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Buchstaber Invariants of Universal Complexes*

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Abstract Davis and Januszkiewicz introduced (real and complex) universal complexes to give an equivalent definition of characteristic maps of simple polytopes, which now can be seen as "colorings". The author derives an equivalent definition of Buchstaber invariants of a simplicial complex K, then interprets the difference of the real and complex Buchstaber invariants of K as the obstruction to liftings of nondegenerate simplicial maps from K to the real universal complex or the complex universal complex. It was proved by Ayzenberg that real universal complexes can not be nondegenerately mapped into complex universal complexes when dimension is 3. This paper presents that there is a nondegenerate map from 3-dimensional real universal complex to 4-dimensional complex universal complex.

Keywords Buchstaber invariant, Universal complex, Lifting problem
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1 Introduction

The Buchstaber invariant is an important combinatorial invariant of simplicial complexes. This number was first introduced by Buchstaber [3] to describe the maximal rank of a torus subgroup which acts freely on the moment-angle complex over a simple convex polytope. Later, Fukukawa and Masuda generalized this number to the case of finite simplicial complexes and 2-torus actions in [7]. Therefore there exist two types of Buchstaber invariants: The complex (or ordinary) Buchstaber invariant $s(K)$ and the real Buchstaber invariant $s_{\mathbb{R}}(K)$. In a certain sense they measure the degree of symmetry of moment-angle complexes and real moment-angle complexes respectively.

These two kinds of numbers are closely related to the colorings or characteristic maps of a simplicial complex. In fact, a subgroup which acts freely defines a monomorphism, which corresponds to a coloring in such a way that they satisfy a short exact sequence (cf. Section 2). So Buchstaber invariants can be considered as an invariant of simplicial complexes, not only simple polytopes.

There is a general bound

$$1 \le s(K) \le s_{\mathbb{R}}(K) \le m - n,$$

where K is an $(n-1)$-dimensional simplicial complex with m vertices.

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It is known from [1, Lemma 5] that $s(K) = s_{\mathbb{R}}(K)$ for any simplicial complex K with $\dim K \leq 2$. Moreover, if K is an $(n-1)$-dimensional simplicial sphere with $n = 2, 3$, then $s(K) = s_{\mathbb{R}}(K) = m - n$. For 2-dimensional spheres this follows from the four color theorem.

However, the calculation of Buchstaber invariants in general is quite difficult. Even for the skeleta of a simplex Buchstaber invariants are not completely computed (cf. [7]). The readers are referred to [3, 5] for some basic properties.

It is known that the complex conjugation both induces involutions on $\mathcal{Z}_K$ and torus T^m with $\mathbb{R}\mathcal{Z}_K$ and 2-torus $(\mathbb{Z}_2)^m$ as the fixed point sets respectively, for every K^{n-1} with m vertices. This means that a torus subgroup of maximal rank which acts freely on $\mathcal{Z}_K$ actually determines a 2-torus subgroup which also acts freely on $\mathbb{R}\mathcal{Z}_K$ via the above involutions, although it may not be of the maximal rank. This implies that $s(K) \leq s_{\mathbb{R}}(K)$. Then how about the converse? The following is the lifting problem.

Lifting Problem *Consider the above correspondence from torus subgroups of maximal rank that act freely on $\mathcal{Z}_K$ to 2-torus subgroups that act freely on $\mathbb{R}\mathcal{Z}_K$ as a map Θ. Does its image contain all the 2-torus subgroups of maximal rank?*

For example, if there exist quasitoric manifolds over a simple convex polytope P^n, then there exist small covers, too. Denote all of the small covers and quasitoric manifolds over P by $\mathcal{S}(P)$ and $\mathcal{Q}(P)$ respectively. Passing to the real part, as described above, defines a map from $\mathcal{Q}(P)$ to $\mathcal{S}(P)$, since the fixed point set of the complex conjugation on a quasitoric manifold is a small cover. We may ask if this map is surjective (up to equivariant homeomorphism or equivariant cobordism). This is still unknown.

Remark 1.1 A quasitoric manifold or a small cover over a simple convex polytope P^n with m codimension-1 faces can be seen as the quotient space of $\mathcal{Z}_K$ or $\mathbb{R}\mathcal{Z}_K$ under free actions of some torus subgroup or 2-torus subgroup of rank $m - n$ respectively.

Obviously, $\Delta(K) = s_{\mathbb{R}}(K) - s(K)$ is an obstruction of the lifting problem: If $\Delta(K) \neq 0$, the image of Θ is not 2-torus subgroups of maximal rank. Davis and Januszkiewicz [4] introduced two classes of simplicial complexes $\mathcal{K}_1^n$ and $\mathcal{K}_2^n$ which have universal properties for the category of toric spaces which they studied in their paper. Moreover, they are also closely related to the calculation of the colorings and Buchstaber invariants (cf. [9]). $\Delta(K)$ can be controlled by $\Delta(\mathcal{K}_1^n)$ for some n. That is, if K has a $(\mathbb{Z}_2)^n$-coloring (cf. Section 2), then

$$\Delta(K) \leq \Delta(\mathcal{K}_1^n).$$

Ayzenberg [2] showed that $\Delta(\mathcal{K}_1^4) \geq 1$. Next is the main theorem of this paper about such universal complexes $\mathcal{K}_1^n$ and $\mathcal{K}_2^n$.

Theorem 1.1 *Let $\mathcal{K}_1^n$ be a real universal complex.*
(1) $\Delta(\mathcal{K}_1^n) = 1,\ n = 4$;
(2) $\Delta(\mathcal{K}_1^{n-1}) \leq \Delta(\mathcal{K}_1^n),\ n \geq 5$.

Using this fact, we can deduce the following corollary.

Corollary 1.1 $\Delta(\mathcal{K}_1^n) \geq 1,\ n \geq 4.$

The article is organized as follows. In Section 2, the definitions of Buchstaber invariants and universal complexes are introduced. We investigate their basic properties and relations, which give another description of Buchstaber invariants. We calculate $\Delta(\mathcal{K}_1^n)$ when $n \leq 4$ and prove that $\Delta(\mathcal{K}_1^n)$ is non-decreasing for n in Section 3. In Section 4 we discuss several further problems on Buchstaber invariants.

2 Buchstaber Invariants and Universal Complexes

In this section, we give the original definition of Buchstaber invariants and an equivalent description, which is related to universal complexes.

Let K be an abstract simplicial complex on the vertex set $[m] = \{1, \cdots, m\}$. We can define a (complex) moment-angle complex associated with K:

$$\mathcal{Z}_K = \bigcup_{\sigma \in K} (D^2, S^1)^\sigma,$$

where $(D^2, S^1)^\sigma = \prod_i X_i$, $X_i = D^2$ if $i \in \sigma$ and $X_i = S^1$ if not.

Since (D^2, S^1) is invariant under the action of S^1 and $T^m = (S^1)^m$ acts in coordinate-wise manner on $(D^2)^m$, $\mathcal{Z}_K$ admits a torus action T^m as the restriction to the subspace of $(D^2)^m$.

Definition 2.1 *The (complex) Buchstaber invariant $s(K)$ is the maximal dimension of a subtorus of T^m which acts freely on $\mathcal{Z}_K$.*

Similarly, we can define a (real) moment-angle complex:

$$\mathbb{R}\mathcal{Z}_K = \bigcup_{\sigma \in K} (D^1, S^0)^\sigma,$$

which admits a $(\mathbb{Z}_2)^m$-action. And therefore we can get the following definition.

Definition 2.2 *The (real) Buchstaber invariant $s_{\mathbb{R}}(K)$ is the maximal rank of a subgroup of $(\mathbb{Z}_2)^m$ which acts freely on $\mathbb{R}\mathcal{Z}_K$.*

For any simple convex polytope P^n, there exist corresponding definitions of $s(P)$ and $s_{\mathbb{R}}(P)$ by $s(P) = s(\partial(P^*))$ and $s_{\mathbb{R}}(P) = s_{\mathbb{R}}(\partial(P^*))$. Here $\partial(P^*)$ is the boundary of the simplicial polytope dual of P.

Example 2.1 For n-simplex Δ^n (considered as a simplicial complex), there is no subgroup of T^{n+1} or $(\mathbb{Z}_2)^{n+1}$ which acts freely on $\mathcal{Z}_{\Delta^n} = (D^2)^{n+1}$ or $\mathbb{R}\mathcal{Z}_{\Delta^n} = (D^1)^{n+1}$. Therefore $s(\Delta^n) = s_{\mathbb{R}}(\Delta^n) = 0$.

Consider its boundary. By construction we have $\mathcal{Z}_{\partial\Delta^n} = \partial(D^2)^{n+1} = S^{2n+1}$ and the diagonal subgroup of T^{n+1} acts freely on $\mathcal{Z}_{\partial\Delta^n}$. There are no larger subgroups of T^{n+1} acting freely on $\mathcal{Z}_{\partial\Delta^n}$, thus $s(\partial\Delta^n) = 1$. Similarly, $s_{\mathbb{R}}(\partial\Delta^n) = 1$.

Now let us give the definition of colorings. Let

$$R_d = \begin{cases} \mathbb{Z}_2, & \text{if } d = 1, \\ \mathbb{Z}, & \text{if } d = 2. \end{cases}$$

Definition 2.3 *Let K be an $(n-1)$-dimensional simplicial complex on the vertex set $[m]$. A R_d^r-coloring of K is a map λ from the vertex set of K to R_d^r, such that for $\sigma = [i_1, \cdots, i_k] \in K$,* *the subspace spanned by* $\lambda(i_1),\cdots,\lambda(i_k)$ *is a direct summand in* R_d^r. $\Lambda=(\lambda(1),\cdots,\lambda(m))$ *is called the coloring matrix.*

Remark 2.1 In the case that K is a simplicial sphere dual to a simple convex n-polytope P, such an R_d^n-coloring is usually called a "characteristic map" of P.

The existence of a characteristic map of P is directly related to Buchstaber invariants.

Theorem 2.1 (cf. [3, Proposition 7.34]) P^n *admits a characteristic map if and only if* $s(P)=m-n$, *where* m *is the number of the facets of* P.

Actually, let $H\subset T^m$ be a subtorus of dimension l. We can write it in the form

$$H=\{(\mathrm{e}^{2\pi\mathrm{i}(s_{11}t_1+\cdots+s_{1l}t_l)},\cdots,\mathrm{e}^{2\pi\mathrm{i}(s_{m1}t_1+\cdots+s_{ml}t_l)})\in T^m\},\tag{2.1}$$

where $t_i\in\mathbb{R}$, $i=1,\cdots,l$. The integer $m\times l$-matrix $S=(s_{ij})$ defines a monomorphism $\mathbb{Z}^l\to\mathbb{Z}^m$, whose image is a direct summand in $\mathbb{Z}^m$. H acts freely if and only if $S_{\widetilde{i_1},\cdots,\widetilde{i_n}}$ which is obtained by deleting the rows $i_1,\cdots,i_n$ of S defines a monomorphism to a direct summand for every vertex $v=F_{i_1}\cap\cdots\cap F_{i_n}$ (cf. [3, Lemma 7.32]). And if $l=s(P)=m-n$, there is a short exact sequence

$$0\longrightarrow\mathbb{Z}^{m-n}\xrightarrow{S}\mathbb{Z}^m\xrightarrow{\Lambda}\mathbb{Z}^n\longrightarrow 0,$$

where $\Lambda=(\lambda_1,\cdots,\lambda_m)$, λ_i is the facet vector for each facet.

Then for the same reason there holds the following proposition.

Proposition 2.1 *Let* $H\subset T^m$ *be a subtorus of dimension* $m-r$, *written in the form of* (2.1). *The corresponding matrix is denoted by* S. *If* H *acts freely on* $\mathcal{Z}_K$, *there is a* $\mathbb{Z}^r$*-coloring of* K *whose coloring matrix* $\Lambda_{r\times m}$ *fits in the short exact sequence*

$$0\longrightarrow\mathbb{Z}^{m-r}\xrightarrow{S}\mathbb{Z}^m\xrightarrow{\Lambda}\mathbb{Z}^r\longrightarrow 0.$$

Conversely, if K *admits a* $\mathbb{Z}^r$*-coloring* $(r\le m)$ *and* Λ *is the coloring matrix, then there is a subtorus* H *of dimension* $m-r$ *which acts freely, such that the above exact sequence holds.*

This result also holds for the real case. So we can get an equivalent statement of the Buchstaber invariants.

Proposition 2.2 (cf. [9]) *Denote the minimal integer for which there is an* R_d^r*-coloring by* $r_{\mathbb{R}}(K)$ *if* $d=1$ *and* $r(K)$ *if* $d=2$. *Then*

$$s(K)=m-r(K),$$
$$s_{\mathbb{R}}(K)=m-r_{\mathbb{R}}(K).$$

Let us give some examples.

Example 2.2 Suppose that P is a simple convex polytope. The subtorus

$$H=\{(\mathrm{e}^{2\pi\mathrm{i}t},\cdots,\mathrm{e}^{2\pi\mathrm{i}t})\}$$

surely acts freely. So $s(P)\ge 1$.

Example 2.3 For $m > n$, there is an n-dimensional simplicial convex polytope with m vertices denoted by $C^n(m)$ and called a cyclic polytope, defined as the convex hull of m points on the curve $v(t) = (t, t^2, \cdots, t^n)$. Its boundary is a simplicial complex, still denoted by $C^n(m)$. Let $v_1, \cdots, v_m$ be its vertices, where $v_i = v(t_i)$ for $t_1 < \cdots < t_m$. Shephard [11] showed that $[v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}] \in C^n(m)$, $i_1 < i_2 < i_3 < i_4$ if and only if $i_1 + 1 = i_2$, $i_3 + 1 = i_4$ or $i_1 = 1$, $i_4 = m$, $i_2 + 1 = i_3$. We can give a $\mathbb{Z}^4$-coloring matrix $\Lambda(C^4(m))$ of $C^4(m)$ for $m = 6, 7$.

$$\Lambda(C^4(6)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\Lambda(C^4(7)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

So $s(C^4(6)) = 2$, $s(C^4(7)) = 3$.

Moreover, by the works of Erokhovets [6], we have some more general results about cyclic polytopes:

$$s(C^n(m)) = 2 \quad \text{if and only if} \quad 2 \le m - n \le 2 + \frac{n-13}{48}.$$

Next is the main part of this paper. Associating to R_d^n, $(n-1)$-dimensional simplicial complex $\mathcal{K}_d^n$ is defined as follows:

(1) The vertex set of $\mathcal{K}_d^n$ is PR_d^n, the set of lines in R_d^n.

(2) A k-simplex in $\mathcal{K}_d^n$ is a collection of $k+1$ lines $\{l_0, l_1, \cdots, l_k\}$, $l_i \in PR_d^n$, which span a $(k+1)$-dimensional unimodular subspace of R_d^n.

Remark 2.2 A subspace A of R_d^n is unimodular if there exists another subspace B such that $R_d^n = A \oplus B$.

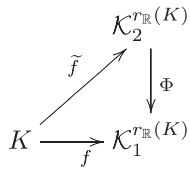
As shown in [4, p. 429], $\mathcal{K}_d^n$ determines a universal G_d^n-space ($G_d = \mathbb{Z}_2$ if $d = 1$ and $G_d = S^1$ if $d = 2$), so $\mathcal{K}_d^n$ is also called the real universal complex if $d = 1$, or the complex universal complex if $d = 2$.

According to the definitions, an R_d^r-coloring of K is equivalent to a nondegenerate simplicial map from K to $\mathcal{K}_d^r$, so we have $r_{\mathbb{R}}(\mathcal{K}_1^n) = n$ and $r(\mathcal{K}_2^n) = n$.

Remark 2.3 A nondegenerate simplicial map is a simplicial map which restricts to an isomorphism on each simplex. There is a natural nondegenerate simplicial map from $\mathcal{K}_2^n$ to $\mathcal{K}_1^n$, denoted by Φ, which is induced by the mod 2 map from $\mathbb{Z}^n$ to $(\mathbb{Z}_2)^n$ mapping each integer vector $(a_1, \cdots, a_n)$ to $(a_1, \cdots, a_n) \mod 2$.

Now we can describe the lifting problem as the following statement (also cf. [10, Remark 6]).

Let K be a simplicial complex of dimension $n-1$. Then there is a nongenerate simplicial map $f : K \longrightarrow \mathcal{K}_1^{r_{\mathbb{R}}(K)}$. Does there exist a nongenerate simplicial map $\widetilde{f} : K \longrightarrow \mathcal{K}_2^{r_{\mathbb{R}}(K)}$ such that the following diagram is commutative?



So we can see that $\Delta(K) = s_{\mathbb{R}}(K) - s(K) = r(K) - r_{\mathbb{R}}(K)$ is an obstruction of the lifting problem in the sense that $\Delta(K) \neq 0$ implies that there is no lifting in the above diagram. On the other hand, we may have $\Delta(K) = 0$ but the lifting still does not exist for some particular maps from K to $\mathcal{K}_1^{r_{\mathbb{R}}(K)}$.

3 Computations of $\Delta(\mathcal{K}_1^n)$

We see that if there is a cross-section $\pi : \mathcal{K}_1^n \longrightarrow \mathcal{K}_2^n$ of Φ for any n (i.e., π is a nondegenerate simplicial map such that $\Phi \circ \pi$ is an identity of $\mathcal{K}_1^n$), then any nondegenerate simplicial map $f : K \longrightarrow \mathcal{K}_1^{r_{\mathbb{R}}(K)}$ always admits a lifting $\widetilde{f} = \pi \circ f$.

This is true for $n \leq 3$.

Theorem 3.1 $\Delta(\mathcal{K}_1^n) = 0$ *for* $n = 1, 2, 3$.

Proof Define the map $\pi : \mathcal{K}_1^n \longrightarrow \mathcal{K}_2^n$ on the vertices by sending each binary vector to the corresponding integral vector with 0/1-coefficients. It is well known that whenever $A \in \mathrm{GL}(n, \mathbb{Z}_2)$ and $n = 1, 2, 3$, then the corresponding integral 0/1-matrix lies in $\mathrm{GL}(n, \mathbb{Z})$. Thus π is a well-defined nondegenerate map of simplicial complexes. Obviously, it is a lift of Φ.

This statement was originally observed by Ayzenberg in [1, Lemma 5]. My result is a special case.

For $n = 4$ there does not exist such a cross-section $\pi : \mathcal{K}_1^n \longrightarrow \mathcal{K}_2^n$ by the following lemma.

Lemma 3.1 (cf. [2, Thereom 1]) *There is no nondegenerate simplicial map from* $\mathcal{K}_1^4$ *to* $\mathcal{K}_2^4$.

Now let us begin to compute $\Delta(\mathcal{K}_1^4)$.

We know that according to Hadamard's maximum determinant problem (cf. [8]), for $A \in \mathrm{GL}(n, \mathbb{Z}_2)$, if we regard A as an integral matrix, its determinant does not exceed $\frac{(n+1)^{\frac{n+1}{2}}}{2^n}$. For the case of $n = 4$, $\det A = \pm 1$ or ± 3.

Let $V_i \subset (\mathbb{Z}_2)^4$ be the set of all vectors which have exactly i non-zero coordinates, $i = 1, 2, 3, 4$.

Lemma 3.2 *For* $A \in \mathrm{GL}(4, \mathbb{Z}_2)$, *regard it as an integral matrix. If* $\det A = \pm 3$, *then there are only two possible cases:*

(1) *The row vectors of* A *are just the four vectors of* V_3;

(2) *One row vector of* A *belongs to* V_3 *and the others belong to* V_2.

Proof Firstly, A has no row vector in V_1. Otherwise, $\det A$ is equal to the determinant of some matrix in $\mathrm{GL}(3, \mathbb{Z}_2)$, which is certainly ± 1 if regarded as an integral matrix. Next, A has no row vector in V_4. Otherwise, suppose that the 1st row vector of A is $(1, 1, 1, 1)$. For the same reason, other three row vectors do not belong to V_3. If not so, the 1st row of A can be turned into a vector in V_1 by a row transformation with the determinant unchanged. So the three row vectors of A except $(1,1,1,1)$ must belong to V_2. However, this is also impossible, since $\det A$ will be even.

Therefore according to the above arguments, the row vectors of A can be only contained in V_2 and V_3. We only need to eliminate the two cases:

(1) Two row vectors belong to V_2 and other two belong to V_3.

(2) Only one row vector belongs to V_2 and other three belong to V_3.

Consider the case (1). Without loss of generality, suppose that the 1st and 2nd rows of A are $(1,1,1,0)$ and $(1,1,0,1)$ respectively. There is only one vector in V_2 whose 3rd and 4th coordinates are 1. So one of the remaining row vectors can necessarily remove two non-zero coordinates of $(1,1,1,0)$ or $(1,1,0,1)$ by a row transformation, which is in contradiction with the 1st statement. Similarly, the case (2) is also impossible.

Remark 3.1 Actually, we can obtain the specific matrix of the case: One row vector of A belongs to V_3 and the others belong to V_2 as the following:

Without loss of generality, suppose that the 1st row of A is $(1,1,1,0)$. Furthermore, suppose that three other row vectors' 4th coordinates are 1, otherwise by subtracting the one whose 4th coordinate is 0 from the 1st row, the resulting matrix does not change the determinant but its 1st row belongs to V_1. This is in contradiction with Lemma 3.2. So

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ & & & 1 \\ & B & & 1 \\ & & & 1 \end{pmatrix},$$

B is a 0/1-matrix in $\mathrm{GL}(3,\mathbb{Z})$ whose rows are all vectors with only one nonzero coordinate. Multiply the sum of the 2nd, 3rd and 4th row by -1 and add it to the 1st row, we get a matrix

$$\begin{pmatrix} 0 & 0 & 0 & -3 \\ & & & 1 \\ & B & & 1 \\ & & & 1 \end{pmatrix}.$$

Its determinant is $3 \cdot \det B = \pm 3$.

Now we can calculate $\Delta(\mathcal{K}_1^4)$.

Theorem 3.2 $\Delta(\mathcal{K}_1^4) = 1$.

Proof From Lemma 3.1, it is sufficient to find a nondegenerate simplicial map from $\mathcal{K}_1^4$ to $\mathcal{K}_2^5$, which is equivalent to a map from $(\mathbb{Z}_2)^4$ to $\mathbb{Z}^5$ such that every basis of $(\mathbb{Z}_2)^4$ is mapped to a part of a basis of $\mathbb{Z}^5$. A natural idea is to add a 5th coordinate to vectors in $(\mathbb{Z}_2)^4$, and then regard them as vectors in $\mathbb{Z}^5$. We try to find a function $f : (\mathbb{Z}_2)^4 \longrightarrow \mathbb{Z}$, such that for any basis $\{\alpha_1, \cdots, \alpha_4\}$ of $(\mathbb{Z}_2)^4$, $\left\{\binom{c\alpha_1}{f(\alpha_1)}, \cdots, \binom{c\alpha_4}{f(\alpha_4)}\right\}$ is a part of a basis of $\mathbb{Z}^5$, which means that there exists $\beta = (b_1, \cdots, b_5)^{\mathrm{T}} \in \mathbb{Z}^5$, such that

$$\det\left(\begin{pmatrix} \alpha_1 \\ f(\alpha_1) \end{pmatrix}, \cdots, \begin{pmatrix} \alpha_4 \\ f(\alpha_4) \end{pmatrix}, \beta\right) = 1.$$

Expand this determinant along the 5th column, and let A denote the matrix $(\alpha_1, \cdots, \alpha_4)$. Replace the i-th row of A by $(f(\alpha_1), \cdots, f(\alpha_4))$, and denote the derived matrix by A_i. Then

$$\det\left(\begin{pmatrix}\alpha_1\\ f(\alpha_1)\end{pmatrix}, \cdots, \begin{pmatrix}\alpha_4\\ f(\alpha_4)\end{pmatrix}, \beta\right)$$
$$= b_5 \det A - b_1 \det A_1 - b_2 \det A_2 - b_3 \det A_3 - b_4 \det A_4.$$

Denote it by Γ. It is easy to see that $\Gamma = 1$ if and only if $\det A$, $\det A_1$, $\cdots$, $\det A_4$ are relatively prime, i.e.,

$$\text{g.c.d}(\det A, \det A_1, \cdots, \det A_4) = 1. \tag{3.1}$$

Claim If we set $f(\alpha) = 1$ for all $\alpha \in (\mathbb{Z}_2)^4$, then (3.1) holds for any $A \in \mathrm{GL}(4, \mathbb{Z}_2)$.

We have mentioned that the determinants of matrices in $\mathrm{GL}(4, \mathbb{Z}_2)$, regarded as integral matrices, must be ± 1 or ± 3.

If $\det A = \pm 1$, (3.1) naturally holds.

If $\det A = \pm 3$, by Lemma 3.2 and Remark 3.1, there are exactly two possible cases and they can be uniquely determined up to permutation of rows and columns. That is,

$$\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}, \quad \begin{pmatrix}1&1&1&0\\1&0&0&1\\0&1&0&1\\0&0&1&1\end{pmatrix}.$$

There must exist one of $\det A_1, \cdots, \det A_4$ that is relatively prime to $\det A$ in the above two cases, for

$$\begin{pmatrix}1&1&1&1\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix} = -1, \quad \begin{pmatrix}1&1&1&1\\1&0&0&1\\0&1&0&1\\0&0&1&1\end{pmatrix} = 2.$$

So (3.1) holds.

For $n \geq 5$, we can not determine the exact value of $\Delta(\mathcal{K}_1^n)$. But a non-decreasing relation holds for general situations.

Theorem 3.3 $\Delta(\mathcal{K}_1^n) \leq \Delta(\mathcal{K}_1^{n+1})$, $n \geq 1$.

Proof It is sufficient to prove that $r(\mathcal{K}_1^n) \leq r(\mathcal{K}_1^{n+1}) - 1$. Let $r(\mathcal{K}_1^{n+1})$ be N. Then there exists some nondegenerate simplicial map f from $\mathcal{K}_1^{n+1}$ to $\mathcal{K}_2^N$. Without loss of generality, we can assume that f maps $e_1 = (1, 0, \cdots, 0)^{\mathrm{T}} \in (\mathbb{Z}_2)^{n+1}$ to $(1, 0, \cdots, 0)^{\mathrm{T}} \in \mathbb{Z}^N$. So f can be written as the following form:

$$f(v) = \begin{pmatrix}g(v)\\ h(v)\end{pmatrix}, \quad g(v) \in \mathbb{Z},\ h(v) \in \mathbb{Z}^{N-1} \quad \text{for } v \in (\mathbb{Z}_2)^{n+1}.$$

Similarly, vectors of $\mathbb{Z}^N$ are supposed to be written as

$$u = \begin{pmatrix}u^1\\ u^2\end{pmatrix}, \quad u^1 \in \mathbb{Z},\ u^2 \in \mathbb{Z}^{N-1}.$$

Let W denote the set of the vectors in $(\mathbb{Z}_2)^{n+1}$ whose 1st coordinates are zero. The full subcomplex K_W of $\mathcal{K}_1^{n+1}$ spanned by the vertex set W is isomorphic to $\mathcal{K}_1^n$.

For every $(k-2)$-simplex $v_2v_3\cdots v_k \in K_W$, $e_1v_2v_3\cdots v_k$ is a $(k-1)$-simplex of $\mathcal{K}_1^{n+1}$. $f(e_1)$, $f(v_2)$, $f(v_3)$, $\cdots$, $f(v_k)$ form a $(k-1)$-simplex of $\mathcal{K}_2^N$, which implies that $\exists u_{k+1}$, u_{k+2}, $\cdots$, $u_N \in \mathbb{Z}^N$, such that

$$\begin{aligned}
&\det(f(e_1), f(v_2), \cdots, f(v_k), u_{k+1}, \cdots, u_N)\\
&= \det\begin{pmatrix} 1 & g(v_2) & \cdots & g(v_k) & u_{k+1}^1 & \cdots & u_N^1 \\ 0 & h(v_2) & \cdots & h(v_k) & u_{k+1}^2 & \cdots & u_N^2 \end{pmatrix}\\
&= 1.
\end{aligned}$$

Expanding the above determinant along the 1st column, we have

$$\det\big(h(v_2), \cdots, h(v_k), u_{k+1}^2, \cdots, u_N^2\big) = 1.$$

Namely, $h(v_2), \cdots, h(v_k)$ must be a part of a basis of $\mathbb{Z}^{N-1}$. Hence h is a nondegenerate simplicial map from K_W to $\mathcal{K}_2^{N-1}$.

Remark 3.2 This theorem can also be deduced from [1, Proposition 8] from another point of view.

Using Lemma 3.1, we can directly get the following corollary.

Corollary 3.1 *There is no nondegenerate simplicial map from $\mathcal{K}_1^n$ to $\mathcal{K}_2^n$ for $n \geq 4$, i.e.,*

$$\Delta(\mathcal{K}_1^n) \geq 1, \quad n \geq 4.$$

4 Conclusion and Further Problems

For a given family $\mathcal{F}$ of simplicial complexes, we may ask whether $\Delta(K) = 0$ for every $K \in \mathcal{F}$. The answer is negative for the family of all simplicial complexes since we have a counterexample $\mathcal{K}_1^n$ for $n \geq 4$. As proved in [4], $\mathcal{K}_1^n$ is Cohen-Macaulay, thus the answer for the family of all Cohen-Macaulay complexes is negative as well. This question for the family of simplicial spheres is open.

Problem 4.1 *Whether $\Delta(K) = 0$ or not for any simplicial spheres K?*

We know that any simple convex polytope determines a dual simplicial sphere. If a simple convex polytope P^n with m codimension-1 faces admits a small cover (equivalently, $s_{\mathbb{R}}(P) = m-n$), then the above problem is just equivalent to the existence of quasitoric manifolds over it (equivalently, $s(P) = m-n$). A further discussion is the lifting problem mentioned in Section 1. A special case follows.

Problem 4.2 *Can we obtain a quasitoric manifold over a simple convex polytope with a given small cover as a real part?*

Another problem is the estimation of the upper bound of $\Delta(\mathcal{K}_1^n)$ when n goes to $+\infty$.

It follows easily from the definition.

Proposition 4.1 *If a simplicial complex K admits a $(\mathbb{Z}_2)^n$-coloring for some n, then*

$$\Delta(K) \leq \Delta(\mathcal{K}_1^n).$$

Proof $r = r_{\mathbb{R}}(K) \leq n$. There exists a nondegenerate simplicial map from K to $\mathcal{K}_1^r$ and a nondegenerate simplicial map from $\mathcal{K}_1^r$ to $\mathcal{K}_2^{r+\Delta(\mathcal{K}_1^r)}$. Their composition is then a nondegenerate simplicial map from K to $\mathcal{K}_2^{r+\Delta(\mathcal{K}_1^r)}$. So

$$\Delta(K) \leq \Delta(\mathcal{K}_1^r) \leq \Delta(\mathcal{K}_1^n).$$

Hence, the estimation of $\Delta(\mathcal{K}_1^n)$ helps to estimate the general cases. By Theorem 3.3, we know that $\Delta(\mathcal{K}_1^n)$ is a non-decreasing function of n. Thus, its upper bound is significant.

Finally we give a conjecture on this upper bound. For more discussion the reader is referred to [2, 6–7].

Conjecture 4.1 $\Delta(\mathcal{K}_1^n)$ *is unbounded when* n *goes to* $+\infty$.

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