculation works in full generality, namely, if $P = Pe^n$, $n \geq 2$ and the classes $\alpha_i \in H^3(\mathcal{Z}_P)$, $1 \leq i \leq n+1$ are represented by $(n+1)$ pairs of the parallel permutohedra facets (see Figure 2), then $\langle \alpha_1, \cdots, \alpha_{n+1} \rangle$ is defined and trivial. Similarly, if $P = St^n$, $n \geq 2$ and the classes $\alpha_i \in H^3(\mathcal{Z}_P)$, $1 \leq i \leq n$ are represented by n pairs of the parallel stellahedra facets (see Figure 3), then $\langle \alpha_1, \cdots, \alpha_n \rangle$ is defined and trivial.

We next consider a particular family of 2-truncated n-cubes $\mathcal{P}$, one for each dimension n, for which $\mathcal{Z}_{\mathcal{P}}$ has a nontrivial Massey product of order n.

Definition 4.1 *Suppose that $I^n = [0,1]^n, n \geq 2$ is an n-dimensional cube with facets $F_1, \cdots, F_{2n}$, such that $F_i, 1 \leq i \leq n$ contain the origin 0, a unit inner normal vector to F_i, $1 \leq i \leq n$ is $(0, \cdots, 1, \cdots, 0)$ with 1 in the ith position, F_i and F_{n+i}, $1 \leq i \leq n$ being parallel. Then we define $\mathcal{P}$ as a result of a consecutive cut of faces of codimension 2 from I^n, having the following Stanley-Reisner ideal:*

$$I = (v_k v_{n+k+i},\ 0 \leq i \leq n-2,\ 1 \leq k \leq n-i, \cdots),$$

where v_i correspond to F_i, $1 \leq i \leq 2n$ and in the dots are the monomials corresponding to the new facets (i.e., facets obtained after performing truncations). This determines uniquely the combinatorial type of $\mathcal{P}$.

Example 4.2 For $n = 2$ we get a 2-dimensional cube (the square) $\mathcal{P}$ and its Stanley-Reisner ideal is the following one:

$$I = (v_1 v_3, v_2 v_4).$$

For $n = 3$ we get a simple polytope $\mathcal{P}$ from Figure 1, for which $K = K_{\mathcal{P}}$ is a simplicial complex with a nontrivial triple Massey product in $H^*(\mathcal{Z}_K)$ due to the result of Baskakov (see [2]). Moreover, using the computer software Plantri it can be seen that K is the only one of the 14 combinatorially different 2-spheres with 8 vertices giving nontrivial higher Massey products in $H^*(\mathcal{Z}_K)$ (see [9]).

The Stanley-Reisner ideal of $\mathcal{P}$ can be written as follows (see Figure 1):

$$I = (v_1v_4, v_2v_5, v_3v_6, v_1v_5, v_2v_6, w_1v_3, w_1v_5, w_2v_2, w_2v_4, w_1w_2).$$

Remark 4.3 The 2-truncated cube $\mathcal{P}$ is not a graph-associahedron as its number of facets $f_0(\mathcal{P}) = \frac{n(n+3)}{2} - 1 < f_0(As^n) = \frac{n(n+3)}{2}$ (see [7, Theorem 9.2]). However, we can easily construct the building set B for $\mathcal{P}$ on the vertex set $[n+1]$ by identifying F_i with $\{1, \cdots, i\}$ for $1 \leq i \leq n$ and identifying F_i with $\{i - n + 1\}$ for $n + 1 \leq i \leq 2n$. Then, by Theorem 2.2, we consecutively cut the following faces:

$$\begin{aligned}
&\{1\} \sqcup \{3\},\ \{1,2\} \sqcup \{4\},\ \cdots, \{1, \cdots, n-1\} \sqcup \{n+1\},\\
&\cdots,\\
&\{1\} \sqcup \{n\},\ \{1,2\} \sqcup \{n+1\}.
\end{aligned}$$

Thus, $\mathcal{P} = P_B$ for the building set B consisting of the building set B_0 of an n-cube from Example 2.1, the above subsets of $[n+1]$ and all the subsets of $[n+1]$ which are the unions of nontrivially intersecting elements in B.

Theorem 4.2 *Let $\alpha_i \in H^3(\mathcal{Z}_{\mathcal{P}})$ be represented by a 3-cocycle v_iu_{n+i} for $1 \leq i \leq n$ and $n \geq 2$. Then all Massey products of consecutive elements from $\alpha_1, \cdots, \alpha_n$ are defined and the whole n-product $\langle \alpha_1, \cdots, \alpha_n \rangle$ is nontrivial.*

Proof Let us prove the theorem by induction on n. The base case $n = 2$ is trivial: α_1 and α_2 are the classes of 3-dimensional spheres in $\mathcal{Z}_{\mathcal{P}} \cong S^3 \times S^3$ and their cup-product (i.e., Massey 2-product) is the dual to the fundamental class of $\mathcal{Z}_{\mathcal{P}}$.

We first note that all Massey products of orders less than n vanish simultaneously in $H^{*,*}(\mathcal{Z}_{\mathcal{P}}) \cong H\big[\Lambda[u_1, \cdots, u_m] \otimes \Bbbk[\mathcal{P}], d\big]$, i.e., contain coboundaries. Starting with the representing cocycles v_iu_{i+n} of α_i, it can be seen by induction on the dimension n of $\mathcal{P}$ that if a defining system C for the n-product $\langle \alpha_1, \cdots, \alpha_n \rangle$ can be extended from ith diagonal of the matrix C to its $(i+1)$th diagonal for all $2 \leq i \leq n$, then $c_{lm}, m - l = i \geq 2$ have either a form $v_ku_{j_1} \cdots u_{j_{2i-1}}$ or a form $v_ku_{j_1} \cdots u_{j_{2i-1}} + d(u_ku_{j_1} \cdots u_{j_{2i-1}})$ (up to the signs). The latter can be checked as the differential in the cohomology algebra preserves multigrading and by using the codimension 2 face cuts from the definition of $\mathcal{P}$ (see also the example below).

Then the Massey n-product $\langle \alpha_1, \cdots, \alpha_n \rangle$ is defined and any cohomology class belonging to it lies in the multigraded component $H^{-(2n-2),(2,\cdots,2,0,\cdots,0)}(\mathcal{Z}_{\mathcal{P}})$ of the moment-angle manifold $\mathcal{Z}_{\mathcal{P}}$ with one of its representatives being the class of the cocycle $v_1v_{2n}u_2 \cdots u_{2n-1}$. Up to sign we have the following equality for any representative c of an element in $\langle \alpha_1, \cdots, \alpha_n \rangle$ for any defining system C (see [17]):

$$c = d(c_{1,n+1}) - (-1)^3 v_1u_{n+1}c_{2,n+1} - \overline{c}_{1,3}c_{3,n+1} - \cdots - \overline{c}_{1,n}v_nu_{2n},$$

where $(n+1)\times(n+1)$-matrix C is upper triangular with zeros on the diagonal and $c_{i,i+1} = -v_i u_{n+i}$ for $1 \le i \le n$, such that the following condition holds:

$$cE_{1,n+1} = d(C) - \overline{C} \cdot C$$

and $\overline{c}_{ij} = (-1)^{|c_{ij}|} c_{ij}$ depends on the degree $|c_{ij}|$ of a matrix element c_{ij}.

By definition of higher Massey operations (see [17]), one has: $d(c_{2,n+1})$ is a representative in $\langle \alpha_2, \cdots, \alpha_n \rangle$ and $d(c_{1,n})$ is a representative in $\langle \alpha_1, \cdots, \alpha_{n-1} \rangle$. To prove that $v_1 v_{2n} u_2 \cdots u_{2n-1}$ is the only representing cocycle for the n-product, we use induction on n, the represnting monomials for the indeterminacies and the multigrading in $H^*(\mathcal{Z}_{\mathcal{P}})$, see Theorem 1.3. For instance, the indeterminacy for the first of the $(n-1)$-products above is lying in the multigraded component of $v_2 u_3 \cdots u_n u_{n+2} \cdots u_{2n}$ and the only cocycle in that component is the coboundary $d(u_2 \cdots u_n u_{n+2} \cdots u_{2n})$. The indeterminacy for the second of the $(n-1)$-products above is lying in the multigraded component of $v_1 u_2 \cdots u_{n-1} u_{n+1} \cdots u_{2n-1}$ and the only cocycle in that component is the coboundary $d(u_1 \cdots u_{n-1} u_{n+1} \cdots u_{2n-1})$.

Thus, the Massey n-product $\langle \alpha_1, \cdots, \alpha_n \rangle$ is defined and nontrivial, consisting only of the cohomology class of $v_1 v_{2n} u_2 \cdots u_{2n-1}$.

Remark 4.4 Note that the nontrivial n-product constructed above is decomposable. Namely, one has $[v_1 v_{2n} u_2 \cdots u_{2n-1}] = \pm[v_1 u_{n+1} \cdots u_{2n-1}] \cdot [v_{2n} u_2 \cdots u_n]$.

Example 4.3 Consider the case $n = 4$. Then the Stanley-Reisner ideal of $\mathcal{P}$ is

$$I = (v_1v_5, v_2v_6, v_3v_7, v_4v_8, v_1v_6, v_2v_7, v_3v_8, v_1v_7, v_2v_8, \cdots)$$

and the cohomology classes $\alpha_i, 1 \le i \le 4$ are represented by the cocycles $a_i = v_i u_{4+i}, 1 \le i \le 4$. One has (up to sign)

$$\begin{aligned}
a_1a_2 &= d(v_1u_2u_5u_6) = d(c_{1,3}),\\
a_2a_3 &= d(v_2u_3u_6u_7) = d(c_{2,4}),\\
a_3a_4 &= d(v_2u_4u_7u_8) = d(c_{3,5}).
\end{aligned}$$

Then one has the following cocycle representing a class in $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ (here the Massey 2-product of a and b is equal to $\overline{a} \cdot b$, $\overline{a} = (-1)^{|a|} a$):

$$v_1u_5 \cdot (-v_2u_3u_6u_7) - v_1u_2u_5u_6 \cdot v_3u_7 = d(v_1u_2u_3u_5u_6u_7) = d(c_{1,4})$$

and the following cocycle representing a class in $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$:

$$v_2u_6 \cdot (-v_3u_4u_7u_8) - v_2u_3u_6u_7 \cdot v_4u_8 = d(v_2u_3u_4u_6u_7u_8) = d(c_{2,5}).$$

Alternatively, one has (up to sign)

$$\begin{aligned}
a_1a_2 &= d(v_2u_1u_5u_6 - v_5u_2u_1u_6 + v_6u_2u_1u_5) = d(c_{1,3}),\\
a_2a_3 &= d(v_3u_2u_6u_7 - v_6u_3u_2u_7 + v_7u_3u_2u_6) = d(c_{2,4}),\\
a_3a_4 &= d(v_4u_2u_7u_8 - v_7u_4u_2u_8 + v_8u_4u_2u_7) = d(c_{3,5}).
\end{aligned}$$

The representing cocycle for $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ will be $d(v_3u_1u_2u_5u_6u_7) = d(c_{1,4})$ and for $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$, one gets $d(v_4u_2u_3u_6u_7u_8) = d(c_{2,5})$.

Thus, the Massey products $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ and $\langle\alpha_2,\alpha_3,\alpha_4\rangle$ vanish simultaneously and the 4-product $\langle\alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle$ is defined. More precisely, the representing cocycle c for $\langle\alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle$ is equal to

$$d(c_{1,5})-\overline{a}_1c_{2,5}-\overline{c}_{1,3}c_{3,5}-\overline{c}_{1,4}a_4.$$

Considering the multigrading in $H^*(\mathcal{Z}_P)$ it is easy to see that the latter 4-fold product consists of the only class with a representative (up to sign) $v_1v_8u_2\cdots u_7$ in $H^{-6,(2,\cdots,2,0,\cdots,0)}(\mathcal{Z}_P)\subset H^{-6,16}(\mathcal{Z}_P)\subset H^{10}(\mathcal{Z}_P)$, where $\mathcal{Z}_P$ is a closed smooth 17-dimensional manifold.

Finally, one has $[v_1v_8u_2\cdots u_7]=-[v_1u_5u_6u_7]\cdot[v_8u_2u_3u_4]$.

Using Theorem 4.2 we can construct a smooth closed 2-connected manifold M with a compact torus action, such that there are nontrivial higher Massey products of any prescribed orders $n_1,\cdots,n_r,\ r\geq 2$ in $H^*(M)$. Namely, consider the building sets $B_i, 1\leq i\leq r$ for $\mathcal{P}^{n_i}, 1\leq i\leq r$. Let $M=\mathcal{Z}_P$, where $P=P_{B'}, B'=B(B_1,\cdots,B_r)$ (see [6, Construction 1.5.19]) and B be a connected building set of a $(r-1)$-dimensional cube. Then P is a flag polytope combinatorially equivalent to $I^{r-1}\times\mathcal{P}^{n_1}\times\cdots\times\mathcal{P}^{n_r}$ (see [6, Lemma 1.5.20]) and $H^*(\mathcal{Z}_P)$ contains nontrivial Massey products of orders $n_i, 1\leq i\leq r$ as the functor $\mathcal{Z}$ preserves products for simple polytopes. Note that $P=P_{B'}$ is still a flag nestohedron and, therefore, can be realized as a 2-truncated cube.

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