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Torsions of 3-Dimensional Small Covers

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Abstract In this paper, it is shown that for a 3-dimensional small cover M over a polytope P, there are only 2-torsions in $H_1(M;\mathbb{Z})$. Moreover, the mod 2 Betti number growth of finite covers of M is studied.

Keywords Mod 2 Betti number growth, Small cover, Hyperbolic manifolds
2000 MR Subject Classification 57M10, 20F55, 51M10

1 Introduction

1.1 Small covers

Small covers, or Coxeter orbifolds, were studied by Davis and Januszkiewicz [7]. They are a class of manifolds which admit locally standard $\mathbb{Z}_2^n$-actions, such that the orbit spaces are n-dimensional simple polyhedra. The algebraic and topological properties of a small cover are closely related to the combinatorics of the orbit polyhedron and the coloring on its boundary. In this paper, we focus on the 3-dimensional case.

Definition 1.1 *Let P be a 3-dimensional polytope, Γ be a trivalent graph in ∂P which gives a cell decomposition of ∂P. A $\mathbb{Z}_2^3$-coloring is a map $\lambda : \partial P - \Gamma \longrightarrow \mathbb{Z}_2^3$ such that $\lambda(f_{i_1})$, $\lambda(f_{i_2})$ and $\lambda(f_{i_3})$ generate $\mathbb{Z}_2^3$ when $\overline{f}_{i_1}$, $\overline{f}_{i_2}$ and $\overline{f}_{i_3}$ are sharing a common vertex, where f_{i_j} is a connected component in $\partial P - \Gamma$ and $\overline{f}_{i_j}$ is the corresponding closure.*

From a $\mathbb{Z}_2^3$-coloring λ and the trivial principal $\mathbb{Z}_2^3$-bundle over P, we can get a 3-manifold which depends only on the coloring λ. Preparing eight copies of P, namely $P \times \mathbb{Z}_2^3$, then a quotient space $M(P, \lambda)$ can be constructed under the following equivalent relation:

$$(x, \alpha_1) \sim (y, \alpha_2) \Leftrightarrow \begin{cases} x = y, \quad \alpha_1 = \alpha_2, & \text{if } x \in \operatorname{Int} P, \\ x = y, \quad \alpha_1\alpha_2^{-1} \in G_f, & \text{if } x \in \partial P. \end{cases} \tag{1.1}$$

Here G_f is the subgroup generated by $\lambda(f_{i_1}), \cdots, \lambda(f_{i_k})$, where $f = \overline{f}_{i_1} \cap \cdots \cap \overline{f}_{i_k}$ is the only i-face, $0 \le i \le 2$, that contains x as an interior point. It is easy to see that $M(P, \lambda)$ is a closed 3-manifold and we call it a small cover over P.

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For example, if we consider a coloring on a tetrahedron that the four faces are colored by e_1, e_2, e_3 and $e_1+e_2+e_3$ respectively, then following the construction above, we can get a closed orientable 3-manifold $\mathbb{RP}^3$. It should be noticed that a tetrahedron admits a unique right-angled spherical structure. And those spherical structures on the four copies of the tetrahedron are glued together to form the unique spherical structure on $\mathbb{RP}^3$.

Choi-park [6] once discussed the torsions of real topological toric manifolds. For any positive odd number q, they constructed a real topological toric manifold N whose integral cohomology has a q-torsion. For a large q, the manifold being constructed will be of large dimension. What's more, they gave a formula for the cohomology groups of real topological toric manifolds with coefficient $\mathbb{Z}_p$. Even though not being stated explicitly, from [6, Theorem 4.6] we can see that there are no odd torsions in a 3-dimensional small cover. Letting P be a 3-dimensional polytope and $M=M(P,\lambda)$ be any small cover over P with at most 2-torsion in cohomology, Trevisan [22] gave out all the possible integral homology and cohomology groups. And it is still unknown about the existence of any 2^k-torsion for $k\geq 2$ in a 3-dimensional small cover. In this paper, we show the following result.

Theorem 1.1 *Let $M=M(P,\lambda)$ be a 3-dimensional small cover. Then there are only $\mathbb{Z}_2$-torsions in $H_1(M;\mathbb{Z})$.*

1.2 Asymptotic behaviors of mod 2 Betti numbers of finite covers

Let G be an infinite group, $G_i<G$ be a sequence of finite-index subgroups of G. If $G_{i+1}<G_i$, then we say $\{G_i\}_{i=1}^{\infty}$ is a tower of G. If $\bigcap\limits_{i=1}^{\infty} G_i=1$, then $\{G_i\}_{i=1}^{\infty}$ is co-final. If $G_i\lhd G$, then $\{G_i\}_{i=1}^{\infty}$ is a regular sequence of G.

The asymptotic behavior of algebraic invariants in finite covers M_i of a 3-manifold M depends on the sequence $\{M_i\}_{i=1}^{\infty}$ of M. For any co-final regular tower $\{M_i\}_{i=1}^{\infty}$ of a hyperbolic 3-manifold M, $\lim\limits_{i\to\infty}\frac{b_1(M_i)}{[M:M_i]}$ equals to the L^2-Betti number of $\mathbb{H}^3$. And it is zero as shown in [17], namely the normalized first Betti number converges to zero for co-final regular towers. But this is not true for all the co-final sequences of M, see Girão [11, Theorem 3.1] as well as [9, 13] for related topics. And Girão [10–11] also studied the rank gradients of some hyperbolic 3-manifolds.

There are many works on the asymptotic behavior of homology torsions in finite covers of a 3-manifold, see [4, 14, 16, 18, 21]. In particular, it is conjectured that torsion growth of a co-final normal subgroup sequence of a hyperbolic 3-manifold M is related to the volume of M (see [4, 18]). It is also conjectured that exponential torsion growth for any sequence (might not be normal, even not co-final) of a fibered 3-manifold N_ϕ is related to the virtual homology entropy of ϕ (see [14]). We show the following theorem.

Theorem 1.2 *Let M be a small cover over a right-angled hyperbolic polytope. Then M has a co-final finite-cover sequence M_i such that $H_1(M_i;\mathbb{Z}_2)$ has exponential growth.*

Theorem 1.2 can be compared with Theorem 1.2 of [15], where Lackenby proved that any finitely generated, discrete, non-elementary subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with torsions has a tower of subgroups with linear-increasing mod p homologies for some prime p. See also [19, Theorem 2.2] as well as an example in [5] where the closed hyperbolic 3-manifold M has a normal co-final sequence M_i such that $H_1(M_i;\mathbb{Z}_3)$ is always $\mathbb{Z}_3^3$. In [8, p. 64], it is stated that "At the same time, very deep recent work of Wise on quasi-convex hierarchies combined with a theorem of Lackenby implies that for every hyperbolic 3-manifold group G and every prime p, the p-gradient of G is zero". The p-gradient of G is defined as $RG_p(G) = \inf\limits_{H} \frac{d_p(H)-1}{[G:H]}$, where H runs over all subnormal subgroups of finite p-power index in G and $d_p(H)$ is the rank of $\frac{H}{[H,H]H^p}$. So Theorem 1.2 shows that there are differences between subnormal sequence and general sequence in considering their p-gradients.

2 Preliminaries

2.1 Small cover

For an n-dimensional simple polytope P in Lobachevski n-space $\mathbb{H}^n$, Davis and Januszkiewicz showed that if there is a 2^n-index torsion-free subgroup Γ of the Coxeter group over P, then the manifold corresponding to this subgroup, namely the Clifford-Klein space form $\mathbb{H}^n/\Gamma$, is a small cover over P. It is a G-manifold with group action $\mathbb{Z}_2^n$.

Moreover, there is another equivalent but more practical way in describing small cover by using the language of coloring: Let $\mathcal{F}(P) = \{F_1, F_2, \cdots, F_m\}$ be the set of all co-dimensional one faces of P. Such face is named as facet. Then we define a $\mathbb{Z}_2^n$-coloring characteristic function

$$\lambda : \mathcal{F}(L) = \{F_1, F_2, \cdots, F_m\} \longrightarrow \mathbb{Z}_2^n,$$

where $\lambda(F_1), \lambda(F_2), \cdots, \lambda(F_n)$ generate $\mathbb{Z}_2^n$ when the facets $F_1, F_2, \cdots, F_n$ share a common vertex. This condition is called the non-singular condition. And the corresponding characteristic matrix is defined to be the matrix obtained by placing the image of facets $F_1, \cdots, F_m$ under λ column by column. By the way, such function λ is not destined to exist and its existence is concerned with the Buchstaber invariant.

If characteristic function λ is defined successfully, then we can construct manifold $M(P, \lambda) := P \times \mathbb{Z}_2^n / \sim$ by the following equivalent relation:

$$(x, g_1) \sim (y, g_2) \Leftrightarrow \begin{cases} x = y, \quad g_1 = g_2, & \text{if } x \in \operatorname{Int} P, \\ x = y, \quad g_1^{-1} g_2 \in G_f, & \text{if } x \in \partial P, \end{cases} \tag{2.1}$$

where $f = F_{i_1} \cap \cdots \cap F_{i_{n-k}}$, $0 \le k \le n-1$, is the unique co-dimensional $(n-k)$-face that contains x as an interior point, and G_f is the subgroup generated by $\lambda(F_{i_1}), \lambda(F_{i_2}), \cdots, \lambda(F_{i_{n-k}})$. $M(P, \lambda)$ is called a small cover over P. For example, defining a $\mathbb{Z}_2^2$-coloring characteristic function λ on the square as show in Figure 1, where $(1, 0) = e_1$, $(0, 1) = e_2$ are the standard basis of $\mathbb{Z}_2^2$.

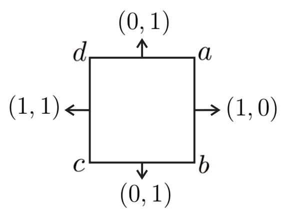

Figure 1 A coloring on the square.

Then by gluing the four pieces together along the facets according to the equivalent relation, we can finally get the Klein bottle.

3 Torsions in Small Covers

In this section, we show that there are only 2-torsions in $H_1(M;\mathbb{Z})$ for a 3-dimensional small cover M, which can be viewed as a refinement of Theorem 3.1 of [7]. We start from a construction in [7].

Lemma 3.1 *There is a presentation matrix H for $H_1(M;\mathbb{Z})$, whose non-trivial entries are either 2 or -2. Moreover, there are at most two non-trivial entries in each row. If a row has exactly two non-trivial entries, then they must be 2 and -2.*

Proof Let P be a 3-dimensional polytope and we embed P in $\mathbb{R}^3$. Choosing a vector μ in $\mathbb{R}^3$ which is generic to P. Then adopting a function $\phi:\mathbb{R}^3\to\mathbb{R}$ by $\phi(x)=\langle x,\mu\rangle$, where $\langle x,\mu\rangle$ is the inner product. Now using $\phi(x)$ as a height function, we can get a directed graph on the 1-skeleton of P.

There is a unique vertex, such that all the three adjacent edges point away from it. We denote this vertex by I and this is the unique "bottom" vertex. There is also a unique vertex, such that all the three adjacent edges point towards it. We denote this vertex by T and this is the unique "top" vertex. For other vertices, say O, there are two possibilities:

Case 1 We have two of the three edges adjacent to O point away from it while one points towards it. By h-vector and some simple combinatorial analysis, it is easy to see there are totally $m-3$ such type of vertices, where $m=|\mathcal{F}(P)|$. We denote these vertices by $V_1,V_2,V_3,\cdots,V_{m-3}$.

Case 2 We have one of the three edges adjacent to O points away from it while the other two point towards it. It is easy to see that there are still $m-3$ vertices of this type. We denote them by $W_1,W_2,W_3,\cdots,W_{m-3}$.

For each vertex V_i, we take E_i to be the unique closed edge that runs towards V_i. Defining G to be the union of all of these E_i. Then G is a connected graph in the 1-skeleton of P which contains I and does not contain T. We use a cube $\mathcal{D}$ to explain all these notions in Figure 2.

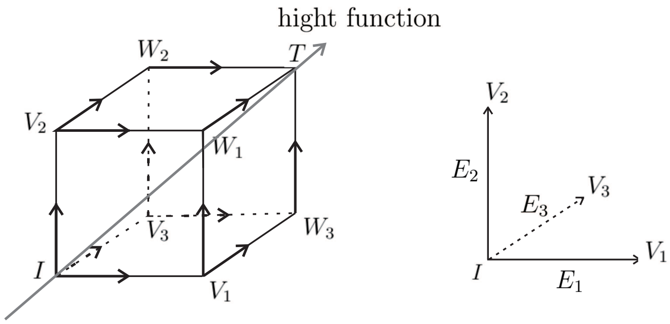

Figure 2 Directed graph of a cube $\mathcal{D}$ and its union of E_i.

Considering $\pi^{-1}(G)$ in M, where $\pi:M\to P$ is the projection map. Now in the small cover $M=M(P,\lambda)$, $\pi^{-1}(G)$ is a graph which is a double of G along the vertex set (see [7, Lemma 1.3]. Namely, as shown in Figure 3, there are two copies of G, which are denoted by G' and G'' respectively, in the corresponding small cover M. Their vertices are marked by V_1', V_2', V_3',$\cdots$,V_{m-3}' and V_1'', V_2'', V_3'',$\cdots$,V_{m-3}''. And their edges are labeled by E_1', E_2', E_3',$\cdots$,E_{m-3}' and E_1'', E_2'', E_3'',$\cdots$,E_{m-3}''. Then $\pi^{-1}(G)=G'\bigsqcup G''/V_i'\sim V_i''$, $1\le i\le m-3$.

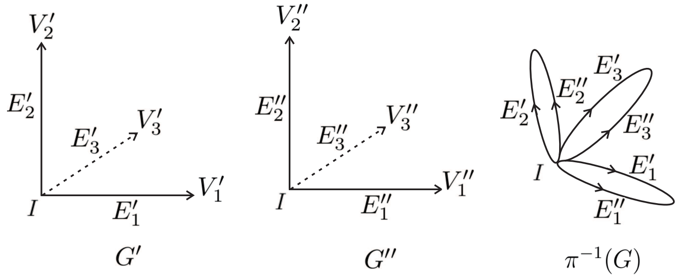

Figure 3 G', G'' and $\pi^{-1}(G)$ of cube $\mathcal{D}$.

For each edge E_i with respect to V_i, we label the colorings of faces adjacent to V_i by α, β and γ. Among them, α and β are the two colorings of faces that are adjacent to E_i. Then $E_i \times \{1\}$, $E_i \times \{\alpha\}$, $E_i \times \{\beta\}$ and $E_i \times \{\alpha+\beta\}$ are glued together to form an edge in $\pi^{-1}(G)$. This edge is what we denote by E_i' in $\pi^{-1}(G)$. Another edge in $\pi^{-1}(G)$ with the representative $E_i \times \{\gamma\}$ is exactly what we mean by E_i'' in $\pi^{-1}(G)$.

Now for each vertex W_j, there is a 2-cell corresponding to it. Assume that the two edges running towards W_j are l' and l''. The face containing l' and l'' is F_j. And the colorings of the two faces which are adjacent to l' and l'' respectively are μ and ν. In 3-manifold M, four copies of $F_j - (\partial F_j - l' - l'')$ are glued together to build an open embedded disk along pre-images of l' and l'' under π, then we denoted this disk by D_j. Illustrations about all these descriptions are shown in Figures 4–5.

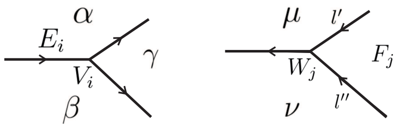

Figure 4 Denotation illustrations.

Furthermore $M - \big(\pi^{-1}(G) \bigcup\limits_{j=1}^{m-3} D_j\big)$ is an open 3-ball, which is the union of eight copies of $P - \mathrm{Cl}(\partial P - U_1 - U_2 - U_3)$, where U_1, U_2, U_3 are the three faces adjacent to the vertex T.

Now $H_1(M;\mathbb{Z})$ can be obtained by quotienting D_j out from $H_1(\pi^{-1}(G))$, where $\{(E_i' \bigsqcup - E_i''/V_i' \sim V_i'')\}_{i=1}^{m-3}$ is a basis of $H_1(\pi^{-1}(G);\mathbb{Z})$ and each D_j gives a relation. For a vertex V_i in F_j, where F_j is the face corresponding to the vertex W_j, we have $\frac{\partial W_j}{\partial V_i} = E_i \times \{1\} - E_i \times \{\mu\} + E_i \times \{\mu+\nu\} - E_i \times \{\nu\}$, following the locating relations as shown in Figure 5.

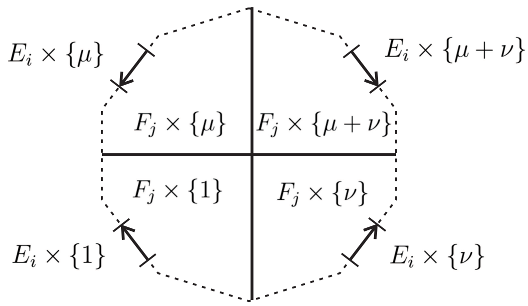

Figure 5 Building up a disk.

Now the matrix $\left(\frac{\partial W_j}{\partial V_i}\right)_{1\leq i,j\leq m-3}$ is a presentation matrix of $H_1(M;\mathbb{Z})$. We will furtherly figure out that the non-trivial entries of $\frac{\partial W_j}{\partial V_i}$ are either $2(E_i'-E_i'')$ or $-2(E_i'-E_i'')$.

We picture the relative locations of E_i and F_j as well as some related colorings in Figure 6, here E_i is on the boundary of F_j, namely F_j would contribute to the relation for quotient. We firstly adjust the colorings of the three facets adjacent to V_i to be e_1, e_2 and e_3. This can be realized by simply performing a suitable coordinate transformation. And the other two faces adjacent to W_j are denoted as d and e.

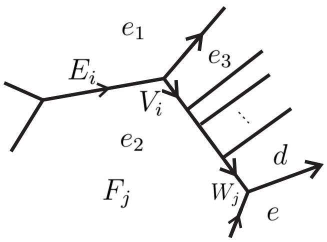

Figure 6 Relative locations with fixed coloring basis.

We list out all the possible colorings on face d and face e in Table 1 based on the non-singular condition. There are totally 24 cases. The colorings of face e placed on the right of a certain row are the only four choices when face d is colored by the coloring placed in the left column of that row. For example, when d is colored by e_3, then e can only be colored by e_1, e_1+e_2, e_1+e_3 and $e_1+e_2+e_3$.

Table 1 All possible colorings for $(d,\, e)$ and corresponding $\left|\frac{\partial W_j}{\partial V_i}\right|$.

Colorings on face d	Colorings on face c and corresponding $\left\|\frac{\partial W_j}{\partial V_i}\right\|$			
e_3	e_1	e_1+e_2	e_1+e_3	$e_1+e_2+e_3$
	0	0	$2(E'-E'')$	$2(E'-E'')$
e_1	e_3	e_1+e_3	e_2+e_3	$e_1+e_2+e_3$
	0	0	0	0
e_1+e_2	e_3	e_1+e_3	e_2+e_3	$e_1+e_2+e_3$
	0	0	0	0
e_1+e_3	e_1	e_3	e_2+e_3	e_1+e_2
	0	$2(E'-E'')$	$2(E'-E'')$	0
e_2+e_3	e_1	e_1+e_2	e_1+e_3	$e_1+e_2+e_3$
	0	0	$2(E'-E'')$	$2(E'-E'')$
$e_1+e_2+e_3$	e_1	e_3	e_2+e_3	e_1+e_2
	0	$2(E'-E'')$	$2(E'-E'')$	0

And then we can discuss all the possible $\left|\frac{\partial W_j}{\partial V_i}\right|$. The results are placed just below the coloring cases respectively as shown in Table 1.

Therefore, using the notations claimed before and marking them in Figure 7, we can make the conclusion as follows.

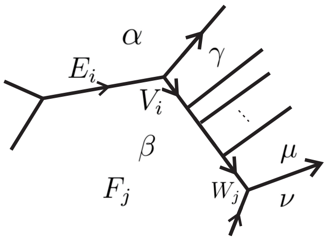

Figure 7 Relative locations for general case.

(1) If $\mu \in \{\alpha, \beta, \alpha+\beta\}$, then $\frac{\partial W_j}{\partial V_i} = 0$.

(2) If $\nu \in \{\alpha, \beta, \alpha+\beta\}$, then $\frac{\partial W_j}{\partial V_i} = 0$ as well.

(3) If $\{\mu, \nu\} \cap \{\alpha, \beta, \alpha+\beta\} = \emptyset$, μ and ν both lie in $\{\gamma, \alpha+\gamma, \beta+\gamma, \alpha+\beta+\gamma\}$, so $\mu+\nu \in \{\alpha, \beta, \alpha+\beta\}$. And then in $\pi^{-1}(G)$, $E_i \times \{1\} = E_i \times \{\mu+\nu\}$, $E_i \times \{\mu\} = E_i \times \{\nu\}$. Thus $E_i \times \{1\} - E_i \times \{\mu\} + E_i \times \{\mu+\nu\} - E_i \times \{\nu\} = 2(E_i \times \{1\} - E_i \times \{\mu\})$.

We always see P from the outside, which means that we always orient the boundary of F_j, the face corresponds to W_j, anti-o'clockly. For the edge E_i, which corresponds to a vertex V_i in ∂F_j, its orientation may or may not be the same with the orientation derived form ∂F_j. So we should add either plus or minus sign to the absolute value of $\frac{\partial W_j}{\partial V_i}$. Namely we have

$$\frac{\partial W_j}{\partial V_i} = \begin{cases} 0, & \text{if } \{\mu, \nu\} \cap \{\alpha, \beta, \alpha+\beta\} \neq \emptyset, \\ \pm 2, & \text{if } \{\mu, \nu\} \cap \{\alpha, \beta, \alpha+\beta\} = \emptyset. \end{cases} \tag{3.1}$$

Therefore, in the presentation matrix $H = \big(\frac{\partial W_j}{\partial V_i}\big)_{1 \le i, j \le m-3}$, the non-trivial entries are either 2 or -2. There are only two faces, denoted by F_{j_1} and F_{j_2}, that are adjacent to E_i. If the orientation of E_i agrees with the orientation of ∂F_{j_1}, then it will definitely disagree with the orientation of ∂F_{j_2}. Thus there are at most two non-trivial entries in each row. Furthermore, if a row possesses two non-trivial entries, then they must be 2 and -2.

Proof of Theorem 1.1 By transposing H and rearranging the rows, we get a new matrix $(A_{(m-3)\times m_1} \mid B_{(m-3)\times m_2} \mid C_{(m-3)\times m_3})$, where $m_1+m_2+m_3 = m-3$. Here A is a zero matrix, B has only one non-trivial entry, 2 or -2 in each column, and C is a matrix with exactly two non-trivial entries, 2 and -2, in every column.

We first multiply (-1) if necessary to make all the non-trivial entries in B to be 2. Furthermore we suitably replace some column i by column $i + (\pm 1)\times$column j, where $m_1+1 \le i \le m-3$ $m_1+1 \le j \le m_1+m_2$, and reorder the columns to obtain a new matrix, also called by H, in the following form

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\mathrm{I} & 0 \\ 0 & 0 & C_1 \end{pmatrix},$$

where each column of C_1 has exactly two non-trivial entries (see [20, Chapter 8]).

Let $C_1 = (c_{i,j})$, and assuming $c_{1,1} = 2$ and $c_{1,2} = -2$. We add C_1's first row to its second row and get a matrix C_2. Moreover by adding the first column of C_2 (might times with -1) to some other columns of C_2, we obtain a matrix, denoted also by C_2, of the following form

$$C_2 = \begin{pmatrix} 2 & 0 \\ 0 & C_3 \end{pmatrix}.$$

Now we can easily see that, in each column of C_3, there are still at most two non-trivial entries, 2 and -2, in each column. By reperforming the processes that were applied for B and C_1, we can finally get a presentation matrix of the form

$$H = \begin{pmatrix} 0 & 0 \\ 0 & 2\mathrm{I} \end{pmatrix}.$$

Thus there are only 2-torsions in $H_1(M;\mathbb{Z})$.

Remark 3.1 Our proof above only holds in 3-dimensional small covers, and it is not true for higher dimensions.

4 Mod 2 Betti Numbers of Finite Covers of Small Covers.

The following simple lemma is well-known, for example, see [12].

Lemma 4.1 *Let P be a 3-dimensional right-angled hyperbolic polytope. Then P has at least one pentagon face. Moreover, there is no face F of P, such that every pentagon in P is adjacent to F.*

Proof Since P is a right-angled hyperbolic polytope, by Andreev's theorem (see [2]), there is no triangle or quadrilateral in $\mathcal{F}(P)$. Denoted by f_k the number of k-gons among the faces of P, $k \geq 5$, then a simple calculation by means of Euler's formula implies that f_5 is non-zero.

Moreover, if P has a face F which is adjacent to every pentagon in P, then by doubling P along F, we can get a right-angled hyperbolic polytope Q such that every face of it has at least six edges, contradicting the previous fact.

Theorem 4.1 *Let P be a 3-dimensional right-angled hyperbolic polytope, $G(P)$ be the Coxeter group associated to P. Then there are hyperbolic polytopes P_i, where each P_i is a doubling of P_{i-1} along a face of P_{i-1}, such that*

$$\lim_{n\to\infty} \frac{\log(H_1(G(P_i)))}{|P_i : P|} = (m-7)\log 2, \tag{4.1}$$

where m is the number of faces of P, and P_0 is defined to be P.

Proof The Coxeter group of P is given by

$$G(P) = \{x_1, x_2, \cdots, x_m \mid x_i^2 = 1,\ x_i x_j = x_j x_i,\ \text{if } F_i \cap F_j \neq \emptyset\}.$$

So $H_1(G(P)) = \mathbb{Z}_2^m$. Now let $\#\{P^{(2)}\}$ be the number of faces of P, and P_{i+1} be a doubling of P_i along a pentagon. We have $\#\{P_{i+1}^{(2)}\} = 2(\#\{P_i^{(2)}\}) - 7$. So $\#\{P_k^{(2)}\} = 2^k m - 7(2^k - 1)$. Then

$$\lim_{n\to\infty} \frac{\log(2^k m - 7(2^k - 1))}{2^k} = (m-7)\log 2. \tag{4.2}$$

Remark 4.1 Comparing to Girão's approaches (see [10–11]) on rank gradients of small covers, Atkinson's result (see [3]) on the relationship between volume and the number of vertices of a hyperbolic polytope is not necessary in our proof.

We now make a refinement of the proof and result of Theorem 4.1.

Theorem 4.2 *Let P be a 3-dimensional right-angled hyperbolic polytope, $G(P)$ be the Coxeter group associated to P. Then there are hyperbolic polytopes P_i, where each P_i is a doubling of P_{i-1} along a face of P_{i-1}, such that $G(P_i)$ is a co-final sequences in $G(P)$ with*

$$\lim_{n\to\infty}\frac{\log(H_1(G(P_i)))}{|P_i:P|}=(m-7)\log 2, \tag{4.3}$$

where m is the number of faces of P and P_0 is defined to be P. That is, the homology torsion of finite covers grows exponentially.

Proof We embed P in $\mathbb{H}^3$ and fix a point x in the interior of P. Denoted by $F_1, F_2, \cdots, F_m$ the faces of P and it is satisfied that $d(F_1,x)\le d(F_2,x)\cdots\le d(F_m,x)$. We assume that F_1 has a edges. If F_1 is a pentagon, then we double P along F_1 and denote the resulting polyhedron by Q_1. Otherwise, from Lemma 4.1, there will be a minimum i such that F_i is a pentagon and not adjacent to F_1.

We now double P along F_i while the initial F_1 remains in Q_1. Then there is another face of Q_1 which is a pentagon and is not adjacent to F_1. We double Q_1 along that face and get a polyhedron Q_2. Denote the polyhedron, that results from doubling P for k times, by Q_k. For an arbitrary ϵ, we can make k large enough to satisfy $\frac{a+7}{2^k}\le\frac{\epsilon}{4}$. It can be calculated that Q_k has $2^k m-(2^k-1)7$ faces. We now double Q_k along F_1 and denote the resulting polyhedron by P_1. Then P_1 has $2(2^k m-(2^k-1)7)-a-2$ faces. We have

$$\frac{2(2^k m-(2^k-1)7)-a-2}{2^{k+1}}\ge(m-7)-\frac{\epsilon}{2}. \tag{4.4}$$

Now x is also in the interior of P_1, and the interior of F_1 lies in the interior of P_1. We take the minimum i when F_i lies in the boundary of P_1 and double P_1 along pentagons many times as above to obtain a polytope Q. Furthermore we double Q along the face that contains F_1 and obtain a polytope P_2, such that

$$\frac{\#(P_2^{(2)})}{\mathrm{vol}(P_2):\mathrm{vol}(P)}\ge(m-7)-\Big(\frac{\epsilon}{4}+\frac{\epsilon}{8}\Big). \tag{4.5}$$

As $d(x,\partial P)\ge d(x,\partial P_1)\ge d(x,\partial P_2)$, we can get a polytope R by repeating the above process for at most m times, such that

$$\frac{\#(R^{(2)})}{\mathrm{vol}(R):\mathrm{vol}(P)}\ge(m-7)-\frac{\epsilon}{2}. \tag{4.6}$$

Now we have $d(x,\partial P)\ge d(x,\partial R)$. Then by taking R as the initial P and applying previous operations, we can have a polytope S, such that

$$\frac{\#(S^{(2)})}{\mathrm{vol}(S):\mathrm{vol}(P)}\ge(m-7)-\Big(\frac{\epsilon}{2}+\frac{\epsilon}{4}\Big). \tag{4.7}$$

In fact, the distance between x and the boundary of above polytopies diverges to infinite by repeating the above process, then as in [11, Section 5] (which is contributed by Agol [1]), the Coxeter groups related to the polytopies we construct form a co-finial sequence. Now the sequence above have torsion growth $(m-7)\log 2$ by the arbitrariness of ϵ.

Proof of Theorem 1.2 We proved in Theorem 4.1 that for a right-angled hyperbolic polytope P and the Coxeter group $G(P)$ associated to P, there are hyperbolic polytopes P_i, where each P_i is a doubling of P_{i-1} along a face of P_{i-1} and $P_0 = P$, such that $G(P_i)$ is a co-final sequence in $G(P)$ with the numbers of facets of P_i growing exponentially. Thus for any small cover M_i over P_i, by [7, Theorem 3.1], $H_1(M_i; \mathbb{Z}_2)$ has an exponential torsion growth with $\pi_1(M_i)$ a co-final sequence.

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