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Wedge Operations and Doubling Operations of Real Toric Manifolds

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Abstract This paper deals with two things. First, the cohomology of canonical extensions of real topological toric manifolds is computed when coefficient ring G is a commutative ring in which 2 is unit in G. Second, the author focuses on a specific canonical extensions called doublings and presents their various properties. They include existence of infinitely many real topological toric manifolds admitting complex structures, and a way to construct infinitely many real toric manifolds which have an odd torsion in their cohomology groups. Moreover, some questions about real topological toric manifolds related to Halperin's toral rank conjecture are presented.

Keywords Real toric manifold, Small cover, Real topological toric manifold, Cohomology ring, Doubling, Simplicial wedge, Rational homology sphere
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1 Introduction

The real topological toric manifold is a topological generalization of the real toric manifold. Every n-dimensional real topological toric manifold M is characterized by a pure $(n-1)$-dimensional simplicial complex K, called the base complex of M, and a map $\lambda\colon V(K) \to \mathbb{Z}_2^n$, satisfying certain conditions, called a characteristic map when $V(K)$ is the set of vertices of K. The pair (K, λ) is called a characteristic pair and we write $M = M(K, \lambda)$.

For a real topological toric manifold M, one can define new real topological toric manifolds called canonical extensions of M. This concept is originally due to Ewald (the namer of "canonical extension") [6] for toric manifolds and Bahri-Bendersky-Cohen-Gitler [1] for (real) topological toric manifolds (their original work is for quasitoric manifolds and small covers, but their method extends well without change for topological toric manifolds and real topological toric manifolds). We remark that the canonical extension is deeply related with a classical operation on simplicial complexes called the simplicial wedge operation, which is performed at the base complex.

The integral cohomology ring of a real topological toric manifold M is difficult to compute in general unlike that of a topological toric manifold. Recently Suciu-Trevisan [9–10] have found a formula to compute the homology $H_*(M; \mathbb{Q})$, and Choi and the author [5] have strengthened their result for the cohomology $H^*(M; G)$, where G is a commutative ring in which 2 is a unit. Using their result, we provide a method to compute the cohomology of canonical extensions of a real topological toric manifold (see Section 3).

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There is a special kind of canonical extension called doubling. Roughly speaking, the doubling is the canonical extension performed at each vertex of the base complex. The doubling operation is especially intriguing because it has numerous interesting topological and geometric properties (see [1, 11]). Some of them, in the category of real topological toric manifolds, are covered in this paper. The main result is as follows.

Theorem 1.1 *For a real topological toric manifold $M = M(K, \lambda)$, its double M' satisfies the following properties:*

(1) *There is a graded ring isomorphism*

$$H^*(M'; G) \cong H^*(\mathcal{Z}_K; G),$$

where $\mathcal{Z}_K$ is the moment-angle complex of K, and G is a commutative ring such that 2 is a unit. In particular, M' is orientable.

(2) *If M' has even dimension, then there is a complex structure on M'.*

(3) *If there exists a characteristic map $\widetilde{\lambda}\colon V(K) \to \mathbb{Z}^n$ such that its* mod 2 *reduction is λ, then M' admits a free T^{m-n} action whose orbit space is the topological toric manifold $M(K, \widetilde{\lambda})$.*

Recall that the manifold $M(K, \widetilde{\lambda})$ is called a lifting of M. An immediate corollary is as follows. This gives an easier way to construct infinitely many real topological toric manifolds with arbitrary odd torsion, which was previously accomplished in [5].

Corollary 1.1 *Let $M = M(K, \lambda)$ be a real topological toric manifold such that $H^*(\mathcal{Z}_K)$ has a q-torsion element for some odd integer q. Then $H^*(M')$ has a q-torsion element.*

Moreover, we present questions about real topological toric manifolds related to Halperin's toral rank conjecture.

This paper is organized as follows. In Section 2, we review the definition of topological toric manifolds and real topological toric manifolds, and characteristic pairs corresponding to them, and simplicial wedge construction and canonical extensions. In Section 3, we compute cohomology of canonical extensions of a real topological toric manifold. In Section 4, we present various properties of doublings of real topological toric manifolds. Finally, in Section 5, we give some questions and conjectures related to Halperin's toral rank conjecture and the sum of Betti numbers of real topological toric manifolds.

2 Topological Toric Manifolds and Characteristic Pairs

A topological toric manifold defined in [7] is a closed smooth $2n$-manifold M with an effective smooth $(\mathbb{C}^*)^n$-action such that there is an open and dense orbit and M is covered by finitely many invariant open subsets each of which is equivariantly diffeomorphic to a smooth representation space of $(\mathbb{C}^*)^n$. Similarly, we say that a closed smooth manifold M of dimension n with an effective smooth action of $(\mathbb{R}^*)^n$ having an open dense orbit is a real topological toric manifold if it is covered by finitely many invariant open subsets each of which is equivariantly diffeomorphic to a direct sum of real one-dimensional smooth representation spaces of $(\mathbb{R}^*)^n$.

For a simplicial complex K, the vertex set is denoted by $V(K)$. A pure simplicial complex K of dimension $n-1$ is called star-shaped if there is a geometric realization $|K| \subset \mathbb{R}^n$ such that any ray in $\mathbb{R}^n$ from the origin intersects $|K|$ once and only once. A star-shaped simplicial

complex is automatically a triangulated sphere. The following is implied by [7, Theorem 7.2]. As a manifold with the restricted action of the compact torus $T^n = (S^1)^n \subset (\mathbb{C}^*)^n$, a topological toric manifold is characterized by the following information:

(1) a star-shaped simplicial complex K;

(2) a map $\lambda\colon V(K) \to \mathbb{Z}^n$ such that for any $\sigma \in K$, $\{\lambda(i) \mid i \in \sigma\}$ is a subset of an integral basis of $\mathbb{Z}^n$.

Similarly, as a $\mathbb{Z}_2^n$-manifold, any real topological toric manifold is determined by the pair (K, λ) such that

(1) a star-shaped simplicial complex K;

(2) a map $\lambda\colon V(K) \to \mathbb{Z}_2^n$ such that for any $\sigma \in K$, $\{\lambda(i) \mid i \in \sigma\}$ is a linearly independent set in $\mathbb{Z}_2^n$.

In either case, the condition (2) is called the non-singularity condition, K is the base complex of M, and λ is called a characteristic map. The pair (K, λ) is called a characteristic pair. We denote by $M(K, \lambda)$ the (real) topological toric manifold determined by (K, λ). In this paper, we regard topological toric manifolds as T^n-manifolds and real topological toric manifolds as $\mathbb{Z}_2^n$-manifolds.

Now we define the notion of canonical extensions of $M(K, \lambda)$. We briefly present the construction of [1] here. Let K be a simplicial complex of dimension $n-1$ on vertices $V(K) = [m] := \{1, 2, \cdots, m\}$. A subset of $V(K)$ is called a non-face of K if it is not a face of K. A non-face is minimal if any proper subset is a face of K. Note that a simplicial complex is determined by its minimal non-faces.

In the setting above, let $J = (j_1, \cdots, j_m) \in \mathbb{N}^m$ be a vector of positive integers. For the vertex set $V(K) = \{1, \cdots, m\}$, $V(K(J))$ is a set

$$V(K(J)) := \{\underbrace{1_1, 1_2, \cdots, 1_{j_1}}, \underbrace{2_1, 2_2, \cdots, 2_{j_2}}, \cdots, \underbrace{m_1, \cdots, m_{j_m}}\}$$

of $\sum_{i=1}^{m} j_i$ elements. For a subset $\omega = \{i_1, \cdots, i_k\}$ of $[m]$, $\omega(J)$ is denoted as

$$\omega(J) := \{\underbrace{(i_1)_1, \cdots, (i_1)_{j_{i_1}}}, \underbrace{(i_2)_1, \cdots, (i_2)_{j_{i_2}}}, \cdots, \underbrace{(i_k)_1, \cdots, (i_k)_{j_{i_k}}}\} \subset V(K(J)). \tag{2.1}$$

Denote by $K(J)$ the simplicial complex with vertex set $V(K(J))$ with minimal non-faces $\omega(J)$ for each minimal non-face ω of K. When $J = (1, \cdots, 1, 2, 1, \cdots, 1)$ is the m-tuple with 2 as the i-th entry, it is evident that

$$K(J) = (I \star \mathrm{Lk}_K\{i\}) \cup (\partial I \star (K \setminus \{i\})),$$

where $K \setminus \{i\}$ is the full subcomplex with $m-1$ vertices except i and $\mathrm{Lk}_K\{i\}$ is the link of K at i. We sometimes use the notation $K(J) = \mathrm{wed}_i(K)$ and call it the simplicial wedge of K at i or simply the wedge. The operation itself is called the simplicial wedge construction or wedging. By consecutive application of wedgings to K, one can construct $K(J)$ for any J.

Any characteristic map $\lambda\colon [m] \to \mathbb{Z}_2^n$ can be regarded as an $(n \times m)$-matrix written as Λ again,

$$\Lambda = \begin{pmatrix} \lambda(1) & \cdots & \lambda(m) \end{pmatrix}_{(n \times m)},$$

which is called a characteristic matrix. For convenience of notation, we write

$$C_i := \begin{pmatrix} 0 & \cdots & 0 & \lambda(i) \end{pmatrix}_{n\times j_i}$$

for $i = 1, \cdots, m$ and

$$S_k := \begin{pmatrix} 1 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 1 \end{pmatrix}_{(k-1)\times k}$$

for $k \geq 1$. If $k = 1$ then S_k is an empty matrix. The following definition is the mod 2 version of that introduced in [1].

Definition 2.1 *Let K be an $(n-1)$-dimensional star-shaped simplicial complex on $[m]$. For a given real topological toric manifold $M = M(K, \lambda)$ and an m-tuple of positive integers $J = (j_1, \cdots, j_m)$, the canonical extension $M(J)$ is the real topological toric manifold determined by the characteristic pair $(K(J), \lambda(J))$, where $\lambda(J)$ is given by the block matrix*

$$\Lambda(J) = \begin{pmatrix} S_{j_1} & O & \cdots & O \\ O & S_{j_2} & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & S_{j_m} \\ C_1 & C_2 & \cdots & C_m \end{pmatrix},$$

whose columns are indexed by $1_1, 1_2, \cdots, 1_{j_1}, 2_1, 2_2, \cdots, 2_{j_2}, \cdots, m_1, \cdots, m_{j_m}$.

One can check the non-singularity condition for $\lambda(J)$, For details see [1]. Refer to [3–4] for general topological toric manifolds and real topological toric manifolds over $K(J)$.

3 Cohomology of Real Topological Toric Manifolds and Wedge Operations

First of all, we need some notations of [2, 5] which will be used throughout this paper. Let K be a star-shaped simplicial complex of dimension $n-1$ with vertex set $[m]$ and $M(K, \lambda)$ a real topological toric manifold over K. For $\omega \subseteq [m]$, denote by $K_\omega = \{\sigma \in K \mid \sigma \subseteq \omega\}$ the full subcomplex of K with respect to ω. Let $\Lambda = (\lambda(1) \ \cdots \ \lambda(m))$ be the characteristic matrix of λ. The $(n \times m)$-matrix Λ can be regarded as the linear map

$$\cdot\Lambda \colon \mathbb{Z}_2^n \to \mathbb{Z}_2^m, \quad s \mapsto s\Lambda$$

by matrix multiplication when the vector spaces are understood as row vector spaces.

Let us use the notation 2^X for the power set of X. For any integer $k \geq 0$, there is a natural identification $\phi \colon 2^{[k]} \to \mathbb{Z}_2^k$ determined by the following:

(1) $\phi(\{i\})$ is the i-th coordinate vector of $\mathbb{Z}_2^k$, $1 \leq i \leq k$, and

(2) for $A, B \subseteq [k]$, $\phi(A \Delta B) = \phi(A) + \phi(B)$, where $A \Delta B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference.

In this paper, we identify the power set $2^{V(K)}$ and the vector space $\mathbb{Z}_2^{|V(K)|}$, so $2^{V(K)}$ is equipped with $\mathbb{Z}_2$-vector space structure. Note that, with this identification, one can define a map $\phi^{-1} \circ (\cdot\Lambda) \circ \phi \colon 2^{[n]} \to 2^{[m]}$, written again as $\cdot\Lambda$.

Throughout this paper, let us assume that G is a ring such that 2 is a unit.

Theorem 3.1 (see [9] for $G = \mathbb{Q}$ and [5, Theorem 4.6]) *There is a graded group isomorphism*

$$H^p(M(K,\lambda);G) \cong \bigoplus_{s\subseteq[n]} \widetilde{H}^{p-1}(K_{s\Lambda};G).$$

Remark 3.1 Actually, the ring structure of $H^*(M(K,\lambda);G)$ is described in [5]. Moreover, as a graded ring, it is isomorphic to a graded subring of the graded ring $H^*(\mathbb{R}\mathcal{Z}_K;G)$, where $\mathbb{R}\mathcal{Z}_K$ is the real moment-angle complex of K. For simplicity, we are going to focus only on the group structure of the cohomology in this paper. It is well-known that there is a graded group isomorphism:

$$H^p(\mathbb{R}\mathcal{Z}_K) \cong \bigoplus_{\omega\subseteq[m]} \widetilde{H}^{p-1}(K_\omega). \tag{3.1}$$

See (2) of [2] for references. It is worthwhile to remark that, with coefficients in G, the cohomology of $M(K,\lambda)$ is a subgroup of the cohomology of $\mathbb{R}\mathcal{Z}_K$.

Once we have a general formula for the rational cohomology of a real topological toric manifold $M(K,\lambda)$, it looks natural to apply the formula for its canonical extensions.

For $s \subseteq [n]$, the set $s\Lambda \subseteq [m]$ can be understood as the vector in $\mathbb{Z}_2^m$ which is the sum of i-th row vectors of Λ whenever $i \in s$.

Let $\alpha \subseteq V(K(J))$ be a set of vertices of $K(J)$. The following is a simple lemma to help to compute cohomology of real topological toric manifolds. Recall (2.1) for the notation $\omega(J)$.

Lemma 3.1 *For a given subset $\alpha \subseteq V(K(J))$, let $K(J)_\alpha$ be the full subcomplex of $K(J)$ with respect to α. If $K(J)_\alpha$ is not contractible, then $\alpha = \omega(J)$ for some $\omega \subseteq [m]$.*

Proof Suppose that $\alpha \neq \omega(J)$ for any $\omega \subseteq [m]$. Then after a suitable relabeling and reindexing, $1_1 \in \alpha$ but $1_2 \notin \alpha$. Then, by the definition of $K(J)$, for any face τ of $K(J)_{\alpha\setminus\{1_1\}}$, $\tau\cup\{1_1\}$ becomes a face of $K(J)_\alpha$. So, $K(J)_\alpha$ is a cone with apex $\{1_1\}$, which proves the lemma.

Theorem 3.1 and (3.1) tell us that within the vector space $2^{[m]}$, only the subspace $S(\Lambda) := \{s\Lambda \in \mathbb{Z}_2^m \mid s \in \mathbb{Z}_2^n\}$, which is the image of $\cdot\Lambda$, contributes to the cohomology $H^*(M(K,\lambda);G)$. Although one could consider the set $S(\Lambda(J))$ for the cohomology of the canonical extension $M(J)$, we can use Lemma 3.1 to further reduce $S(\Lambda(J))$ to

$$S(\Lambda,J) := S(\Lambda(J)) \cap \{\omega(J) \mid \omega \subseteq V(K)\},$$

which is a linear subspace in $2^{V(K(J))}$, since it is the intersection of two subspaces.

Proposition 3.1 *Let $J = (j_1, j_2, \cdots, j_m)$ be an m-tuple of positive integers. Then the following vectors span $S(\Lambda, J)$:*

(1) *$\{i\}(J)$, when $1 \le i \le m$ and j_i is even;*

(2) *$s\Lambda(J)$ for some $s \subseteq [n]$, if j_i is odd whenever the i-th component of $s\Lambda$ is nonzero.*

Example 3.1 Let

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

and $J = (2, 1, 3, 2, 3)$. Then by Definition 2.1,

$$\Lambda(J) = \left(\begin{array}{cc|c|ccc|cc|ccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

To compute $S(\Lambda, J)$, let us use the language of row vectors. Then by definition, $S(\Lambda, J)$ is precisely the set of row vectors r which satisfy the following two conditions:

(1) r is in the row space of $\Lambda(J)$, and

(2) r is a linear combination of

$$\begin{aligned} \{1\}(J) &= \left(\begin{array}{cc|c|ccc|cc|ccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \\ \{2\}(J) &= \left(\begin{array}{cc|c|ccc|cc|ccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \\ \{3\}(J) &= \left(\begin{array}{cc|c|ccc|cc|ccc} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \\ \{4\}(J) &= \left(\begin{array}{cc|c|ccc|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right), \\ \{5\}(J) &= \left(\begin{array}{cc|c|ccc|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right). \end{aligned}$$

First, observe that $\{1\}(J)$ and $\{4\}(J)$ are in the row space of $\Lambda(J)$ since the first and fourth entries of J are even. On the other hand, $\{i\}(J)$ need not be in the row space of $\Lambda(J)$ if the i-th entry of J is odd. Instead, one checks that

$$\begin{aligned} r_2 &:= \left(\begin{array}{cc|c|ccc|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \\ r_3 &:= \left(\begin{array}{cc|c|ccc|cc|ccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \end{aligned}$$

and

$$r_5 := \left(\begin{array}{cc|c|ccc|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

are in the row space of $\Lambda(J)$. Next, consider

$$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

which is the second row of Λ and observe that non-zero entries of $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \Lambda$ are second and fifth ones, and the second and fifth entries of J are all odd. The second row of the block matrix

$$\begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 \end{pmatrix} = \left(\begin{array}{cc|c|ccc|cc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

is

$$t := \left(\begin{array}{cc|c|ccc|cc|ccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

and finally check that $t + r_2 + r_5 = \{2\}(J) + \{5\}(J) = \{2, 5\}(J) \in S(\Lambda, J)$. This generally holds for any $s\Lambda(J)$ satisfying Proposition 3.1(2). In fact, one can check that these are all: that is, $S(\Lambda, J) = \operatorname{span}\{\{1\}(J), \{4\}(J), \{2, 5\}(J)\}$. See the proof below.

Proof of Proposition 3.1 The proof is straightforward once one searches for sums of row vectors of $\Lambda(J)$ of the form $\omega(J)$ for $\omega \subseteq V(K)$. Note that the sum of rows of the block

$$S_k = \begin{pmatrix} 1 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 1 \end{pmatrix}_{(k-1)\times k}$$

is $\begin{pmatrix} 1 & \cdots & 1 & 1 \end{pmatrix}$ if k is even and $\begin{pmatrix} 1 & \cdots & 1 & 0 \end{pmatrix}$ if k is odd. More precisely, $S(\Lambda(J)) = S \oplus C$, where

$$S = \operatorname{row} \begin{pmatrix} S_{j_1} & O & \cdots & O \\ O & S_{j_2} & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & S_{j_m} \end{pmatrix}$$

and

$$C = \operatorname{row} \begin{pmatrix} C_1 & C_2 & \cdots & C_m \end{pmatrix}.$$

Suppose that $r \in S(\Lambda, J)$ is decomposed to $r = r_S + r_C$ for $r_S \in S$ and $r_C \in C$. If $r_C = 0$, then $r = r_S$ is certainly a linear combination of vectors satisfying (1). If $r_C \neq 0$, then one easily checks that r_C contributes exactly to a vector satisfying (2).

Since the wedge operation is topologically iterated suspensions (see for example the proof of Proposition 2.2 of [3]), $\widetilde{H}^*(K(J)_{\omega(J)})$ is isomorphic to $\widetilde{H}^*(K_\omega)$ as graded groups, up to degree shift, for any J. Combining this fact with Proposition 3.1 and Theorem 3.1, one concludes the following corollary.

Corollary 3.1 *Let us assume that G is a ring such that 2 is a unit. For a given real topological toric manifold M, let J and J' be m-tuples of positive integers and $M(J)$ and $M(J')$ corresponding real topological toric manifolds. If $J \equiv J' \mod 2$, then $H^*(M(J); G)$ and $H^*(M(J'); G)$ are isomorphic as ungraded groups. As two special cases, we have*

(1) *if every entry of J is even, then $H^*(M(J); G) \cong H^*(\mathbb{R}\mathcal{Z}_K; G)$ as ungraded groups, and*

(2) *if every entry of J is odd, then $H^*(M(J); G) \cong H^*(M; G)$ as ungraded groups.*

Proof Let us write $M = M(K, \lambda)$ and assume that $J \equiv J' \mod 2$. A direct application of Theorem 3.1 and Lemma 3.1 leads to

$$H^p(M(J); G) \cong \bigoplus_{\omega(J) \in S(\Lambda, J)} \widetilde{H}^{p-1}(K(J)_{\omega(J)}; G). \tag{3.2}$$

Proposition 3.1 guarantees that $S(\Lambda, J)$ and $S(\Lambda, J')$ are canonically isomorphic. Since the wedge operation is topologically iterated suspensions (see for example the proof of Proposition 2.2 of [3]), $\widetilde{H}^*(K(J)_{\omega(J)})$ is isomorphic to $\widetilde{H}^*(K_\omega)$ as graded groups, up to degree shift, for any J. Hence the first assertion is proven. The two special cases can be shown again using Proposition 3.1. If every entry of J is odd, then $S(\Lambda, J)$ is isomorphic to the image of $\cdot\Lambda$. If every entry of J is even, then $\{\{i\}(J) \mid 1 \leq i \leq m\}$ spans $S(\Lambda, J)$.

Even though our argument so far is for canonical extensions, in particular cases like the following example there is a good chance of computing the ungraded cohomology groups of real topological toric manifolds over $K(J)$, not only canonical extensions.

Example 3.2 Think of the pentagon $K = \partial P_5$ whose vertices are labeled by $1, 2, 3, 4, 5$ in counterclockwise direction and consider real topological toric manifolds over the $K(J) = K(a_1, a_2, \cdots, a_5)$. According to [3, Theorem 8.3], a real topological toric manifold $M(\Lambda)$ over $K(J)$ is given by the characteristic matrix

$$\Lambda = \begin{pmatrix} S_{a_1} & 0 & 0 & 0 & N_1 \\ 0 & S_{a_2} & 0 & 0 & 0 \\ 0 & 0 & S_{a_3} & 0 & 0 \\ 0 & 0 & 0 & S_{a_4} & N_4 \\ 0 & 0 & 0 & 0 & S_{a_5} \\ C_1 & C_2 & C_3 & C_4 & C_5 \end{pmatrix}_{(\sum_i a_i - 3) \times \sum_i a_i}$$

up to rotational symmetry of the pentagon. Here, the columns are labeled as $1_1, \cdots, 1_{a_1}, \cdots, 5_1, \cdots, 5_{a_5}$ and the block matrices are as follows:

$$C_i := \begin{pmatrix} 0 & \cdots & 0 & v_i \end{pmatrix}_{2 \times a_i},$$

where $i = 1, \cdots, 5$ and v_i is the i-th column of

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & d \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

for $d = 0$ or 1. Moreover,

$$S_{a_i} := \begin{pmatrix} 1 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 1 \end{pmatrix}_{(a_i - 1) \times a_i}$$

and

$$N_i := \begin{pmatrix} 0 & \cdots & 0 & n_i \end{pmatrix}_{(a_i - 1) \times a_5}$$

for arbitrary choice of vectors n_1 and n_4. Let us denote the number of nonzero entries of n_i by k_i for $i = 1$ or 4.

By an easy application of Theorem 3.1 and Lemma 3.1, one can see that the sum of $\mathbb{Q}$-Betti numbers of $M(\Lambda)$, written as $\mathrm{trk}(M(\Lambda))$, is determined by $a_1, a_2, \cdots, a_5, k_1, k_4$, and d mod 2. Assume that $d = 0$ (the case $d = 1$ goes similarly). If k_1 and k_4 are even, then we can treat Λ like a canonical extension for computing $\mathrm{trk}(M(\Lambda))$ and use Proposition 3.1 and Corollary 3.1, because even number of ones are added to be zero. Note that K has the following non-contractible full subcomplexes

$$K_\emptyset, K_{13}, K_{14}, K_{24}, K_{25}, K_{35}, K_{124}, K_{134}, K_{135}, K_{235}, K_{245}, K_{12345},$$

all of which contibute 1 to $\mathrm{trk}(\mathbb{R}\mathcal{Z}_K)$ (the empty set was added for convention). Therefore $\mathrm{trk}(\mathbb{R}\mathcal{Z}_{K(J)}) = 12$ for any J. In the following incomplete table, the first column is for some selected parities of $a_1, a_2, \cdots, a_5$ and the second column indicates ω so that the corresponding

$K(J)_{\omega(J)}$ contributes to $\mathrm{trk}(M(\Lambda))$ in Theorem 3.1.

parity of $a_1, a_2, \cdots, a_5$	ω	$\mathrm{trk}(M(\Lambda))$
00000	$\emptyset, 13, 14, 24, 25, 35, 124,$ $134, 135, 235, 245, 12345$	12
00001	$\emptyset, 13, 14, 24, 124, 134$	6
00011	$\emptyset, 13$	2
01001	$\emptyset, 13, 14, 134$	4
10110	$\emptyset, 25, 134, 12345$	4
01101	$\emptyset, 14, 235, 12345$	4
10101	$\emptyset, 24$	2
11110	$\emptyset, 134$	2
11011	$\emptyset, 12345$	2
11111	$\emptyset, 134, 235$	3

If either k_1 or k_4 is odd, then we cannot directly apply Proposition 3.1, but similar work can be done still and it is left to the reader. The point is even in these cases, only the parities of $a_1, \cdots, a_5, k_1, k_4$, and d matter. The complete calculation will be done in future work.

Remark 3.2 The cohomology of canonical extensions of topological toric manifolds are well covered in [1]. Its analogue for $\mathbb{Z}_2$-cohomology of real topological toric manifolds holds as well. Note that this paper deals with cohomology of real topological toric manifolds with coefficient G such that 2 is a unit in G.

4 Doubling Operation of Real Topological Toric Manifolds

In previous section, we have seen that the cohomology $H^*(M(J); G)$ is determined by J mod 2. In this section, we are going to focus a specific kind of canonical extensions of real topological toric manifolds called doubles and study their basic properties. The terms "double" and "doubling" for the simplicial complex were originally used in the literature such as [11].

Definition 4.1 *Let K be a simplicial complex on $[m]$, not necessarily star-shaped. Let $J = (2, 2, \cdots, 2)$ be the m-tuple every component of which is two. The simplicial complex $K(J)$ is called the double of K and is denoted by K'. Assume that K is star-shaped with dimension $n-1$ and we are given a characteristic map $\lambda\colon [m] \to \mathbb{Z}_2^n$ over K, defining a real topological toric manifold $M = M(K, \lambda)$. Then the canonical extension $M(J)$ is called the double of M and is denoted by $M' = M(K', \lambda')$. In either case, the operation itself is called the doubling.*

Let us recall a definition of moment-angle complexes. Let (X, A) be a CW-pair of spaces and K a given simplicial complex on $[m]$. For $\sigma \in K$, write

$$D(\sigma; (X, A)) := \prod_{i=1}^{m} Y_i, \quad \text{where } Y_i = \begin{cases} X, & i \in \sigma, \\ A, & i \in [m] \setminus \sigma, \end{cases} \tag{4.1}$$

and define a polyhedral product or a generalized moment-angle complex as

$$Z(K; (X, A)) := \bigcup_{\sigma \in K} D(\sigma; (X, A)) \subseteq X^m.$$

Note that $Z(K; (D^2, S^1)) = \mathcal{Z}_K$ and $Z(K; (D^1, S^0)) = \mathbb{R}\mathcal{Z}_K$. If (X, A) is equipped with a Γ-action for a group Γ, then there is a canonical Γ^m-action on $Z(K; (X, A))$. For example,

there is a canonical T^m-action on $\mathcal{Z}_K$ and a canonical $\mathbb{Z}_2^m$-action on $\mathbb{R}\mathcal{Z}_K$ respectively. When $J = (2, 2, \cdots, 2)$, for $\omega \subseteq [m]$, we use the notation $\omega' := \omega(J) = \{i_1, i_2 \mid i \in \omega\}$ for simplicity. For $\mathbb{R}\mathcal{Z}_{K'}$, one has an additional natural action of $\mathbb{Z}_2^m$ as a subgroup of $\mathbb{Z}_2^{2m}$, called the diagonal action (written as $\cdot_d$ to avoid confusion), given by

$$\omega \cdot_d x = \omega' \cdot x$$

for $\omega \in 2^{[m]} = \mathbb{Z}_2^m$ and $\omega' \in 2^{\{1_1, \cdots, m_1, 1_2, \cdots, m_2\}} \cong \mathbb{Z}_2^{2m}$.

We remark that $\mathbb{R}\mathcal{Z}_{K'} \subset (D^1)^{2m}$ is equipped with a $\mathbb{Z}_2^{2m}$-action and there is an action of a subgroup $\mathbb{Z}_2^m \subset \mathbb{Z}_2^{2m}$ which is induced by the $\mathbb{Z}_2^m$-action of $\mathbb{R}\mathcal{Z}_K$.

An important implication of [1, Section 7] is the following theorem.

Theorem 4.1 (see [1, Section 7]) *There is a T^m-action on $\mathbb{R}\mathcal{Z}_{K'}$ extending the diagonal action such that there is an equivariant homeomorphism between $\mathbb{R}\mathcal{Z}_{K'}$ and $\mathcal{Z}_K$ for this action.*

To roughly see how the theorem works, note that the 2-disc $D^2 \subset \mathbb{C}$ has a natural S^1-action and $D^1 \times D^1 \subset \mathbb{R} \times \mathbb{R}$ has a natural $\mathbb{Z}_2^2$-action. Fix an identification $D^2 \to D^1 \times D^1$ so that the $\mathbb{Z}_2$-action on $D^1 \times D^1$ given by

$$t \cdot (x, y) = (t \cdot x, t \cdot y)$$

is compatible with the $\mathbb{Z}_2$-action on D^2 as the subgroup $\mathbb{Z}_2 \leq S^1$. Hence we can equip $D^1 \times D^1$ with the induced S^1-action. Recalling (4.1), let us write $D(\sigma; (D^2, S^1)) = D(\sigma)$ for a cell of $\mathcal{Z}_K$ and $D(\sigma; (D^1, S^0)) = \mathbb{R}D(\sigma)$ for that of $\mathbb{R}\mathcal{Z}_{K'}$. For every simplex $\sigma \in K$, one can easily see that

$$D(\sigma) = \bigcup_{\substack{\tau \supset \sigma' \\ \tau \in K'}} \mathbb{R}D(\tau)$$

under our identification.

Let K be a simplicial complex on $[m]$. We already know that $\mathbb{Z}_2^m$ acts on $\mathbb{R}\mathcal{Z}_K$. If $M = M(K, \lambda)$ is a real topological toric manifold with characteristic matrix $\Lambda \colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$, then an important fact in toric topology states that $\ker \Lambda \cong \mathbb{Z}_2^{m-n}$ freely acts on $\mathbb{R}\mathcal{Z}_K$ and the quotient $\mathbb{R}\mathcal{Z}_K / \ker \Lambda$ with the quotient action of $\mathbb{Z}_2^m / \ker \Lambda \cong \mathbb{Z}_2^n$ is equivariantly homeomorphic to M. Assume that we are given $M' = M(K', \lambda')$, which is the double of a real topological toric manifold M, and denote by Λ' its characteristic matrix. The matrix Λ' can be regarded as a linear map $\Lambda' \colon \mathbb{Z}_2^{2m} \to \mathbb{Z}_2^{n+m}$ and we have $M' \cong \mathbb{R}\mathcal{Z}_{K'} / \ker \Lambda' \cong \mathcal{Z}_K / \ker \Lambda'$. The diagonal action of $\mathbb{Z}_2^m \leq T^m$ on $\mathcal{Z}_K$ induces a quotient action of $\mathbb{Z}_2^m / \ker \Lambda' \cong \mathbb{Z}_2^n$ on $M' \cong \mathcal{Z}_K / \ker \Lambda'$, which is again called the diagonal action.

Proposition 4.1 *Every vector of the subspace $\ker \Lambda'$ has the form ω' for $\omega \subset [m]$. More precisely,*

$$\ker \Lambda' = \{\omega' \mid \omega \in \ker \Lambda\}.$$

Proof This is done by direct calculation. First of all, for a linear map defined by the following matrix (I_n is the identity matrix of size n)

$$L = \left(\begin{array}{c|c} \mathrm{I}_n & A \end{array} \right)_{n \times m}, \tag{4.2}$$

note that $\ker L$ is given by the column space

$$\ker L = \operatorname{col}\left(\begin{array}{c} -A \\ \hline \mathrm{I}_{m-n} \end{array}\right)_{m\times(m-n)}. \tag{4.3}$$

By the non-singularity condition, we can assume that Λ has the form of (4.2) for a suitable order of columns and up to row operations. In other words, $\Lambda = \left(\begin{array}{c|c} \mathrm{I}_n & A \end{array}\right)_{n\times m}$. One observes that the double Λ' can be written as

$$\Lambda' = \left(\begin{array}{c|c} \mathrm{I}_m & \mathrm{I}_m \\ \hline 0 & \Lambda \end{array}\right) = \left(\begin{array}{c|c|c} & \mathrm{I}_n & 0 \\ \cline{2-3} \mathrm{I}_m & 0 & \mathrm{I}_{m-n} \\ \hline 0 & \mathrm{I}_n & A \end{array}\right),$$

when the columns are labeled as $1_1, \cdots, m_1, 1_2, \cdots, m_2$. To have the form of (4.2), we perform some row operations. Namely, one adds the $(i+m)$-th row to the i-th row for $1 \le i \le n$. After that, we obtain

$$\left(\begin{array}{c|c|c} & 0 & A \\ \cline{2-3} \mathrm{I}_m & 0 & \mathrm{I}_{m-n} \\ \hline 0 & \mathrm{I}_n & A \end{array}\right)$$

or

$$\left(\begin{array}{c|c} & A \\ \cline{2-2} \mathrm{I}_{m+n} & \mathrm{I}_{m-n} \\ \cline{2-2} & A \end{array}\right).$$

Applying (4.3), we conclude

$$\ker \Lambda' = \operatorname{col}\left(\begin{array}{c} A \\ \hline \mathrm{I}_{m-n} \\ \hline A \\ \hline \mathrm{I}_{m-n} \end{array}\right),$$

and so, recalling that

$$\ker \Lambda = \operatorname{col}\left(\begin{array}{c} A \\ \hline \mathrm{I}_{m-n} \end{array}\right)$$

completes the proof.

The following definition is due to Zhi Lü.

Definition 4.2 *Let K be a star-shaped simplicial complex of dimension $n-1$ with m vertices. Let M be a real topological toric manifold over K. If there is a topological toric manifold N such that M is the fixed point set of the conjugation on N, then N is called a lifting of M. Equivalently, for a characteristic map $\lambda\colon V(K) \to \mathbb{Z}_2^n$ over $\mathbb{Z}_2$, if there is a characteristic map $\widetilde{\lambda}$ over $\mathbb{Z}$ such that the following diagram*

$$\begin{array}{ccc} & & \mathbb{Z}^n \\ & \overset{\widetilde{\lambda}}{\nearrow} & \big\downarrow \text{mod } 2 \\ V(K) & \xrightarrow[\lambda]{} & \mathbb{Z}_2^n \end{array}$$

commutes, $\widetilde{\lambda}$ is called a lifting of λ.

Proof of Theorem 1.1 Observe that (1) is a stronger form of part (2) of Corollary 3.1 when $M(J) = M'$. The equation (3.2) becomes

$$H^p(M';G) = \bigoplus_{\omega\subseteq[m]} \widetilde{H}^{p-1}(K'_{\omega'};G)$$

when every entry of J is even. On the other hand, by (3.1) and Lemma 3.1,

$$H^p(\mathbb{R}\mathcal{Z}_{K'}) \cong \bigoplus_{\omega\subseteq[m]} \widetilde{H}^{p-1}(K'_{\omega'}).$$

Therefore $H^*(M';G) \cong H^*(\mathbb{R}\mathcal{Z}_{K'};G)$ as graded groups. For the cup product, an application of [5, Remark 4.7] gives

$$\begin{aligned} H^*(M';G) &\cong H((R/\mathcal{I}\otimes G)|_{\Lambda'}, d) \\ &\cong H(R/\mathcal{I}\otimes G, d) \\ &\cong H^*(\mathbb{R}\mathcal{Z}_{K'};G), \end{aligned}$$

where the second identity is because of the fact all entries of J are even, and the third one is due to [2, Theorem 5.1]. See [5, Theorem 3.1] for details of the formula. The last step to prove (1) is to recall $\mathbb{R}\mathcal{Z}_{K'}$ is diffeomorphic to $\mathcal{Z}_K$.

By Theorem 4.1, $\mathcal{Z}_K$ and $\mathbb{R}\mathcal{Z}_{K'}$ are T^m-equivariantly homeomorphic and the diagonal action can be considered to be an action on $\mathcal{Z}_K$ as a subgroup $\mathbb{Z}_2^m \subset T^m$. Moreover, by Proposition 4.1, $\ker\Lambda' \cong \mathbb{Z}_2^{m-n}$ acts freely on $\mathcal{Z}_K$ as a subgroup of $\mathbb{Z}_2^m$. Since K is a star-shaped simplicial complex, there is a T^m-invariant complex structure on $\mathcal{Z}_K$ by [8, Theorem 3.3]. Since $\ker\Lambda'$ is a discrete group and its free action on $\mathcal{Z}_K$ preserves the complex structure, (2) is proved. We remark that M' is invariant under the diagonal action. For (3), for the free T^{m-n}-action such that $\mathcal{Z}_K/T^{m-n} = M(K,\widetilde{\lambda})$, observe that its subgroup $\mathbb{Z}_2^{m-n}$ acts freely on $\mathcal{Z}_K$ and it coincides with $\ker\Lambda'$. Therefore, the quotient space $\mathcal{Z}_K/\ker\Lambda'$ has the quotient group action of $T^{m-n}/\mathbb{Z}_2^{m-n} \cong (\mathbb{R}P^1)^{m-n} \cong T^{m-n}$ which is still free, proving (3).

One can apply (1) of Theorem 1.1 to construct real topological toric manifolds whose cohomology rings have odd torsion elements (see also [5]).

Proof of Corollary 1.1 Apply of Theorem 1.1(1) for $G = \mathbb{Q}$ and $G = \mathbb{Z}_q$ and use the universal coefficient formula. See the proof of Theorem 5.10 of [5] for details.

5 Further Questions and Open Problems

Motivated with (3) of Theorem 1.1, we present the following question. It is a version of the famous Halperin's toral rank conjecture, applied for real topological toric manifolds. Related work for moment-angle complexes can be found in [12].

Question 5.1 Let us assume that K is a star-shaped simplicial complex of dimension $n-1$ with m vertices and $M = M(K,\lambda)$ is a real topological toric manifold. Suppose that a compact torus T^k freely acts on M. Then the following is true: $k \le m-n$ and the equality holds if and only if $M = N' \times T^\ell$, where N' is the double of a real topological toric manifold N which has a lifting and T^ℓ is the compact torus of rank ℓ.

For a real topological toric manifold $M(K,\lambda)$, observe that

$$\operatorname{trk}(M(K,\lambda)) \leq \operatorname{trk}(\mathbb{R}\mathcal{Z}_K)$$

by Theorem 3.1 and (3.1). Moreover, it is easy to see that $\operatorname{trk}\mathbb{R}\mathcal{Z}_{K(J)}$ is independent of J. We say that $M(K,\lambda)$ is maximal if $\operatorname{trk}(M(K,\lambda)) = \operatorname{trk}\mathbb{R}\mathcal{Z}_K$. Note that Corollary 3.1(2) implies that M' is maximal for any real topological toric manifold M. Also note that $M(j_1,\cdots,j_m)$ is a double if j_i is even for $1 \leq i \leq m$.

Question 5.2 Does every maximal real topological toric manifold have the form $N' \times T^\ell$ for another real topological toric manifold N and a torus T^ℓ?

In a way similar to the case of maximal real topological toric manifolds, one can also consider "minimal" real topological toric manifolds. By definition, $\operatorname{trk} M$ is minimal if and only if M is ($\mathbb{Q}$-)acyclic. Besides acyclic ones, we can also consider rational homology spheres. Note that a real topological toric manifold can neither be $\mathbb{Z}$-acyclic nor be an integral homology sphere because of 2-torsions.

Question 5.3 Describe real topological toric manifolds which are $\mathbb{Q}$-acyclic or rational homology spheres.

Note that $\mathbb{R}P^n$ is $\mathbb{Q}$-acyclic if n is even and is a rational homology sphere if n is odd. So one could ask about the existence of odd-dimensional $\mathbb{Q}$-acyclic manifolds or even-dimensional rational homology spheres in the realm of real topological toric manifolds. The following observation is due to Suyoung Choi.

Proposition 5.1 *There is no odd-dimensional real topological toric manifold which is $\mathbb{Q}$-acyclic.*

Proof The alternating sum of $\mathbb{Q}$-Betti numbers is one, and that of $\mathbb{Z}_2$-Betti numbers is zero because of Poincaré duality with $\mathbb{Z}_2$ coefficients. They are the same invarient called the Euler characteristic, so a contradiction.

On the other hand, there does exist an even dimensional rational homology sphere. Consider the cyclic polytope $C^4(7)$. It admits a unique real topological toric manifold M of dimension 4 up to weakly equivariant diffeomorphism as can be seen in [3, Section 8]. Actually one can check M is orientable and furthermore it is a rational homology sphere.

A simple argument based on Corollary 3.1 implies the following fact.

Lemma 5.1 *Let M be a real topological toric manifold which is a rational homology sphere. Then $M(J)$ is also a rational homology sphere for any $J = (j_1,\cdots,j_m)$ such that every j_i is odd.*

In summary, we have the following proposition.

Proposition 5.2 *For every $n \in \mathbb{Z}_+$ other than 2, there exists an n-dimensional real topological toric manifold which is a rational homology sphere.*

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