Chin. Ann. Math. Ser. B
38(6), 2017, 1353–1364
DOI: 10.1007/s11401-017-1043-3

Chinese Annals of
Mathematics, Series B

On the Equivalence of Integral T^k-Cohomology Chern Numbers and T^k-K-Theoretic Chern Numbers*

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Abstract This paper mainly deals with the question of equivalence between equivariant cohomology Chern numbers and equivariant K-theoretic Chern numbers when the transformation group is a torus. By using the equivariant Riemann-Roch relation of Atiyah-Hirzebruch type, it is proved that the vanishing of equivariant cohomology Chern numbers is equivalent to the vanishing of equivariant K-theoretic Chern numbers.

Keywords Equivariant K-theoretic Chern number, Equivariant cohomology Chern number, Equivariant bordism
2000 MR Subject Classification 57R85, 57R20, 55N22, 57R91

1 Introduction

Let W be a compact smooth manifold. One knows that the boundary of W is a closed smooth manifold. Conversely, given a closed smooth manifold M, how to know M is the boundary of some compact smooth manifold?

Thom [1] solved this question completely in 1950's. In fact, he proved the following theorem.

Theorem 1.1 *Let M be a closed smooth manifold. M is the boundary of a compact smooth manifold if and only if all Stiefel-Whitney characteristic numbers of M are zero.*

This theorem leads to the connection between characteristic numbers and the question whether a manifold is the boundary of some compact smooth manifold with boundary.

It is known that there are various kinds of characteristic classes and characteristic numbers defined in ordinary cohomology ring, for example, Chern classes and Chern numbers. It is natural to consider the relation between vanishing of Chern numbers and being the boundary of some manifold with some stably complex structure.

Recall that a unitary manifold (or a stably complex manifold) is a smooth manifold whose stable tangent bundle admits a complex structure. Let M be a unitary manifold with stable tangent bundle TM. The corresponding Chern classes (resp. Chern numbers) of TM can be defined in integral cohomology ring $H^*(M;\mathbb{Z})$ (resp. $H^*(pt;\mathbb{Z})$). In his paper [10], Milnor proved the following theorem.

Theorem 1.2 *Let M be a unitary manifold. M is the boundary of a compact unitary manifold W with the induced unitary structure of W if and only if all Chern numbers of M are zero.*

Manuscript received February 14, 2017.
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*This work was supported by the National Natural Science Foundation of China (No. 11301335).

We can further consider this question in other generalized cohomology. In complex K-theory, if M is a unitary manifold, then one can define K-theoretic Chern classes (resp. K-theoretic Chern numbers) in $K^*(M)$ (resp. $K^*(pt)$). In Section 2, we will use the Riemann-Roch relation of Atiyah-Hirzebruch type to show the following proposition.

Proposition 1.1 *Let M be a closed unitary manifold. All integral cohomology Chern numbers of M vanish if and only if all K-theoretic Chern numbers of M vanish.*

Therefore, it follows that a unitary closed M is the boundary of some compact unitary manifold if and only if all K-theoretic Chern numbers of M vanish.

In equivariant case, suppose that the transformation group is G. We can still ask this question: Is it true that the vanishing of some equivariant characteristic numbers implies being the boundary of some G-manifold?

When the transformation group $G = T^k$ is a torus, recall that M is a unitary T^k-manifold provided that M is a T^k-manifold whose stable tangent bundle TM admits a complex structure comparable with the T^k-action. In his paper [12], tom Dieck proved the following theorem.

Theorem 1.3 (see [12]) *Let M be a unitary T^k-manifold. M is the boundary of a compact unitary T^k-manifold with the induced unitary T^k-structure if and only if all the equivariant K-theoretic characteristic numbers vanish.*

In tom Dieck's theorem, equivariant K-theoretic characteristic numbers play an important role. We want to know the question whether it is possible the vanishing of equivariant cohomology characteristic numbers can still determine that M is the boundary of a compact unitary T^k-manifold with the induced unitary T^k-structure.

Inspired by Proposition 1.1, if we could prove the equivalence of the vanishing of these two kinds of equivariant characteristic numbers, then the answer of the question above would be yes. Hence, our main theorem in this paper is the following result.

Theorem 1.4 *Let M be a unitary T^k-manifold with stable complex tangent bundle $TM \in K_{T^k}(M)$. Then all integral equivariant cohomology Chern numbers of M vanish if and only if all equivariant K-theoretic Chern numbers of M vanish.*

Therefore, one obtains the answer of the question above.

Theorem 1.5 *Let M be a unitary T^k-manifold. M is the boundary of a compact unitary T^k-manifold with the induced unitary T^k-structure if and only if all the equivariant cohomology characteristic numbers vanish.*

Remark 1.1 In [9], Zhi Lü and the author gave the first proof of Theorem 1.5 directly and also consider more cases. The proof in this paper is another aspect of the understanding of the relation between various kinds of equivariant Chern numbers.

This paper is organized as follows. In Section 2, we will recall ordinary Chern numbers and K-theoretic Chern numbers in nonequivariant case and will prove Proposition 1.1. In Section 3, we turn to equivariant ordinary Chern numbers and K-theoretic Chern numbers and prove the T^k-version Riemann-Roch relation of Atiyah-Hirzebruch type in Section 3.3. Combining the results in Section 4, we will finish the proof of our main theorem in Section 4.

2 *K*-Theoretic Chern Classes and Integral Cohomology Chern Numbers: Nonequivariant Case

2.1 *K*-theoretic Chern classes and Chern numbers

First let us recall the definitions of K-theoretic Chern classes and Chern numbers and some standard facts (see [2]). Let ξ be a complex vector bundle over a space X and $\lambda_t(\xi) \in K^*(X)[[t]]$ be the power series

$$\sum_{i=0}^{\infty} \lambda^i(\xi) t^i,$$

where $\lambda^i(\xi)$ denotes the i-th exterior power of the vector bundle ξ. Then λ_t can be extended to a homomorphism

$$\lambda_t : K^*(X) \longrightarrow K^*(X)[[t]],$$
$$x \mapsto \sum_{i=0}^{\infty} \lambda^i(x) t^i.$$

Furthermore, one can define the operations $\gamma^i(x)$ by putting

$$\gamma_t(x) := \lambda_{\frac{t}{1-t}}(x) = \sum_{i}^{\infty} \gamma^i(x) t^i \in K(X)[[t]], \quad x \in K(X).$$

Proposition 2.1 (see [2]) *Let ξ and η be two complex bundles over a finite CW complex X. One has*

(a) *If ξ is a line bundle, then $\gamma^1(\xi) = \xi - 1$ and $\gamma^k(\xi) = 0$ for $k > 1$.*

(b) $\gamma^k(\xi \oplus \eta) = \sum\limits_{p+q=k} \gamma^p(\xi) \cdot \gamma^q(\eta)$.

(c) *Let $f : Y \to X$ be a continuous map. Then $f^! \gamma^k(\xi) = \gamma^k(f^!(\xi))$.*

Now we can use these operations γ^i to define K-theoretic Chern classes. Let M be a closed unitary manifold with stable complex tangent bundle $TM \in K(M)$. Then the total K-theoretic Chern class of M is defined to be

$$c^K(M) := \sum_{i=0} \gamma^i(TM),$$

with the i-th K-theoretic Chern class $c_i^K(M) = \gamma^i(TM)$.

Remark 2.1 In [5], the i-th K-theoretic Chern class of M is also defined by $(-1)^i \gamma^i(TM)$. In this paper, it is more convenient to use the definition of $\gamma^i(TM)$ as above.

Using the Gysin map, we can define the K-theoretic Chern numbers of M. Namely, let $p : M \longrightarrow pt$ be the constant map and $p_!^K : K^*(M) \longrightarrow K^*(pt)$ be the Gysin map induced by p in complex K-theory. Then the K-theoretic Chern numbers of M are defined to be

$$c_\omega^K[M] := p_!^K(c_\omega^K(M)),$$

where each $\omega = (i_1, i_2, \cdots, i_s)$ is a partition of $|\omega| = i_1 + i_2 + \cdots + i_s$, and $c_\omega^K(M)$ means $c_{i_1}^K(M) c_{i_2}^K(M) \cdots c_{i_s}^K(M)$.

2.2 Riemann-Roch relation of Atiyah-Hirzebruch type

In this part, we will review the Riemann-Roch relation of Atiyah-Hirzebruch type (see [3, 8]).

Definition 2.1 *For any topological space X, we denote by $H^{**}(X;R)$ the direct product of $H^i(X;R)$ with coefficient ring R. More precisely,*

$$H^{**}(X;R) := \{(x_0, x_1, \cdots, x_n, \cdots) \mid x_i \in H^i(X;R)\}.$$

For any two elements $a = \{x_i\}$, $b = \{y_i\}$ in $H^{**}(X;R)$, the product $a \cdot b = \{z_i\}$ is defined to be

$$z_n = \sum_{i=0}^{n} x_i y_{n-i}.$$

Remark 2.2 When we discuss the Borel construction $EG \times_G X$ (which is not compact) of X, it is better to use the direct product $H_G^{**}(X;\mathbb{Q})$ instead of the cohomology ring $H_G^*(X;\mathbb{Q}) = \oplus H_G^i(X;\mathbb{Q})$.

We know that the Chern character ch is a natural transformation from complex K-theory to ordinary cohomology theory with rational coefficients

$$ch : K^*(X) \longrightarrow H^{**}(X;\mathbb{Q}).$$

Let X be a compact space and ξ be a complex vector bundle over X. Then one has Thom isomorphisms in complex K-theory and ordinary cohomology theory:

$$\psi_! : K^*(X) \longrightarrow K^*(D(\xi), S(\xi)),$$
$$\psi_* : H^*(X) \longrightarrow H^*(D(\xi), S(\xi)).$$

With respect to these Thom isomorphisms $\psi_!$ and ψ_*, one has the following diagram:

$$\begin{array}{ccc} K^*(X) & \xrightarrow{ch} & H^{**}(X;\mathbb{Q}) \\ {\scriptstyle\psi_!}\big\downarrow & & {\scriptstyle\psi_*}\big\downarrow \\ K^*(D(\xi), S(\xi)) & \xrightarrow{ch} & H^{**}(D(\xi), S(\xi);\mathbb{Q}) \end{array}$$

This diagram is not commutative. However, we have the following nonequivariant Riemann-Roch relation.

Proposition 2.2 *Let X be a compact space and ξ be a complex vector bundle over X. Then for each $\alpha \in K^*(X)$,*

$$ch(\psi_!(\alpha)) = \psi_*(ch(\alpha) \cdot Td^{-1}(\xi)),$$

where $Td(\xi) \in H^{**}(X;\mathbb{Q})$ is the total Todd class of ξ.

Proof We refer to [8, p. 182] or [4, Proposition 3.5].

With respect to the Gysin maps, Proposition 2.2 leads to the Riemann-Roch relation of Atiyah-Hirzebruch type (see [3] or [8, Theorem 26.5.2]). Indeed, this kind of Riemann-Roch relation can be stated as follows.

Proposition 2.3 *Let M and N be two closed unitary manifolds and $f : M \longrightarrow N$ be a smooth map. Then for each $\alpha \in K^*(M)$,*

$$ch(f_!^K(\alpha) \cdot Td(TN)) = f_!(ch(\alpha) \cdot Td(TM)),$$

where $f_!^K : K^*(M) \longrightarrow K^{*-(m-n)}(N)$ and $f_! : H^*(M;\mathbb{Q}) \longrightarrow H^{*-(m-n)}(N;\mathbb{Q})$ are the Gysin maps induced by f in complex K-theory and ordinary cohomology theory respectively with $m = \dim M$ and $n = \dim N$.

Remark 2.3 As a special case of Proposition 2.3, let $p : M \longrightarrow pt$ be the constant map. Then for each $\alpha \in K^*(M)$,

$$ch(p_!^K(\alpha)) = p_!(ch(\alpha) \cdot Td(TM)).$$

2.3 *K*-theoretic Chern numbers and ordinary cohomology Chern numbers

In this part, we will use the Riemann-Roch relation of Atiyah-Hirzebruch type to prove the following proposition.

Proposition 2.4 *Let M be a closed unitary manifold with stable complex tangent bundle TM. All integral cohomology Chern numbers of M vanish if and only if all K-theoretic Chern numbers of M vanish.*

Proof First, assume that all integral cohomology Chern numbers of M vanish. Then for any K-theoretic Chern number $p_!^K(c_\omega^K(M))$ with partition $\omega = (i_1, i_2, \cdots, i_s)$, one has

$$ch(p_!^K(c_\omega^K(M))) = p_!(ch(c_\omega^K(M)) \cdot Td(TM)),$$

where $p_! : H^{**}(M;\mathbb{Q}) \longrightarrow H^{**}(pt;\mathbb{Q})$ is the Gysin map induced by p in ordinary cohomology. Since $p_!(ch(c_\omega^K(M)) \cdot Td(TM))$ is represented by a combinations of ordinary cohomology Chern numbers, it follows that

$$p_!(ch(c_\omega^K(M)) \cdot Td(TM)) = 0 = ch(p_!^K(c_\omega^K(M))).$$

Moreover, since the Chern character $ch : K^*(pt) \to H^{**}(pt;\mathbb{Q})$ is injective, one has

$$p_!^K(c_\omega^K(M)) = 0.$$

Conversely, assume that all K-theoretic Chern numbers of M vanish. Then for any cohomology Chern number $p_!(c_\omega(M))$, consider the corresponding K-theoretic Chern class $c_\omega^K(M)$. One has

$$ch(c_\omega^K(M)) = c_\omega(M) + \text{terms of higher degree} \in H^{**}(M;\mathbb{Q}).$$

By the Riemann-Roch relation, one has

$$\begin{aligned} 0 &= ch(p_!^K(c_\omega^K(M))) \\ &= p_!(ch(c_\omega^K(M)) \cdot Td(TM)) \\ &= p_!(c_\omega(M) + \text{terms of higher degree}) \\ &= p_!(c_\omega(M)) + \text{terms of higher degree}, \end{aligned}$$

which implies that $p_!(c_\omega(M)) = 0$ in $H^{**}(pt;\mathbb{Q})$. This means that $p_!(c_\omega(M)) = 0$ in $H^{**}(pt) \cong \mathbb{Z}$. The proof is completed.

3 K-Theoretic Chern Classes and Integral Cohomology Chern Numbers: Equivariant Case

3.1 Equivariant Chern classes and Chern numbers

In this subsection, we recall the definitions of three kinds of equivariant Chern classes. Let M be a unitary G-manifold and E be a unitary G-vector bundle over M. Then applying the Borel construction to $E \longrightarrow M$ gives a unitary G-vector bundle $EG \times_G E$ over $EG \times_G M$. First, recall that the total equivariant cohomology Chern class of E is defined to be the total Chern class of $EG \times_G E$ over $EG \times_G M$:

$$c^G(E) := c(EG \times_G E) \in H^{**}(EG \times_G M).$$

Similarly, in equivariant K-theory $K(EG \times_G -)$, the total equivariant Chern class of E over M is defined to be

$$c^{G,K}(E) := c^K(EG \times_G E) \in K^*(EG \times_G M).$$

In equivariant K-theory $K_G(-)$ (see [11]), the total equivariant Chern class of E is defined to be

$$c^{G,K_G}(E) := \sum_{i=0} \gamma^i(E) \in K_G^*(M),$$

where the operations $\gamma^i(E)$ are defined in the same way as was done in the nonequivariant case (see [7, 12]).

In particular, let M be a unitary G-manifold with stable complex G-vector bundle TM. The corresponding total equivariant Chern class of M is defined by the corresponding total equivariant Chern class of TM as follows:

$$\begin{aligned} c^G(M) &:= c(EG \times_G TM) \in H^{**}(EG \times_G M), \\ c^{G,K}(M) &:= c^K(EG \times_G TM) \in K^*(EG \times_G M), \\ c^{G,K_G}(M) &:= \sum_{i=0} \gamma^i(TM) \in K_G^*(M). \end{aligned}$$

We know that the equivariant Gysin maps are well-defined in these equivariant cohomology theories. Let $p : M \longrightarrow pt$ be the constant map. Then we have three corresponding equivariant Gysin maps

$$\begin{aligned} p_!^G &: H^*(EG \times_G M) \longrightarrow H^*(BG), \\ p_!^{G,K} &: K^*(EG \times_G M) \longrightarrow K^*(BG), \\ p_!^{G,K_G} &: K_G^*(M) \longrightarrow K_G^*(pt). \end{aligned}$$

Hence, the corresponding equivariant Chern numbers of M are defined to be

$$\begin{aligned} c_\omega^G[M]_G &:= p_!^G(c_\omega^G(M)), \\ c_\omega^{G,K}[M]_G &:= p_!^{G,K}(c_\omega^{G,K}(M)), \\ c_\omega^{G,K_G}[M]_G &:= p_!^{G,K_G}(c_\omega^{G,K_G}(M)), \end{aligned}$$

respectively, where $\omega = (i_1, i_2, \cdots, i_s)$ is a partition of $|\omega| = i_1 + i_2 + \cdots + i_s$, and $c_\omega^G(M)$ (resp. $c_\omega^{G,K}(M)$, $c_\omega^{G,K_G}(M)$) means the product $c_{i_1}^G(M)c_{i_2}^G(M)\cdots c_{i_s}^G(M)$ (resp. $c_{i_1}^{G,K}(M)c_{i_2}^{G,K}(M)\cdots c_{i_s}^{G,K}(M)$, $c_{i_1}^{G,K_G}(M)c_{i_2}^{G,K_G}(M)\cdots c_{i_s}^{G,K_G}(M)$).

3.2 Inverse limits in equivariant cohomology theory of Borel type

Assume that G is a compact Lie group. Let M be a compact unitary G-manifold and $EG \times_G M$ be the Borel construction of M. $EG \times_G M$ admits a filtration:

$$\cdots \subset EG(n) \times_G M \subset EG(n+1) \times_G M \subset \cdots \subset EG \times_G M,$$

where $EG(n)$ can be some compact smooth manifold. The following proposition is well-known.

Proposition 3.1 *In ordinary cohomology theory, one has*

$$\varprojlim H^i(EG(n) \times_G M) = H^i(EG \times_G M).$$

In complex K-theory, one also has

$$\varprojlim K^i(EG(n) \times_G M) = K^i(EG \times_G M).$$

For a unitary G-vector bundle $EG \times_G E$ over $EG \times_G M$, let $EG \times_G D(E)$ be the disk bundle of $EG \times_G E$ and $EG \times_G S(E)$ be the sphere bundle of $EG \times_G E$. There is a diagram

$$\begin{tikzcd} K^*(EG \times_G M) \arrow[r, "ch"] \arrow[d, "\psi_!"', "\cong"] & H^{**}(EG \times_G M; \mathbb{Q}) \arrow[d, "\psi_*"', "\cong"] \\ K^*(EG \times_G D(E), EG \times_G S(E)) \arrow[r, "ch"] & H^{**}(EG \times_G D(E), EG \times_G S(E); \mathbb{Q}) \end{tikzcd}$$

where $\psi_!$ and ψ_* are Thom isomorphisms in K-theory and cohomology theory, respectively.

Proposition 3.2 *For any $x \in K(EG \times_G M)$, one has*

$$ch(\psi_!(x)) = \psi_*(ch(x) \cdot {Td_G}^{-1}(E)),$$

where $Td_G(E) := Td(EG \times_G E) \in H^{**}(EG \times_G M; \mathbb{Q})$ is the total Todd class of the unitary vector bundle $EG \times_G E$.

Proof By Proposition 3.1, we will use the finite approximation method to finish the proof. Let $i_n : EG(n) \times_G M \hookrightarrow EG \times_G M$ be the inclusion induced by $EG(n) \longrightarrow EG$. For the Todd genus $Td_G(E)$, since $i_n^*(Td_G(E)) = i_n^*(Td(EG \times_G E)) = Td(EG(n) \times_G E)$, one has

$$\varprojlim Td(EG(n) \times_G E) = Td_G(E).$$

Taking the inverse, one has $\varprojlim Td^{-1}(EG(n) \times_G E) = {Td_G}^{-1}(E)$.

Let $x \in K^*(EG \times_G M)$ and $x_n = i_n^!(x) \in K^*(EG(n) \times_G M)$. One has the following commutative diagram:

$$\begin{tikzcd} K^*(EG \times_G M) \arrow[r, "i_n^!"] \arrow[d, "\psi_!"', "\cong"] & K^*(EG(n) \times_G M) \arrow[d, "\psi_{n!}"', "\cong"] \\ K^*(EG \times_G D(E), EG \times_G S(E)) \arrow[r, "j_n^!"] & K^*(EG(n) \times_G D(E), EG(n) \times_G S(E)) \end{tikzcd}$$

where j_n is the bundle map induced by i_n. The vector bundle $EG(n) \times_G E$ is the pull-back of $EG \times_G E$, so the Thom class $t(EG(n) \times_G E)$ of $EG(n) \times_G E$ satisfies $j_n^!(t(EG \times_G E)) = t(EG(n) \times_G E)$. It follows that $\varprojlim t(EG(n) \times_G E) = t(EG \times_G E)$.

By the definition of Thom isomorphism, we have that $\psi_{n!}(x_n) = \pi_n^!(x_n) \cdot t(EG(n) \times_G E)$ and $\psi_!(x) = \pi^!(x) \cdot t(EG \times_G E)$, where π_n (resp. π) denotes the projective map $EG(n) \times_G (D(E)/S(E)) \longrightarrow EG(n) \times_G M$ (resp. $EG \times_G (D(E)/S(E)) \longrightarrow EG \times_G M$). Since $\varprojlim \pi_n^!(x_n) = \pi^!(x)$ and $\varprojlim t(EG(n) \times_G E) = t(EG \times_G E)$, it follows that

$$\begin{aligned}
\psi_!(x) &= \pi^!(x) \cdot t(EG \times_G E) \\
&= \varprojlim \pi_n^!(x_n) \cdot \varprojlim t(EG(n) \times_G E) \\
&= \varprojlim (\pi_n^!(x_n) \cdot t(EG(n) \times_G E)) \\
&= \varprojlim \psi_{n!}(x_n).
\end{aligned}$$

Similarly, one can show that

$$\psi_*(ch(x) Td_G{}^{-1}(E)) = \varprojlim \psi_{n*}(ch(x_n) \cdot Td^{-1}(EG(n) \times_G E)).$$

On the other hand, since $EG(n) \times_G M$ is compact, by the nonequivariant Riemann-Roch relation (see Proposition 2.2), we have

$$ch(\psi_{n!}(x_n)) = \psi_{n*}(ch(x_n) \cdot Td^{-1}(EG(n) \times_G E)).$$

Taking the inverse limit, one has

$$\begin{aligned}
ch(\psi_!(x)) &= ch(\varprojlim \psi_{n!}(x_n)) \\
&= \varprojlim ch(\psi_{n!}(x_n)) \\
&= \varprojlim \psi_{n*}(ch(x_n) \cdot Td^{-1}(EG(n) \times_G E)) \\
&= \psi_*(\varprojlim ch(x_n) \cdot \varprojlim Td^{-1}(EG(n) \times_G E)) \\
&= \psi_*(ch(x) \cdot Td_G{}^{-1}(E)),
\end{aligned}$$

as desired.

3.3 An equivariant version of Riemann-Roch relation of Atiyah-Hirzebruch type

Let M and N be closed unitary G-manifolds and $f : M \longrightarrow N$ be a G-map. Then f induces equivariant Gysin maps

$$\begin{aligned}
&f_!^G : H^{**}(EG \times_G M) \longrightarrow H^{**}(EG \times_G N), \\
&f_!^{G,K} : K^*(EG \times_G M) \longrightarrow K^*(EG \times_G N), \\
&f_!^{G,K_G} : K_G^*(M) \longrightarrow K_G^*(N)
\end{aligned}$$

in three equivariant cohomology theories $H^*(EG \times_G -)$, $K^*(EG \times_G -)$ and $K_G^*(-)$, respectively.

Since the Chern character is a natural transformation from K-theory to ordinary cohomology, one has the following diagram:

$$\begin{array}{ccc}
K^*(EG \times_G M) & \xrightarrow{ch} & H^{**}(EG \times_G M; \mathbb{Q}) \\
{\scriptstyle f_!^{G,K}} \big\downarrow & & {\scriptstyle f_!^G} \big\downarrow \\
K^*(EG \times_G N) & \xrightarrow{ch} & H^{**}(EG \times_G N; \mathbb{Q})
\end{array}$$

Still, this diagram is not commutative. In a way similar to the nonequivariant case, we have the following equivariant Riemann-Roch relation.

Theorem 3.1 *Let M and N be closed unitary G-manifolds with unitary stable tangent bundles, still denoted by TM and TN, respectively. Let $f : M \longrightarrow N$ be a G-map. Then for any $x \in K^*(EG \times_G M)$,*

$$ch(f_!^{G,K}(x)) \cdot Td_G(TN) = f_!^G(ch(x) \cdot Td_G(TM)).$$

Proof By the definition of equivariant Gysin map, choose an G-embedding $f \times e : M \hookrightarrow N \times V$ with normal bundle η, where V is a G-representation. Then one can obtain the following diagram:

$$\begin{array}{ccc}
K^*(EG\times_G M) & \xrightarrow{ch} & H^{**}(EG\times_G M;\mathbb{Q}) \\
\phi_1^K \Big\downarrow \cong & & \phi_1^H \Big\downarrow \cong \\
K^*(EG\times_G D(\eta), EG\times_G S(\eta)) & \xrightarrow{ch} & H^{**}(EG\times_G D(\eta), EG\times_G S(\eta);\mathbb{Q}) \\
\phi_2^K \Big\downarrow & & \phi_2^H \Big\downarrow \\
K^*(EG\times_G (N\times D(V)), EG\times_G (N\times S(V))) & \xrightarrow{ch} & H^{**}(EG\times_G (N\times D(V)), EG\times_G (N\times S(V));\mathbb{Q}) \\
\phi_3^K \Big\uparrow \cong & & \Phi_3^H \Big\uparrow \cong \\
K^*(EG\times_G N) & \xrightarrow{ch} & H^{**}(EG\times_G N;\mathbb{Q})
\end{array}$$

Note that the middle square in the above diagram is commutative. For any $x \in K^*(EG\times_G M)$, by Proposition 3.2, we have the following relation:

$$ch(\phi_1^K(x)) = \phi_1^H(ch(x) \cdot Td_G{}^{-1}(\eta)).$$

Since $ch\phi_2^K = \phi_2^H ch$, we have

$$ch(\phi_2^K\phi_1^K(x)) = \phi_2^H ch(\phi_1^K(x)) = \phi_2^H\phi_1^H(ch(x) \cdot Td_G{}^{-1}(\eta)).$$

Let $y = f_!^{G,K}(x) = (\phi_3^K)^{-1}\phi_2^K\phi_1^K(x)$. We see that $\phi_3^K(y) = \phi_2^K\phi_1^K(x)$ and

$$ch(\phi_3^K(y)) = \phi_3^H(ch(y) \cdot Td_G{}^{-1}(V)),$$

where $Td_G(V) = Td(EG\times_G V)$ is the equivariant total Todd class of the G-bundle $V\times N \longrightarrow N$. Put these equalities together,

$$\begin{aligned}
ch(f_!^{G,K}(x)) \cdot Td_G^{-1}(V) &= ch(y) \cdot Td_G^{-1}(V) \\
&= (\phi_3^H)^{-1} ch(\phi_3^K(y)) \\
&= (\phi_3^H)^{-1} ch(\phi_2^K\phi_1^K(x)) \\
&= (\phi_3^H)^{-1}\phi_2^H\phi_1^H(ch(x) \cdot Td_G{}^{-1}(\eta)) \\
&= f_!^G(ch(x) \cdot Td_G{}^{-1}(\eta)),
\end{aligned}$$

and one has

$$ch(f_!^{G,K}(x)) = f_!^G(ch(x) \cdot Td_G{}^{-1}(\eta)) \cdot Td_G(V).$$

On the other hand, we see that $TM \oplus \eta = (f \times e)^*T(N \times V) \cong f^*(TN \oplus V)$, where $f^*(TN \oplus V)$ is the pull-back of the bundle $TN \oplus V \longrightarrow N$ via $f : M \longrightarrow N$. It follows that

$$Td_G(\eta) \cdot Td_G(TM) = f^*(Td_G(TN) \cdot Td_G(V)).$$

Hence,

$$\begin{aligned} f_!^G(ch(x)\cdot Td_G(TM)) &= f_!^G(ch(x)\cdot Td_G{}^{-1}(\eta)\cdot f^*(Td_G(TN)\cdot Td_G(V))) \\ &= f_!^G(ch(x)\cdot Td_G{}^{-1}(\eta))\cdot Td_G(V)\cdot Td_G(TN) \\ &= ch(f_!^{G,K}(x))\cdot Td_G(TN), \end{aligned}$$

as desired.

There is a natural transformation from $K_G(-)$ to $K(EG\times_G -)$

$$\begin{gathered} \alpha : K_G(X) \longrightarrow K(EG\times_G X), \\ E \mapsto EG\times_G E \end{gathered}$$

for any G-space X and unitary G-vector bundle E over X. Then we define the equivariant Chern character as follows.

Definition 3.1 *For any G-space X, the equivariant Chern character ch_G is defined by the composition of α and ch:*

$$ch_G := ch\circ\alpha : K_G(X) \longrightarrow K(EG\times_G X) \longrightarrow H^{**}(EG\times_G X;\mathbb{Q}).$$

By choosing the Thom classes, for any G-map $f : M \to N$ between two closed unitary G-manifolds, one has the following commutative diagram:

$$\begin{array}{ccc} K_G(M) & \xrightarrow{\alpha} & K(EG\times_G M) \\ {\scriptstyle f_!^{G,K_G}}\big\downarrow & & {\scriptstyle f_!^{G,K}}\big\downarrow \\ K_G(N) & \xrightarrow{\alpha} & K(EG\times_G N) \end{array}$$

Proposition 3.3 *Let M and N be closed unitary G-manifolds with unitary stable tangent bundles which are still denoted by TM and TN. Let $f : M \longrightarrow N$ be a G-map. Then for any $x \in K_G^*(M)$,*

$$ch_G(f_!^{G,K_G}(x))\cdot Td_G(TN) = f_!^G(ch_G(x)\cdot Td_G(TM)).$$

Proof By Theorem 3.1, for $\alpha(x) \in K^*(EG\times_G M)$, one has

$$ch(f_!^{G,K}(\alpha(x)))\cdot Td_G(TN) = f_!^G(ch(\alpha(x))\cdot Td_G(TM)).$$

Since $\alpha f_!^{G,K_G} = f_!^{G,K}\alpha$, it follows that

$$\begin{aligned} ch_G(f_!^{G,K_G}(x))\cdot Td_G(TN) &= ch(\alpha f_!^{G,K_G}(x))\cdot Td_G(TN) \\ &= ch(f_!^{G,K}\alpha(x))\cdot Td_G(TN) \\ &= f_!^G(ch(\alpha(x))\cdot Td_G(TM)) \\ &= f_!^G(ch_G(x)\cdot Td_G(TM)), \end{aligned}$$

as desired.

4 T^k-K-Theoretic Chern Numbers and T^k-integral Cohomology Chern Numbers

Now we are going to prove our main theorem.

Theorem 4.1 *Let M be a unitary T^k-manifold with stable complex tangent bundle $TM \in K_{T^k}(M)$. Then all integral equivariant cohomology Chern numbers of M vanish if and only if all equivariant K-theoretic Chern numbers of M vanish.*

Proof First, assume that all integral equivariant cohomology Chern numbers of M vanish. Then for any equivariant K-theoretic Chern number

$$c_\omega^{T^k,K_{T^k}}[M]_{T^k} = p_!^{T^k,K_{T^k}}(c_\omega^{T^k,K_{T^k}}(M)),$$

where ω is a partition. By Proposition 3.3, one has

$$\begin{aligned} ch_{T^k}(c_\omega^{T^k,K_{T^k}}[M]_{T^k}) &= ch_{T^k}(p_!^{T^k,K_{T^k}}(c_\omega^{T^k,K_{T^k}}(M))) \\ &= p_!^{T^k}(ch_{T^k}(c_\omega^{T^k,K_{T^k}}(M)) \cdot Td_{T^k}(TM)). \end{aligned}$$

Since $ch_{T^k}(c_\omega^{T^k,K_{T^k}}(M)) \cdot Td_{T^k}(TM)$ is a combination of equivariant cohomology Chern classes, it follows that $ch_{T^k}(c_\omega^{T^k,K_{T^k}}[M]_{T^k})$ can be represented by a combination of equivariant cohomology Chern numbers. By assumption, one has

$$ch_{T^k}(c_\omega^{T^k,K_{T^k}}[M]_{T^k}) = 0.$$

On the other hand, $ch_{T^k} : K_{T^k}(pt) \longrightarrow H^{**}_{T^k}(pt;\mathbb{Q})$ is injective and the equivariant K-theoretic Chern number $c_\omega^{T^k,K_{T^k}}[M]_{T^k} = 0$.

Second, assume that all the equivariant K-theoretic Chern numbers vanish. For any integral equivariant cohomology Chern number $c_\omega^{T^k}[M]_{T^k}$, consider the equivariant K-theoretic Chern class $c_\omega^{T^k,K_{T^k}}(M)$. One has

$$ch_{T^k}(c_\omega^{T^k,K_{T^k}}(M)) = c_\omega^{T^k}(M) + \text{terms of higher degree in } H^{**}_{T^k}(M;\mathbb{Q})$$

and

$$\begin{aligned} 0 &= ch_{T^k}(c_\omega^{T^k,K_{T^k}}[M]_{T^k}) \\ &= ch_{T^k}(p_!^{T^k,K_{T^k}}(c_\omega^{T^k,K_{T^k}}(M))) \\ &= p_!^{T^k}(ch_{T^k}(c_\omega^{T^k,K_{T^k}}(M)) \cdot Td_{T^k}(TM)) \\ &= p_!^{T^k}(c_\omega^{T^k}(M) + \text{terms of higher degree}) \\ &= c_\omega^{T^k}[M]_{T^k} + \text{terms of higher degree in } H^{**}_{T^k}(pt;\mathbb{Q}). \end{aligned}$$

It follows that $c_\omega^{T^k}[M]_{T^k} = 0$ in $H^{**}_{T^k}(pt;\mathbb{Q}) = H^{**}(BT^k;\mathbb{Q})$, which implies that $c_\omega^{T^k}[M]_{T^k} = 0$ in $H^*(BT^k,\mathbb{Z})$ since $H^*(BT^k)$ is torsion-free (see [6]). Thus we have proved that all integral equivariant cohomology Chern numbers of M vanish.

Acknowledgements The author is grateful to Prof. Zhi Lü for very helpful conversations and comments, and he would also like to thank Prof. Peter Landweber for very helpful suggestions and comments.

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