Chin. Ann. Math. Ser. B
38(6), 2017, 1303–1310
DOI: 10.1007/s11401-017-1038-0

Chinese Annals of Mathematics, Series B

On $\mathbb{Z}_3$-Actions on Spin 4-Manifolds*

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Abstract Let X be a closed, simply-connected, smooth, spin 4-manifold whose intersection form is isomorphic to $2k(-E_8) \oplus lH$, where H is the hyperbolic form. In this paper, the authors prove that if there exists a locally linear pseudofree $\mathbb{Z}_3$-action on X, then $\mathrm{Sign}(g, X) \equiv -k \bmod 3$. They also investigate the smoothability of locally linear $\mathbb{Z}_3$-action satisfying above congruence. In particular, it is proved that there exist some nonsmoothable locally linear $\mathbb{Z}_3$-actions on certain elliptic surfaces.

Keywords Group action, Locally linear, Kirby-Siebenmann invariant, Nonsmoothable
2000 MR Subject Classification 57R57, 57M60, 57S25

1 Introduction

Let G be a finite group, and X be an n-dimensional manifold. If $\mathrm{Top}(G, X)$ denotes the set of equivalent classes of topological G-actions on X. Recall that two topological G-actions on X are equivalent if there exists a homeomorphism f of X such that one action is conjugate to the other by f. A topological finite group G-action on an n-dimensional manifold X is called locally linear if for any point $x \in X$, there exists a G_x-invariant neighborhood V_x of x such that V_x is homeomorphic to $\mathbb{R}^n$, and G_x acts on V_x in a linear orthogonal way, where G_x is the isotropy group of x. Similarly, $\mathrm{LL}(G, X)$ denotes the set of equivalent classes of locally linear G-actions on X. If a smooth structure on X is specified, $C^\infty(X, G)$ denotes the set of equivalent classes of smooth G-actions on X with respect to differmorphisms preserving the smooth structure. It is well known that the three classes of group actions have the relation

$$\mathrm{Top}(G, X) \supset \mathrm{LL}(G, X) \xleftarrow{\varphi} C^\infty(X, G),$$

where φ is the map forgetting the smooth structure. For a 4-dimensional manifold X the map φ is not surjective. An action is called pseudofree if it is free outside of a finite set of points.

It is proved by Kwasik and Vogel [11] that the existence of nontrivial locally linear involutions on simply-connected closed topological 4-manifolds implies that the vanishing of the Kirby-Siebenmann obstruction. In the present paper, for a topological $\mathbb{Z}_3$-action on 4-dimensional manifold X, we obtain the following necessarily condition for it to be locally linear.

Manuscript received January 23, 2016.
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*The work was supported by the National Natural Science Foundation of China (Nos. 11371076, 11431009) and the Natural Science Foundation of Hebei Province of China (No. A2014501040).

Theorem 1.1 *Let X be a closed, simply-connected, smooth, spin 4-manifold whose intersection form is isomorphic to $2k(-E_8) \oplus lH$, where H is the hyperbolic form. If a pseudofree, topological $\mathbb{Z}_3$-action on X is locally linear, then* $\mathrm{Sign}(g, X) \equiv -k \bmod 3$.

The proof of the above theorem is based on the properties of Kirby-Sibenmann invariant and Rochlin invariant in [7].

The smoothability of a locally linear orientation preserving pseudofree action on a smooth 4-manifold has been an open question. Kwasik and Lawson [10] provided the first example answering this question in the negative mainly by gauge theory, and in some cases of involutions, Rohlin's μ-invariant is used. In recent years, many nonsmoothable group actions on 4-manifolds are constructed by many authors (see [3–4, 9–14, 16]). For example, the authors proved the existence of nonsmoothable involutions on a large class of spin 4-manifolds in [16], where we use the Rochlin's theorem. In [13–14], Liu and Nakamura constructed groups actions on elliptic surfaces which are not smooth with respect to infinitely many smooth structures including the standard smooth structure. They used the mod p vanishing theorem of Seiberg-Witten invariants in [6] to get nonsmoothable group actions on elliptic surfaces.

In this paper, we restrict our attention to $\mathbb{Z}_3$-actions on spin 4-manifolds and provide an example of nonsmoothable locally linear $\mathbb{Z}_3$-actions on certain elliptic surfaces (see Theorem 4.2).

2 Preliminaries

In this section, a constraint on smooth $\mathbb{Z}_3$-actions and some method of constructing locally linear $\mathbb{Z}_3$-actions are given. We also collect some fomulae which will be used in calculation.

Let G be the cyclic group of order 3 ($G = \mathbb{Z}_3$), and suppose that G acts locally linearly and pseudofreely on a spin 4-manifold X. Now let b_i be the i-th Betti number of X, and b_+ (resp. b_-) be the rank of a maximal positive (resp. negative) definite subspace $H^+(X;\mathbb{R})$ (resp. $H^-(X;\mathbb{R})$) of $H^2(X;\mathbb{R})$. For any G-space V, let V^G be the fixed point set of the G-action. Let $b_\bullet^G = \dim H^\bullet(X;\mathbb{R})^G$, where $\bullet = 2, +, -$. The Euler number of X is denoted by $\chi(X)$ and the signature of X by $\mathrm{Sign}(X)$.

When we fix a generator g of G, the representation at a fixed point can be described by a pair of nonzero integers (a, b) modulo 3 which is well-defined up to order and changing the sign of both together. Hence, there are two types of fixed points:

(1) The type (+): $(1, 2) = (2, 1)$.

(2) The type (−): $(1, 1) = (2, 2)$.

Let m_+ be the number of fixed points of the type (+), and m_- be the number of fixed points of the type (−).

2.1 The realization theorem of locally linear $\mathbb{Z}_3$-actions

To construct locally linear $\mathbb{Z}_3$-actions, we use the following special case of the realization theorem by Edmonds and Ewing [5].

Theorem 2.1 (see [5]) *Suppose that we are given a fixed point data*

$$\mathcal{D} = \{(a_0, b_0), (a_1, b_1), \cdots, (a_n, b_n), (a_{n+1}, b_{n+1})\},$$

where $a_i, b_i \in \mathbb{Z}_3\backslash\{0\}$, and a $\mathbb{Z}_3$-invariant bilinear unimodular even form $\Psi : V \times V \to \mathbb{Z}$, where V is a finitely generated $\mathbb{Z}$-free $\mathbb{Z}[\mathbb{Z}_3]$-module. Then the data $\mathcal{D}$ and the form (V, Ψ) are realizable by a locally linear, pseudofree, G-action on a closed, simply-connected, topological 4-manifold if and only if they satisfy the following two conditions:

(1) *The condition REP: As a $\mathbb{Z}[\mathbb{Z}_3]$-module, $V \cong T \oplus F$, where T is a trivial $\mathbb{Z}[\mathbb{Z}_3]$-module with $\text{rank}_{\mathbb{Z}} T = n$, and F is a free $\mathbb{Z}[\mathbb{Z}_3]$-module.*

(2) *The condition GSF: The G-signature formula is satisfied, i.e.,*

$$\text{Sign}(g, (V, \Psi)) = \sum_{i=0}^{n+1} \frac{(\zeta^{a_i} + 1)(\zeta^{b_i} + 1)}{(\zeta^{a_i} - 1)(\zeta^{b_i} - 1)},$$

where $\zeta = \exp\left(\frac{2\pi \mathrm{i}}{3}\right)$.

Note that the realization theorem for all cyclic groups of prime order provided by Edmonds and Ewing [5] also has a condition TOR. However, the TOR condition is redundant for prime numbers p less than 23. Since the form Ψ is assumed even, the homeomorphism type of X is unique by Freedman's theorem (see [7]).

Set $\bigcup_i C_i$ is invariant under the action of G.

2.2 The Lefschetz fixed points theorem and the *G*-signature theorem

Here we collect some classical formulae. We refer the reader to see [1–2] and the excellent exposition in [4] for more details. Let X be a closed, oriented smooth 4-manifold, and let cyclic group $G \equiv \mathbb{Z}_p$ of prime order act on X effectively via orientation-preserving diffeomorphisms. Then the fixed-point set F, if nonempty, will consist of isolated points and surfaces. If a generator g of G is fixed, each fixed point $m \in F$ is associated with a nonzero integers pair (a_m, b_m), where $-p < a_m, b_m < p$, and they are uniquely determined up to a change of order or a change of sign simultaneously, such that the induced g-action on the tangent space at m is given by the complex linear transformation $(z_1, z_2) \mapsto (\xi^{a_m} z_1, \xi^{b_m} z_2)$, where $\xi = \exp\left(\frac{2\pi \mathrm{i}}{p}\right)$. For each connected surface $Y \subset F$, the action of g on the normal bundle of Y in X is given by $z \mapsto \xi^{c_Y} z$ for an integer c_Y with $0 < c_Y < p$, which is uniquely determined up to a sign modulo p.

Theorem 2.2 (Lefschetz Fixed Point Theorem) *Let $T : X \to X$ generate an action of $\mathbb{Z}_p$ on X, a closed, oriented smooth 4-manifold. Then $L(T, X) = \chi(F)$, where $\chi(F)$ is the Euler characteristic of the fixed-point set F and $L(T, X)$ is the Lefschetz number of the map T, which is defined by*

$$L(T, X) = \sum_{k=0}^{4} (-1)^k \text{tr}(g)|_{H^k(X;\mathbb{R})}.$$

For a simply-connected 4-manifold X , the formula is $\chi(F) = 2 + \text{tr}(g)|_{H^2(X;\mathbb{R})}$.

Theorem 2.3 (*G*-Signature Theorem) *Set*

$$\text{Sign}(g, X) = \text{tr}(g)|_{H^{2,+}(X;\mathbb{R})} - \text{tr}(g)|_{H^{2,-}(X;\mathbb{R})}.$$

Then

$$\text{Sign}(g, X) = \sum_{m \in F} -\cot\left(\frac{a_m \pi}{p}\right) \cdot \cot\left(\frac{b_m \pi}{p}\right) + \sum_{Y \subset F} \csc^2\left(\frac{c_Y \pi}{p}\right) \cdot (Y \cdot Y),$$

where $Y \cdot Y$ denotes the self-intersection number of Y.

The weaker version of the G-signature theorem is used more often since the convenient for calculation.

Theorem 2.4 (G-Signature Theorem—The Weaker Version)

$$|G| \cdot \mathrm{Sign}(X/G) = \mathrm{Sign}(X) + \sum_{m \in F} \mathrm{def}_m + \sum_{Y \subset F} \mathrm{def}_Y,$$

where the terms def_m *and* def_Y *are called signature defects. They are given by the following formulae:*

$$\mathrm{def}_m = \sum_{k=1}^{p-1} \frac{(1+\xi^k)(1+\xi^{kq})}{(1-\xi^k)(1-\xi^{kq})}$$

if the local representation of G at m is given by $(z_1, z_2) \mapsto (\xi^k z_1, \xi^{kq} z_2)$, and

$$\mathrm{def}_Y = \frac{p^2-1}{3} \cdot (Y \cdot Y).$$

Note that the G-signature theorem is also valid for locally linear, topological actions of prime orders in dimension 4. The actions in this paper are pseudofree, i.e., the fixed point set only contains isolated fixed points, so the above formulae will be more concise. Set $\bigcup_i C_i$ is invariant under the action of G.

2.3 Kirby-Siebenmann invariant and Rochlin invariant

Let us review some properties of Kirby-Siebenmann invariant and Rochlin invariant in this part (see [4, 7] for more details).

Let X be a compact topological 4-manifold whose boundary has a unique smooth structure. There is an obstruction in $\mathrm{ks}(X) \in H^4(X, \partial X; \mathbb{Z}_2)$ to extend the smooth structure to a smooth structure on $X \times \mathbb{R}$. Each component of X has $H^4(-, \partial X; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Define $\mathrm{ks}(X)$ to be the sum of these invariants in $\mathbb{Z}_2$ over all components. The $\mathrm{ks}(X)$ is the stable smoothing obstruction if X is connected. Suppose (M, τ) is a closed spin 3-manifold, where τ is a spin structure. There is a smooth spin 4-manifold (W, τ') bounded by (M, τ) since the 3-dimensional smooth bordism group is trivial. Then Rochlin invariant $\mathrm{roc}(M, \tau)$ is defined to be the signature of W, mod 16. This invariant may depend on the spin structure. The invariant is well defined if there is a unique spin structure.

Suppose that a smooth, simply-connected, spin 4-manifold X admits a locally linear, topological action of a finite group G. Then the quotient space X/G is a spin 4-orbifold with only isolated singular points. By removing a regular neighborhood of the singular set, we get a spin 4-manifold with boundary which denoted by N, and the boundary of N inherits a spin structure from that of N which denoted by $\partial\eta$. Then, the Kirby-Siebenmann invariant of N and the Rochlin invariant of $(\partial N, \partial\eta)$ are constrained as follows.

Theorem 2.5 (see [7]) $8 \cdot \mathrm{ks}(N) \equiv \mathrm{Sign}(N) + \mathrm{roc}(\partial \mathrm{N}, \partial\eta) \bmod 16.$

By the results about the Kirby-Siebenmann invariant in [3], it is very easy to get the following theorem.

Theorem 2.6 *The Kirby-Siebenmann invariant* $\mathrm{ks}(N) = 0$ *is a necessary condition for the* G*-action to be smoothable.*

3 The Proof of Theorem 1.1

Let X be a closed, simply-connected, smooth, spin 4-manifold which has the intersection form isomorphic to $2k(-E_8) \oplus lH$, where k and l are positive integers. Therefore $l \geq 2k+1$ by Furuta's inequality (see [8]). Suppose there is a locally linear pseudofree $\mathbb{Z}_3$-acttion on X. The fixed points of a pseudofree $\mathbb{Z}_3$-action on X can be divided into two types by considering their local representation: the type $(+)$ and the type $(-)$, and let k_+, k_- be the numbers of the fixed points of the type$(+)$, type$(-)$ in the fixed point set separately. We can see that the corresponding 4-manifold N has $k_+ + k_-$ boundary components: There are k_+ $L(3,2)$ and k_- $L(3,1)$.

Lemma 3.1 *The Rochlin invariants of* $L(3,1)$ *and* $L(3,2)$ *are* 2 *and* -2, mod 16, *respectively.*

Proof Since lens space $L(3,1)$ bounds a spin 4-manifold Y obtained by plumbing on a chain which has two vertices both weighted by 2, the signature of Y is 2. Note that $L(3,2) = -L(3,1)$. Then, the Rochlin invariants of $L(3,1)$ and $L(3,2)$ can be obtained by definition.

Lemma 3.2 *For any locally linear* $\mathbb{Z}_3$*-acttion on* X, $\mathrm{ks}(N) \equiv 0 \bmod 2$.

Proof Since the obstruction in question is natural for coverings and in this case can be thought of as a multiple of the "top class" in $H^4(N, \partial N; \mathbb{Z}_2)$.

For a pseudofree $\mathbb{Z}_3$-action on X, we have the following constraint on Kirby-Siebenmann invariant and Rochlin invariant.

Theorem 3.1 $8 \cdot \mathrm{ks}(N) \equiv -\frac{16}{3}(k + \mathrm{Sign}(g, X)) \bmod 16$.

Proof By the G-Signature formula, we have

$$\mathrm{Sign}(X/\mathbb{Z}_3) = \frac{1}{3}\Big(\mathrm{Sign}(X) + \sum_{m \in F} \mathrm{def}_m\Big) = \frac{1}{3}[-16k + 2 \cdot \mathrm{Sign}(g, X)].$$

Here we use the facts that the signature defect of a type$(+)$ fixed point is $\mathrm{def}_+ = \frac{2}{3}$, and that of a type$(-)$ fixed point is $\mathrm{def}_- = -\frac{2}{3}$, and $\mathrm{Sign}(g, X) = \frac{1}{3}(k_+ - k_-)$.

By the additivity of the Rochlin invariant, we have

$$\mathrm{roc}(\partial N, \partial \eta) \equiv -2(k_+ - k_-) \mod 16.$$

Hence

$$\mathrm{roc}(\partial N, \partial \eta) \equiv -6 \cdot \mathrm{Sign}(g, X) \mod 16.$$

Note that $\mathrm{Sign}(X/\mathbb{Z}_3) = \mathrm{Sign}(N)$ for the standard choices of orientations. The proof is done.

Taking into account Lemma 3.2, and the formula in Theorem 3.1, then we have the next proposition.

Proposition 3.1 *For a generator* $g \in \mathbb{Z}_3$, $\mathrm{Sign}(g, X) \equiv -k \bmod 3$.

4 Nonsmoothable $\mathbb{Z}_3$-Actions

In this section, an example of nonsmoothable locally linear $\mathbb{Z}_3$-actions on 4-dimensional manifolds X satisfying $k \equiv 1 \bmod 3$ will be provided.

Let X be a 4-manifold as above, its intersection form isomorphic to $2k(-E_8) \oplus lH$, where $k \equiv 1 \bmod 3$. Recall that if there is a locally linear pseudofree $\mathbb{Z}_3$-action on X, $\mathrm{Sign}(g, X) = \frac{1}{3}(m_+ - m_-)$. By Proposition 3.1, $m_+ - m_- \equiv -3 \bmod 9$. Note that $\# X^G = m_+ + m_-$ and $2 + \mathrm{tr}(g|_{H^2(X)}) \leq \chi(X)$. By the Lefschetz fixed point theorem,

$$m_+ + m_- \leq 16k + 2l + 2.$$

For the spin 4-manifold X, the $G = \mathbb{Z}_3$-action can lifts to a G-action on the Spin^c structure naturally. Then, the G-index of the Dirac operator D_X can be written as $\mathrm{ind}_G D_X = \sum\limits_{j=0}^{2} k_j \mathbb{C}_j \in R(G) \cong \mathbb{Z}[t]/(t^3 = 1)$, where $\mathbb{C}_j$ is the complex 1-dimensional weight j representation of G and $R(G)$ is the representation ring of G. The following mod p vanishing theorem can be used to prove the existence of certain nonsmoothable locally linear $\mathbb{Z}_3$-actions.

Theorem 4.1 (see [6]) *Let Y be a smooth closed oriented 4-dimensional $\mathbb{Z}_p$-manifold with $b_1 = 0$ and $b_+ \geq 2$, where p is a prime. Suppose that c is a Spin^c-structure on which $\mathbb{Z}_p$-action lifts, and that $b_+ = b_+^G$. If $2k_j \leq b_+ - 1$ for $j = 0, \cdots, p-1$, then*

$$SW_Y(c) \equiv 0 \mod p.$$

As in [12], the coefficients k_j are calculated by the G-spin theorem. For a generator $g \in G$, the Lefschetz number is calculated by the formula as

$$\mathrm{ind}_g D_X = \sum_{j=0}^{2} \zeta^j k_j,$$

where $\zeta = \exp\big(\frac{2\pi\sqrt{-1}}{3}\big)$. We obtain

$$\mathrm{ind}_g D_X = k_0 + \zeta k_1 + \zeta^2 k_2 = \frac{1}{3}(m_+ - m_-),$$
$$\mathrm{ind}_{g^2} D_X = k_0 + \zeta^2 k_1 + \zeta k_2 = \frac{1}{3}(m_+ - m_-),$$
$$\mathrm{ind}_1 D_X = k_0 + k_1 + k_2 = -\frac{\mathrm{Sign}(X)}{8} = 2k.$$

Then, we have

$$k_0 = \frac{1}{9} 6k + 2(m_+ - m_-),$$
$$k_1 = k_2 = \frac{1}{9} 6k - (m_+ - m_-).$$

We can see that the condition $2k_j \leq b_+ - 1$ in Theorem 3.1 is equivalent to

$$6k - 9(l+1) \leq m_+ - m_- \leq \frac{9}{2}(l+1) - 3k.$$

At last, we give a result that there exist some nonsmoothable locally linear $\mathbb{Z}_3$-actions on certain elliptic surfaces.

An elliptic surface is a compact, complex surface E which comes with a holomorphic projection $\pi : E \to C$ onto a compact, connected complex curve, such that the generic fibers of π are elliptic curves. We will always assume that E is minimal elliptic, i.e., not a blow-up of another elliptic surface.

The projection π (or elliptic fibration) of an elliptic surface has well-understood local behavior. It has only finitely many critical values, and away from these it is a bundle projection with torus fibers (called regular fibers). The singular fibers, or preimages of critical values, come in various types. For minimal elliptic surfaces, we may smoothly change π so that only two types of singular fibers occur: cusp fibers and (smooth) multiple fibers. A cusp fiber is a PL-embedded sphere with a unique non-locally flat point, which is locally a cone on a (right-handed) trefoil knot.

Simply connected minimal elliptic surfaces without multiple fibers are completely classified up to diffeomorphism by a positive integer n. Each such manifold $E(n)$ has a projection with exactly $6n$ cusp fibers and no multiple fibers. It follows that $E(n)$ has Euler characteristic $12n$.

A smooth multiple fiber is a smoothly embedded torus which is multiply covered by nearby regular fibers. In fact, it is essentially a Seifert multiple fiber crossed with S^1. It follows that any elliptic surface can be obtained from one without multiple fibers by a process called logarithmic transform, which is essentially Dehn surgery along a fiber.

Now let $E(n)$ be the relatively minimal simply-connected elliptic surface without multiple fibers, and with geometric genus $p_g = n - 1$. Note that $\mathrm{Sign}(E(n)) = -8n$ and $\chi(E(n)) = 12n$. Thus $E(2) = E(1)\sharp_{\mathbb{T}^2}E(1)$ is the $K3$ surface. To see this just note that the Euler characteristic are additive under taking fiber connected sums over a torus. Hence $\mathrm{Sign}(E(2)) = -16$ and $\chi(E(2)) = 24$ which characterizes $K3$ surface. Besides, the general surface $E(n)$ can be constructed as a fiber connected sum $E(n) = E(n-1)\sharp_{\mathbb{T}^2}E(1)$. So the intersection form of $E(n)$ isomorphic to $n(-E_8) \oplus (2n-1)H$. Suppose n is even and $n \geq 2$. The condition $2k_j \leq b_+ - 1$ in Theorem 3.1 is equivalent to

$$-15n \leq m_+ - m_- \leq \frac{15}{2}n.$$

Recall that the Seiberg-Witten invariant of the $E(n)$ is $c_{n-2} := C_{n-2}^{\frac{n-2}{2}}$. By the mod p vanishing theorem of the Seiberg-Witten invariants, we have the following theorem.

Theorem 4.2 *If $c_{n-2} \not\equiv 0 \bmod 3$ and $n \equiv 2 \bmod 6$, then there exists a locally linear $\mathbb{Z}_3$-action on $E(n)$ stisfying the following conditions, which is nonsmoothable with respect to infinitely many smooth structures on $E(n)$:*

(1) $-15n \leq m_+ - m_- \leq \frac{15}{2}n$,

(2) $m_+ - m_- \equiv -3 \bmod 9$.

The proof of above theorem is divided into two steps. In the first step, to construct a locally linear action, we use the realization theorem due to Edmonds and Ewing [5]. In the second step, we use the mod p vanishing theorem of the Seiberg-Witten invariants to give a constraint on smooth $\mathbb{Z}_3$-action. In fact the proof can be done by imitating the method in [13], so we omit here.

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