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Ehrhart Polynomials of 3-Dimensional Simple Integral Convex Polytopes*

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Abstract The author gives an explicit formula on the Ehrhart polynomial of a 3-dimensional simple integral convex polytope by using toric geometry.

Keywords Integral convex polytopes, Ehrhart polynomials, Toric geometry
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1 Introduction

Let $P \subset \mathbb{R}^d$ be an integral convex polytope of dimension d, that is, a convex polytope whose vertices have integer coordinates. For a non-negative integer l, we write $lP = \{lx \mid x \in P\}$. Ehrhart [2] proved that the number of lattice points in lP can be expressed by a polynomial in l of degree d: $|(lP) \cap \mathbb{Z}^d| = c_d l^d + c_{d-1} l^{d-1} + \cdots + c_0$. This polynomial is called the Ehrhart polynomial of P. It is known that:

(1) $c_0 = 1$.

(2) c_{d-1} is half of the sum of relative volumes of facets of P (see [1, Theorem 5.6]).

(3) c_d is the volume of P (see [1, Corollary 3.20]).

However, we have no formula on other coefficients of Ehrhart polynomials. In particular, we do not know a formula on c_1 for a general 3-dimensional integral convex polytope. In this paper, we find an explicit formula on c_1 of the Ehrhart polynomial of a 3-dimensional simple integral convex polytope (see Theorem 2.1).

Pommersheim [4] gave a method for computing the $(d-2)$-nd coefficient of the Ehrhart polynomial of a d-dimensional simple integral convex polytope P by using toric geometry. He obtained an explicit description of the Ehrhart polynomial of a tetrahedron by using this method. Our formula is obtained by using this method for a general 3-dimensional simple integral convex polytope.

The structure of the paper is as follows. In Section 2, we state the main theorem and give a few examples. In Section 3, we give a proof of the main theorem.

2 The Main Theorem

Let $P \subset \mathbb{R}^3$ be a 3-dimensional simple integral convex polytope, and let $F_1, \cdots, F_n$ be the facets of P. For $k = 1, \cdots, n$, we denote by $v_k \in \mathbb{Z}^3$ the inward-pointing primitive normal

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vector of F_k. For an edge E of P, we denote by $\mathrm{Vol}(E)$ the relative volume of E, that is, the length of E measured with respect to the lattice of rank one in the line containing E.

Definition 2.1 *For each edge $E = F_{k_1} \cap F_{k_2}$ of P, we define an integer $m(E)$ and a rational number $s(E)$ as follows:*

(1) *We define $m(E) = |((\mathbb{R}v_{k_1} + \mathbb{R}v_{k_2}) \cap \mathbb{Z}^3)/(\mathbb{Z}v_{k_1} + \mathbb{Z}v_{k_2})|$.*

(2) *There exists a basis e_1, e_2 for $(\mathbb{R}v_{k_1} + \mathbb{R}v_{k_2}) \cap \mathbb{Z}^3$ such that $v_{k_1} = e_1$ and $v_{k_2} = pe_1 + qe_2$ for some $q > p \geq 0$. Then we define $s(E) = s(p, q)$, where $s(p, q)$ is the Dedekind sum, which is defined by*

$$s(p,q) = \sum_{i=1}^{q} \left(\left(\frac{i}{q}\right)\right)\left(\left(\frac{pi}{q}\right)\right), \quad ((x)) = \begin{cases} x - [x] - \dfrac{1}{2}, & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases}$$

Remark 2.1 We have $q = m(E)$. Although p is not uniquely determined, $s(p, q)$ does not depend on the choice of e_1, e_2. Thus $s(E)$ is well-defined.

Definition 2.2 *For each facet F of P, we define a rational number $C(F)$ as follows. We name vertices and facets around F as in Figure 1. We denote by $v \in \mathbb{Z}^3$ the inward-pointing primitive normal vector of F.*

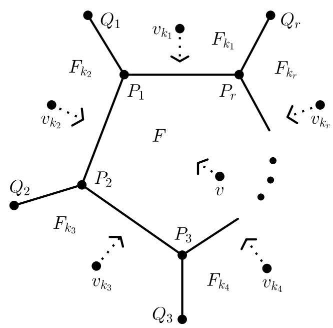

Figure 1 Vertices and facets around F.

For $i = 1, \cdots, r$, we define

$$\varepsilon_i = \det(v, v_{k_{i+1}}, v_{k_i}) > 0, \quad a_i = \frac{\langle \overrightarrow{P_{i-1}Q_{i-1}}, v_{k_{i+1}} \rangle}{\varepsilon_i \langle \overrightarrow{P_{i-1}Q_{i-1}}, v \rangle}, \quad b_i = \frac{\langle \overrightarrow{P_i P_{i+1}}, v_{k_{i-1}} \rangle}{\varepsilon_{i-1} \langle \overrightarrow{P_i P_{i+1}}, v_{k_i} \rangle},$$

where $v_{k_0} = v_{k_r}$, $v_{k_{r+1}} = v_{k_1}$, $\varepsilon_0 = \varepsilon_r$, $P_0 = P_r$, $P_{r+1} = P_1$, $Q_0 = Q_r$ and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^3$. Then we define

$$C(F) = -\sum_{2 \leq i < j \leq r} a_i \begin{vmatrix} b_{i+1} & \varepsilon_{i+1}^{-1} & 0 & \cdots & 0 \\ \varepsilon_{i+1}^{-1} & b_{i+2} & \varepsilon_{i+2}^{-1} & \ddots & \vdots \\ 0 & \varepsilon_{i+2}^{-1} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_{j-2} & \varepsilon_{j-2}^{-1} \\ 0 & \cdots & 0 & \varepsilon_{j-2}^{-1} & b_{j-1} \end{vmatrix} \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-1} \frac{\mathrm{Vol}(P_{j-1}P_j)}{m(P_{j-1}P_j)},$$

where $P_{j-1}P_j$ is the edge whose endpoints are P_{j-1} and P_j, and the determinants above are understood to be one when $j = i + 1$.

Remark 2.2 The proof of Theorem 2.1 below shows that $C(F)$ does not depend on the choice of F_{k_1}.

The following is our main theorem.

Theorem 2.1 *Let $P \subset \mathbb{R}^3$ be a 3-dimensional simple integral convex polytope, and let $E_1, \cdots, E_m$ and $F_1, \cdots, F_n$ be the edges and the facets of P, respectively. Then the coefficient c_1 of the Ehrhart polynomial $|(lP) \cap \mathbb{Z}^3| = c_3 l^3 + c_2 l^2 + c_1 l + c_0$ is given by*

$$\sum_{j=1}^{m} \Big(s(E_j) + \frac{1}{4}\Big)\mathrm{Vol}(E_j) + \frac{1}{12}\sum_{k=1}^{n} C(F_k).$$

Example 2.1 Let a, b, c be positive integers with $\gcd(a, b, c) = 1$ and let $P \subset \mathbb{R}^3$ be the tetrahedron with vertices

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}.$$

We put $A = \gcd(b, c),\ B = \gcd(a, c),\ C = \gcd(a, b)$ and $d = ABC$. Then we have the following table:

Table 1 The values of $\mathrm{Vol}(E)$, $s(E)$ and $C(F)$

edge E	OP_1	OP_2	OP_3	P_1P_2	P_1P_3	P_2P_3
$\mathrm{Vol}(E)$	a	b	c	C	B	A
$m(E)$	1	1	1	$\frac{cC}{d}$	$\frac{bB}{d}$	$\frac{aA}{d}$
$s(E)$	0	0	0	$-s\Big(\frac{ab}{d}, \frac{cC}{d}\Big)$	$-s\Big(\frac{ac}{d}, \frac{bB}{d}\Big)$	$-s\Big(\frac{bc}{d}, \frac{aA}{d}\Big)$

facet F	OP_1P_2	OP_1P_3	OP_2P_3	$P_1P_2P_3$
inward-pointing primitive normal vector of F	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -\frac{bc}{d} \\ -\frac{ac}{d} \\ -\frac{ab}{d} \end{pmatrix}$
$C(F)$	$\frac{ab}{c}$	$\frac{ac}{b}$	$\frac{bc}{a}$	$\frac{d^2}{abc}$

Thus we have

$$\begin{aligned} &\sum_{E:\text{edge}} \Big(s(E) + \frac{1}{4}\Big)\mathrm{Vol}(E) + \frac{1}{12}\sum_{F:\text{facet}} C(F) \\ &= \frac{a}{4} + \frac{b}{4} + \frac{c}{4} + \Big(-s\Big(\frac{ab}{d}, \frac{cC}{d}\Big) + \frac{1}{4}\Big)C + \Big(-s\Big(\frac{ac}{d}, \frac{bB}{d}\Big) + \frac{1}{4}\Big)B \\ &\quad + \Big(-s\Big(\frac{bc}{d}, \frac{aA}{d}\Big) + \frac{1}{4}\Big)A + \frac{1}{12}\Big(\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} + \frac{d^2}{abc}\Big), \end{aligned}$$

which coincides with the formula in [4, Theorem 5].

Example 2.2 Let a and c be positive integers and b be a non-negative integer. Consider the convex hull $P \subset \mathbb{R}^3$ of the six points

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix},$$
$$O' = \begin{pmatrix} b \\ 0 \\ c \end{pmatrix}, \quad A' = \begin{pmatrix} a+b \\ 0 \\ c \end{pmatrix}, \quad B' = \begin{pmatrix} b \\ a \\ c \end{pmatrix}.$$

P is a 3-dimensional simple polytope. We put $g = \gcd(b, c)$. Then we have the following table:

Table 2 The values of $\mathrm{Vol}(E)$, $s(E)$ and $C(F)$

edge E	OA	OB	AB	OO'	AA'	BB'	$O'A'$	$O'B'$	$A'B'$
$\mathrm{Vol}(E)$	a	a	a	g	g	g	a	a	a
$m(E)$	1	$\frac{c}{g}$	$\frac{c}{g}$	1	1	$\frac{c}{g}$	1	$\frac{c}{g}$	$\frac{c}{g}$
$s(E)$	0	$-s\left(\frac{b}{g}, \frac{c}{g}\right)$	$s\left(\frac{b}{g}, \frac{c}{g}\right)$	0	0	$-s\left(1, \frac{c}{g}\right)$	0	$s\left(\frac{b}{g}, \frac{c}{g}\right)$	$-s\left(\frac{b}{g}, \frac{c}{g}\right)$

facet F	OAB	$OAA'O'$	$OBB'O'$	$ABB'A'$	$O'A'B'$
inward-pointing primitive normal vector of F	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \frac{c}{g} \\ 0 \\ -\frac{b}{g} \end{pmatrix}$	$\begin{pmatrix} -\frac{c}{g} \\ -\frac{c}{g} \\ \frac{b}{g} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$
$C(F)$	0	c	$\frac{g^2}{c}$	$\frac{g^2}{c}$	0

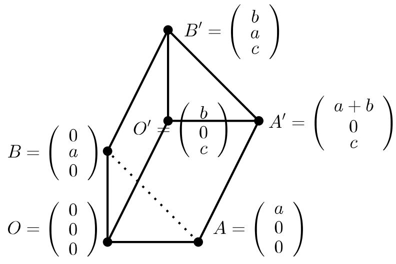

Figure 2 The simple polytope P.

Thus we have

$$\begin{aligned}&\sum_{E:\text{edge}}\Big(s(E)+\frac{1}{4}\Big)\mathrm{Vol}(E)+\frac{1}{12}\sum_{F:\text{facet}}C(F)\\
&=-s\Big(1,\frac{c}{g}\Big)g+\frac{3a}{2}+\frac{3g}{4}+\frac{1}{12}\Big(c+\frac{2g^2}{c}\Big)\\
&=-g\sum_{i=1}^{\frac{c}{g}-1}\Big(\frac{i}{\frac{c}{g}}-\frac{1}{2}\Big)^2+\frac{3a}{2}+\frac{3g}{4}+\frac{c}{12}+\frac{g^2}{6c}\\
&=-g\sum_{i=1}^{\frac{c}{g}-1}\Big(\frac{g^2}{c^2}i^2-\frac{g}{c}i+\frac{1}{4}\Big)+\frac{3a}{2}+\frac{3g}{4}+\frac{c}{12}+\frac{g^2}{6c}\\
&=-\frac{g^3}{c^2}\frac{(\frac{c}{g}-1)\frac{c}{g}(\frac{2c}{g}-1)}{6}+\frac{g^2}{c}\frac{(\frac{c}{g}-1)\frac{c}{g}}{2}-g\frac{\frac{c}{g}-1}{4}+\frac{3a}{2}+\frac{3g}{4}+\frac{c}{12}+\frac{g^2}{6c}\\
&=\frac{3a}{2}+g.\end{aligned}$$

On the other hand, since

$$\#\{(x,y)\in\mathbb{Z}^2 \mid (x,y,z)\in lP\}=\begin{cases}\dfrac{(al+1)(al+2)}{2}, & \Big(\big(\frac{c}{g}\big)\Big| z\Big),\\[2mm] \dfrac{al(al+1)}{2}, & \Big(\big(\frac{c}{g}\big)\not\Big| z\Big)\end{cases}$$

for $z=0,1,\cdots,cl$, we have

$$\begin{aligned}|(lP)\cap\mathbb{Z}^3|&=\frac{(al+1)(al+2)}{2}(gl+1)+\frac{al(al+1)}{2}((cl+1)-(gl+1))\\
&=\frac{a^2c}{2}l^3+\frac{1}{2}(a^2+ac+2ag)l^2+\Big(\frac{3a}{2}+g\Big)l+1.\end{aligned}$$

The coefficient of l is also $\frac{3a}{2}+g$.

3 Proof of Theorem 2.1

First we recall some facts about toric geometry (see [3] for details). Let $P\subset\mathbb{R}^d$ be a d-dimensional integral convex polytope. We define a cone

$$\sigma_F=\{v\in\mathbb{R}^d \mid \langle u'-u,v\rangle\geq 0,\ \forall u'\in P,\ \forall u\in F\}$$

for each face F of P. Then the set

$$\Delta_P=\{\sigma_F \mid F \text{ is a face of } P\}$$

of such cones forms a fan in $\mathbb{R}^d$, which is called the normal fan of P. Let $X(\Delta_P)$ be the associated projective toric variety. We denote by $V(\sigma)$ the subvariety of $X(\Delta_P)$ corresponding to $\sigma\in\Delta_P$. Let $\mathrm{Td}_i(X(\Delta_P))\in A_i(X(\Delta_P))_{\mathbb{Q}}$ be the i-th Todd class in the Chow group of i-cycles with rational coefficients.

Theorem 3.1 *Let $P\subset\mathbb{R}^d$ be a d-dimensional integral convex polytope and $|(lP)\cap\mathbb{Z}^d|=c_dl^d+c_{d-1}l^{d-1}+\cdots+c_0$ be its Ehrhart polynomial. If $\mathrm{Td}_i(X(\Delta_P))$ has an expression of the form $\sum_F r_F[V(\sigma_F)]$ with $r_F\in\mathbb{Q}$, then we have $c_i=\sum_F r_F\mathrm{Vol}(F)$, where $[V(\sigma_F)]$ is the class of $V(\sigma_F)$ in the Chow group and $\mathrm{Vol}(F)$ is the relative volume of F.*

Now we assume that $d = 3$ and P is simple. Then the associated toric variety $X(\Delta_P)$ is $\mathbb{Q}$-factorial and we know the ring structure of the Chow ring $A^*(X(\Delta_P))_{\mathbb{Q}}$ with rational coefficients. Let $E_1, \cdots, E_m$ and $F_1, \cdots, F_n$ be the edges and the facets of P, respectively. We have

$$\sum_{k=1}^{n} \langle u, v_k \rangle [V(\sigma_{F_k})] = 0, \quad \forall u \in (\mathbb{Q}^3)^*. \tag{3.1}$$

If F_{k_1} and F_{k_2} are distinct, then

$$[V(\sigma_{F_{k_1}})][V(\sigma_{F_{k_2}})] = \begin{cases} \dfrac{1}{m(E_j)}[V(\sigma_{E_j})], & 1 \leq \exists j \leq m : F_{k_1} \cap F_{k_2} = E_j, \\ 0, & F_{k_1} \cap F_{k_2} = \emptyset \end{cases} \tag{3.2}$$

in $A^*(X(\Delta_P))_{\mathbb{Q}}$.

Pommersheim gave an expression of $\mathrm{Td}_{d-2}(X(\Delta_P))$ for a d-dimensional simple integral convex polytope $P \subset \mathbb{R}^d$. In the case where $d = 3$, we have the following theorem.

Theorem 3.2 (see [4]) *If $P \subset \mathbb{R}^3$ is a 3-dimensional simple integral convex polytope, then*

$$\mathrm{Td}_1(X(\Delta_P)) = \sum_{j=1}^{m} \Big(s(E_j) + \frac{1}{4} \Big) [V(\sigma_{E_j})] + \frac{1}{12} \sum_{k=1}^{n} [V(\sigma_{F_k})]^2.$$

We use the notation in Definition 2.2. It suffices to show

$$[V(\sigma_F)]^2 = -\sum_{2 \leq i < j \leq r} a_i \begin{vmatrix} b_{i+1} & \varepsilon_{i+1}^{-1} & 0 & \cdots & 0 \\ \varepsilon_{i+1}^{-1} & b_{i+2} & \varepsilon_{i+2}^{-1} & \ddots & \vdots \\ 0 & \varepsilon_{i+2}^{-1} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_{j-2} & \varepsilon_{j-2}^{-1} \\ 0 & \cdots & 0 & \varepsilon_{j-2}^{-1} & b_{j-1} \end{vmatrix} \frac{\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-1}}{m(P_{j-1}P_j)} [V(\sigma_{P_{j-1}P_j})]$$

for each facet F of P.

We put

$$D(s,t) = \begin{vmatrix} b_s & \varepsilon_s^{-1} & 0 & \cdots & 0 \\ \varepsilon_s^{-1} & b_{s+1} & \varepsilon_{s+1}^{-1} & \ddots & \vdots \\ 0 & \varepsilon_{s+1}^{-1} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_{t-1} & \varepsilon_{t-1}^{-1} \\ 0 & \cdots & 0 & \varepsilon_{t-1}^{-1} & b_t \end{vmatrix}$$

for $2 < s \leq t < r$ and $D(s,t) = 1$ for $s > t$. Define $u \in (\mathbb{Q}^3)^*$ by $\langle u, v \rangle = 1$, $\langle u, v_{k_1} \rangle = 0$, $\langle u, v_{k_2} \rangle = 0$. By (3.1) and (3.2), we have

$$[V(\sigma_F)]^2 = -[V(\sigma_F)] \sum_{j=1}^{r} \langle u, v_{k_j} \rangle [V(\sigma_{F_{k_j}})] = -\sum_{j=3}^{r} \frac{\langle u, v_{k_j} \rangle}{m(P_{j-1}P_j)} [V(\sigma_{P_{j-1}P_j})].$$

Hence it suffices to show

$$\langle u, v_{k_j} \rangle = \sum_{i=2}^{j-1} a_i D(i+1, j-1) \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-1} \tag{3.3}$$

for any $j = 3, \cdots, r$.

First we claim that

$$\varepsilon_{j-1}^{-1} v_{k_{j-1}} + \varepsilon_j^{-1} v_{k_{j+1}} = a_j v + b_j v_{k_j} \tag{3.4}$$

for any $j = 2, \cdots, r-1$. By Cramer's rule, we have

$$\begin{aligned} v_{k_{j+1}} &= \frac{\det(v_{k_{j+1}}, v_{k_j}, v_{k_{j-1}})}{\det(v, v_{k_j}, v_{k_{j-1}})} v + \frac{\det(v, v_{k_{j+1}}, v_{k_{j-1}})}{\det(v, v_{k_j}, v_{k_{j-1}})} v_{k_j} + \frac{\det(v, v_{k_j}, v_{k_{j+1}})}{\det(v, v_{k_j}, v_{k_{j-1}})} v_{k_{j-1}} \\ &= \frac{\det(v_{k_{j+1}}, v_{k_j}, v_{k_{j-1}})}{\varepsilon_{j-1}} v + \frac{\det(v, v_{k_{j+1}}, v_{k_{j-1}})}{\varepsilon_{j-1}} v_{k_j} - \frac{\varepsilon_j}{\varepsilon_{j-1}} v_{k_{j-1}}. \end{aligned}$$

So we have

$$\begin{aligned} &\varepsilon_{j-1}^{-1} v_{k_{j-1}} + \varepsilon_j^{-1} v_{k_{j+1}} \\ &= \varepsilon_{j-1}^{-1}\varepsilon_j^{-1} \det(v_{k_{j+1}}, v_{k_j}, v_{k_{j-1}}) v + \varepsilon_{j-1}^{-1}\varepsilon_j^{-1} \det(v, v_{k_{j+1}}, v_{k_{j-1}}) v_{k_j}. \end{aligned} \tag{3.5}$$

Taking the inner product of both sides of (3.5) with $\overrightarrow{P_{j-1}Q_{j-1}}$ gives

$$\varepsilon_j^{-1} \langle \overrightarrow{P_{j-1}Q_{j-1}}, v_{k_{j+1}} \rangle = \varepsilon_{j-1}^{-1}\varepsilon_j^{-1} \det(v_{k_{j+1}}, v_{k_j}, v_{k_{j-1}}) \langle \overrightarrow{P_{j-1}Q_{j-1}}, v \rangle,$$

which means $a_j = \varepsilon_{j-1}^{-1}\varepsilon_j^{-1}\det(v_{k_{j+1}}, v_{k_j}, v_{k_{j-1}})$. Taking the inner product of both sides of (3.5) with $\overrightarrow{P_j P_{j+1}}$ gives

$$\varepsilon_{j-1}^{-1} \langle \overrightarrow{P_j P_{j+1}}, v_{k_{j-1}} \rangle = \varepsilon_{j-1}^{-1}\varepsilon_j^{-1} \det(v, v_{k_{j+1}}, v_{k_{j-1}}) \langle \overrightarrow{P_j P_{j+1}}, v_{k_j} \rangle,$$

which means $b_j = \varepsilon_{j-1}^{-1}\varepsilon_j^{-1}\det(v, v_{k_{j+1}}, v_{k_{j-1}})$. Thus (3.4) follows.

We show (3.3) by induction on j. If $j = 3$, then both sides are $a_2\varepsilon_2$. If $j = 4$, then both sides are $a_2 b_3 \varepsilon_2 \varepsilon_3 + a_3 \varepsilon_3$. Suppose $4 \le j \le r-1$. By (3.4) and the hypothesis of induction, we have

$$\begin{aligned} \langle u, v_{k_{j+1}} \rangle &= \langle u, a_j \varepsilon_j v + b_j \varepsilon_j v_{k_j} - \varepsilon_{j-1}^{-1} \varepsilon_j v_{k_{j-1}} \rangle \\ &= a_j \varepsilon_j + b_j \varepsilon_j \langle u, v_{k_j} \rangle - \varepsilon_{j-1}^{-1} \varepsilon_j \langle u, v_{k_{j-1}} \rangle \\ &= a_j \varepsilon_j + b_j \varepsilon_j \sum_{i=2}^{j-1} a_i D(i+1, j-1) \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-1} \\ &\quad - \varepsilon_{j-1}^{-1} \varepsilon_j \sum_{i=2}^{j-2} a_i D(i+1, j-2) \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\sum_{i=2}^{j} a_i D(i+1, j) \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_j \\ &= a_j \varepsilon_j + a_{j-1} b_j \varepsilon_{j-1} \varepsilon_j + \sum_{i=2}^{j-2} a_i D(i+1, j) \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_j. \end{aligned}$$

Since

$$\sum_{i=2}^{j-2} a_i D(i+1, j) \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_j$$

$$
\begin{aligned}
&= \sum_{i=2}^{j-2} a_i(b_j D(i+1, j-1) - \varepsilon_{j-1}^{-2} D(i+1, j-2))\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_j \\
&= b_j \varepsilon_j \sum_{i=2}^{j-2} a_i D(i+1, j-1)\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-1} \\
&\quad - \varepsilon_{j-1}^{-1} \varepsilon_j \sum_{i=2}^{j-2} a_i D(i+1, j-2)\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-2},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\sum_{i=2}^{j} a_i D(i+1, j)\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_j \\
&= a_j \varepsilon_j + a_{j-1} b_j \varepsilon_{j-1} \varepsilon_j + b_j \varepsilon_j \sum_{i=2}^{j-2} a_i D(i+1, j-1)\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-1} \\
&\quad - \varepsilon_{j-1}^{-1} \varepsilon_j \sum_{i=2}^{j-2} a_i D(i+1, j-2)\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-2} \\
&= a_j \varepsilon_j + b_j \varepsilon_j \sum_{i=2}^{j-1} a_i D(i+1, j-1)\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-1} \\
&\quad - \varepsilon_{j-1}^{-1} \varepsilon_j \sum_{i=2}^{j-2} a_i D(i+1, j-2)\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{j-2} \\
&= \langle u, v_{k_{j+1}} \rangle.
\end{aligned}
$$

Thus (3.3) holds for $j+1$. This completes the proof of Theorem 2.1.

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References

[1] Beck, M. and Robins, S., Computing the Continuous Discretely, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2007.

[2] Ehrhart, E., Polynômes Arithmétiques et Méthode des Polyèdres en Combinatoire, Birkhäuser, Boston, Basel, Stuttgart, 1977.

[3] Fulton, W., Introduction to Toric Varieties, Annals of Mathematics Studies, **131**, Princeton Univ. Press, Princeton, NJ, 1993.

[4] Pommersheim, J. E., Toric varieties, lattice points and Dedekind sums, *Math. Ann.,* **295**, 1993, 1–24.