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Difference Cochains and Reidemeister Traces*

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Abstract The authors consider the difference of Reidemeister traces and difference cochain of given two self-maps, and find out a relation involving these two invariants. As an application, an inductive formula of the Reidemeister traces for self-maps on a kind of CW-complex, including spherical manifolds is obtained.

Keywords Fixed point, Reidemeister trace, Difference cochain, Spherical manifold
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1 Introduction

The Reidemeister trace was introduced by Reidemeister [18] for the study of fixed points of a self-map $f : X \to X$. The Reidemeister trace is a crucial invariant in Nielsen fixed point theory, whose vanishing is a sufficient and necessary condition of deforming f to be fixed point free whenever X is a compact smooth manifold of dimension greater than two. But the computation of Reidemeister traces turns out to be a very hard job in general. The precise formulae of Reidemeister traces are known only on some special spaces, such as Jiang spaces (see [14]), Nil-manifolds, solvmanifolds, and infra-nilmanifolds (see [15]). In this note, we restrict ourselves to a special kind of CW-complexes, including the orbit space $\mathbb{S}^n/G$ of sphere under a free action by a finite group G, over which the computations can be carried out inductively through a relation between the difference of Reidemeister traces and the difference cochain. In geometry, the spaces $\mathbb{S}^n/G$ are characterized by complete Riemannian manifolds of positive constant sectional curvatures, which are kinds of space forms.

The machinery of obstruction theory was introduced for the first time into the context of Nielsen fixed point theory by Fuller [5], where he considered the coincidence problem of two maps from a complex into a manifold by using tools of obstruction theory. In early 1980, Fadell and Husseini re-interpreted the problem of removing the fixed points of a map through homotopy in the language of obstruction theory in [4], where they gave an obstruction theory proof of the theorem that, in the language of the Reidemeister trace, a self-map $f : X \to X$ on a finite complex X can be deformed to be fixed-point-free if and only if its Reidemeister trace $\mathrm{RT}(f)$ vanishes, under the assumption that $\dim(X)$ is greater than two. Varieties of obstruction for fixed points were taken up again by Goncalves and his collaborators (see [2, 7–9]). About the recent development of this topic, the readers can consult [10] and the references therein.

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The viewpoint we take towards the machinery of obstruction theory in current note is distinct from existed ones. The difference cochains are in consideration, which are cohomological invariants lying in a cochain group with local coefficients, while the Reidemeister trace is a K-theoretical invariant which lies in a free Abelian group generated by twisted conjugate classes of an integral group ring of the fundamental group. A fascinating idea is to ask for a straightforward connection between these two invariants. In our settings, we indeed find out such a connection and use it to compute Reidemeister traces and Nielsen numbers. This is the key point of our current work.

Here is an outline of this note. In Sections 2 and 3, we shall recall respectively some notions of Reidemeister trace and difference cochain. In Section 4, we shall formulate our main result Theorem 4.1. In Section 5, we shall confine ourselves to spherical 3-manifolds, and make the computations of the Reidemeister trace $\mathrm{RT}(f)$ and the Nielsen number $N(f)$ for every self-map f on $\mathbb{S}^3/T^{\star}_{24}$.

2 Reidemeister Traces

In this section, we recall from [12–13] some definitions related to the Reidemeister trace. The readers can refer to [6, 14] for an introduction to Nielsen fixed point theory.

Definition 2.1 (see [12–13]) *Let $\psi : G \to G$ be an endomorphism of a group G. Two elements α and β in G are said to be ψ-conjugate if there is an element $\gamma \in G$ such that $\alpha = \gamma\beta\psi(\gamma^{-1})$. We denote by $[\alpha]_\psi$ the ψ-conjugate class of α and by G_ψ the set of ψ-conjugate classes.*

Let $(X, \star)$ be a connected, finite CW-complex with base point $\star$, and $\pi : (\widetilde{X}, \widetilde{\star}) \to (X, \star)$ be the universal covering with base point $\widetilde{\star}$. Then the deck transformation group of $\pi : \widetilde{X} \to X$ can be identified with the fundamental group $\pi_1(X, \star)$ which is assumed to act on $\widetilde{X}$ from the right. By choosing a lifting for each cell in X, the cellular chain complex $C_*(\widetilde{X})$ turns out to be a free right $\mathbb{Z}[\pi_1(X, \star)]$-module.

Let $f : (X, \star) \to (X, \star)$ be a cellular map on X. There is a unique lifting $\widetilde{f}$ of f such that $\widetilde{f}(\widetilde{\star}) = \widetilde{\star}$. As a self-map on $\widetilde{X}$, $\widetilde{f}$ induces a $(\mathbb{Z}[\pi_1(X, \star)], f_\pi)$-chain map $\widetilde{f}_\sharp$ on $C_*(\widetilde{X})$, satisfying $\widetilde{f}_\sharp(\widetilde{\sigma}\alpha) = \widetilde{f}_\sharp(\widetilde{\sigma})f_\pi(\alpha)$ for each $\alpha \in \pi_1(X, \star)$ and each cell $\widetilde{\sigma} \in C_*(\widetilde{X})$, where $f_\pi : \pi_1(X, \star) \to \pi_1(X, \star)$ is the endomomorphism induced by f.

Definition 2.2 (see [12]) *The Reidemeister trace* $\mathrm{RT}(f)$ *is defined to be the alternating sum*

$$\sum_{j=0}^{+\infty}(-1)^j[\mathrm{tr}(\widetilde{f}_\sharp|_{C_j(\widetilde{X})})]_{f_\pi},$$

where $\widetilde{f}_\sharp|_{C_j(\widetilde{X})}$ stands for the module endomorphism induced by $\widetilde{f}$ on the jth-chain group $C_j(\widetilde{X})$, and can be represented by a square matrix with entries in $\mathbb{Z}[\pi_1(X, \star)]$ if we fix a lifting for each cells in X.

The Reidemeister trace $\mathrm{RT}(f)$ takes its value in the free Abelian group generated by f_π-conjugacy classes. The sum of all coefficients is the Lefschetz number $L(f)$ of f. The number of distinct items of $\mathrm{RT}(f)$ is the Nielsen number $N(f)$ of f, which is a crucial invariant in fixed point theory. This number is always finite, since X has finitely many cells.

Actually, $\mathrm{RT}(f)$ is independent of the choice of liftings of cells of X, which constitute a basis of $C_*(\widetilde{X})$ as a free $\mathbb{Z}[\pi_1(X,\star)]$-module. Moreover, one can define the Reidemeister trace for arbitrary self-map by using approximate cellular self-maps, furthermore homotopic self-maps have the same Reidemeister traces.

3 Difference Cochains

In this section, we recall from [1] the notion of difference cochain associated to a pair of maps. For our purpose we focus on pairs of self-maps.

Definition 3.1 *Let $f, f' : (X,\star) \to (X,\star)$ be two cellular self-maps on a connected n-dimensional finite CW-complex such that they are the same on the $(n-1)$-skeleton of X, i.e., $f'|_{X^{(n-1)}} = f|_{X^{(n-1)}}$. The difference cochain $\Delta(f,f')$ associated to the pair (f,f') is defined to be an element of $C^n(X,\pi_n(X))$, satisfying that $\Delta(f,f')(\tau) = [d(f,f',\tau)]$ for each n-cell τ of X, where $\pi_n(X)$ is the local system on X and $d(f,f',\tau) : \mathbb{S}^n \to X$ is given by*

$$d(f,f',\tau)(x) = \begin{cases} f(\tau(x)), & x \in \mathbb{S}^n_+, \\ f'(\tau(x)), & x \in \mathbb{S}^n_-. \end{cases}$$

Here, both $\mathbb{S}^n_+$ and $\mathbb{S}^n_-$ are identified with $\mathbb{D}^n$.

Remark 3.1 If $f'|_{X^{(n-1)}} \simeq f|_{X^{(n-1)}}$, then one can also define difference cochain $\Delta_F(f,f')$, where F is a homotopy from $f|_{X^{(n-1)}}$ to $f'|_{X^{(n-1)}}$. For the details, the readers may refer to [1, Chapter 4].

Lemma 3.1 (see [1, Theorem 4.2.9]) *Let $f, f' : (X,\star) \to (X,\star)$ be two cellular self-maps on a connected n-dimensional finite CW-complex such that $f'|_{X^{(n-1)}} = f|_{X^{(n-1)}}$. Then f and f' are homotopic if and only if the difference cochain $\Delta(f,f')$ vanishes.*

Consider the universal covering $\pi : (\widetilde{X},\widetilde{\star}) \to (X,\star)$ with base point. Similarly, one can define difference cochain $\Delta(\widetilde{f},\widetilde{f'})$ for some self-maps $\widetilde{f}$ and $\widetilde{f'}$ on $\widetilde{X}$. By definition, we obtain immediately the following lemma.

Lemma 3.2 *Let*

$$f, f' : (X,\star) \to (X,\star)$$

be two cellular self-maps on a connected n-dimensional finite CW-complex such that $f'|_{X^{(n-1)}} = f|_{X^{(n-1)}}$. Let $\widetilde{f}$ and $\widetilde{f'}$ be respectively liftings of f and f' with respect to the universal covering space $\pi : (\widetilde{X},\widetilde{\star}) \to (X,\star)$, fixing the base point $\widetilde{\star}$. Then $\pi_\sharp(\Delta(\widetilde{f},\widetilde{f'})(\widetilde{\tau})) = \Delta(f,f')(\tau)$ for any n-cell τ of X and its lifting $\widetilde{\tau}$.

4 Main Result

In this section, we illustrate a relation between difference $\mathrm{RT}(f) - \mathrm{RT}(f')$ of Reidemeister traces and difference cochain $\Delta(\widetilde{f},\widetilde{f'})$ for two given self-maps f and f'. Throughout this section, we always assume the following assumption.

Assumption 4.1 *The CW-complex Y is a connected, n-dimensional finite CW-complex which has a unique 0-cell $\star$, where $n \geq 2$. The universal covering space $\widetilde{Y}$ with base point $\widetilde{\star}$ of Y is $(n-1)$-connected, i.e., $\pi_j(\widetilde{Y},\widetilde{\star}) = 0$ for $j < n$.*

Thus, the universal covering space $\widetilde{Y}$ has a structure of CW-complex which is $\pi_1(Y,\star)$-equivariant.

Lemma 4.1 *Let Y be as in Assumption* 4.1. *Let $f, f' : (Y,\star) \to (Y,\star)$ be two cellular self-maps such that $f_\pi = f'_\pi$. Then $f|_{Y^{(n-1)}} \simeq f'|_{Y^{(n-1)}} : Y^{(n-1)} \to Y$.*

Proof We denote the set of 1-cells of Y by $\{e_1^1, \cdots, e_k^1\}$. Each cell e_i^1 has a characteristic map $\chi_i : (I,0,1) \to (Y,\star,\star)$. It is well-known that $\{[e_1^1], \cdots, [e_k^1]\}$ constitute a basis of generators of $\pi_1(Y,\star)$. By the assumption that $f_\pi = f'_\pi$, we see that $f \circ \chi_i \simeq f' \circ \chi_i$ for each i. Since $Y^{(1)} = \bigvee_{i=1}^{k} e_i^1$, $f|_{Y^{(1)}} \simeq f'|_{Y^{(1)}} : Y^{(1)} \to Y$.

Since $\pi_j(Y) = \pi_j(\widetilde{Y}) = 0$ for $j = 2, 3, \cdots, n-1$, we obtain that $f|_{Y^{(n-1)}} \simeq f'|_{Y^{(n-1)}}$ by a usual argument in homotopy theory.

Lemma 4.2 *Let Y be as in Assumption* 4.1. *Let $f, f' : (Y,\star) \to (Y,\star)$ be two cellular self-maps such that $f|_{Y^{(n-1)}} \simeq f'|_{Y^{(n-1)}}$. Then there exists a self-map $f'' : Y \to Y$, with $f'' \simeq f'$ and $f''|_{Y^{(n-1)}} = f|_{Y^{(n-1)}}$.*

Proof Let $H : Y^{(n-1)} \times I \to Y$ be a homotopy from $f|_{Y^{(n-1)}}$ to $f'|_{Y^{(n-1)}}$. We consider the partial homotopy $H \cup f' : (Y^{(n-1)} \times I) \cup (Y \times \{1\} \to Y)$ whose definition is obvious. Since $Y^{(n-1)}$ is a sub-complex of Y, by using the homotopy extension property, the partial homotopy $H \cup f'$ can be extended into a homotopy $G : Y \times I \to Y$. The self-map f'', defined by $f''(x) = G(x, 0)$, is the desired self-map.

Lemma 4.3 *Let Y as in Assumption* 4.1. *Let $f, f' : (Y,\star) \to (Y,\star)$ be two cellular self-maps such that $f|_{Y^{(n-1)}} = f'|_{Y^{(n-1)}}$. Let $\widetilde{f}$ and $\widetilde{f}'$ be respectively the liftings of f and f' such that $\widetilde{f}(\widetilde{\star}) = \widetilde{f}'(\widetilde{\star}) = \widetilde{\star}$. Then*

$$\theta(\Delta(\widetilde{f}, \widetilde{f}')(\widetilde{\tau})) = [\widetilde{f}(\widetilde{\tau}) - \widetilde{f}'(\widetilde{\tau})]$$

for any n-cell τ of X and its lifting $\widetilde{\tau}$, where $\theta : \pi_n(\widetilde{Y}, \widetilde{\star}) \to H_n(\widetilde{Y}, \mathbb{Z})$ is the Hurewicz homomorphism.

Proof Since $f|_{Y^{(n-1)}} = f'|_{Y^{(n-1)}}$ and $\widetilde{f}(\widetilde{\star}) = \widetilde{f}'(\widetilde{\star})$, we have that $\widetilde{f}|_{\widetilde{Y}^{(n-1)}} = \widetilde{f}'|_{\widetilde{Y}^{(n-1)}}$. Hence, $\widetilde{f}(\widetilde{\tau}) - \widetilde{f}'(\widetilde{\tau})$ is an n-cycle of $\widetilde{Y}$.

By definition of difference cochain (see Definition 3.1), $\Delta(\widetilde{f}, \widetilde{f}')(\widetilde{\tau})$ is the homotopy class of a continuous map $d(\widetilde{f}, \widetilde{f}', \widetilde{\tau}) : \mathbb{S}^n_+ \cup \mathbb{S}^n_- \to \widetilde{Y}$, given by

$$d(\widetilde{f}, \widetilde{f}', \widetilde{\tau})(x) = \begin{cases} \widetilde{f}(\widetilde{\tau}(x)), & x \in \mathbb{S}^n_+, \\ \widetilde{f}'(\widetilde{\tau}(x)), & x \in \mathbb{S}^n_-. \end{cases}$$

Our conclusion holds by definition of the Hurewicz homomorphism.

It should be mentioned that since $\widetilde{Y}$ is $(n-1)$-connected, the Hurewicz homomorphism $\theta : \pi_n(\widetilde{Y}, \widetilde{\star}) \to H_n(\widetilde{Y}, \mathbb{Z})$ is an isomorphism.

Since Y is n-dimensional, the homology $H_n(\widetilde{Y}; \mathbb{Z})$ is a subgroup of $C_n(\widetilde{Y}; \mathbb{Z})$. Let $\{\tau_i \mid i = 1, 2, \cdots, r\}$ be the set of all n-cells of Y, each τ_i is chosen a lifting $\widetilde{\tau}_i$. We have, for each i,

$$C_n(\widetilde{Y}, \mathbb{Z}) = \bigoplus_{k=1}^{r} \mathbb{Z}[\pi_1(Y,\star)] \cdot \widetilde{\tau}_k \xrightarrow{P_i} \mathbb{Z}[\pi_1(Y,\star)] \cdot \widetilde{\tau}_i \xrightarrow{\epsilon_i} \mathbb{Z}[\pi_1(Y,\star)] \to \mathbb{Z}[\pi_1(Y,\star)_{f_\pi}], \tag{4.1}$$

where P_i is the projection onto the ith factor, ϵ_i is the augmentation map and the last arrow is the quotient map.

Theorem 4.1 *Let Y be a connected, n-dimensional finite CW-complex which has a unique 0-cell $\star$ and $\pi_i(Y) = 0$ for $i = 2, 3, \cdots, n-1$. Suppose that $f, f' : Y \to Y$ are two cellular self-maps, which satisfy that $f_\pi = f'_\pi$. Let $\widetilde{f}$ and $\widetilde{f}'$ be respectively the liftings of f and f' such that $\widetilde{f}(\widetilde{\star}) = \widetilde{f}'(\widetilde{\star}) = \widetilde{\star}$. Then*

$$\mathrm{RT}(f) - \mathrm{RT}(f') = (-1)^n \sum_{i=1}^{r} \Phi_i \circ \theta(\Delta(\widetilde{f}, \widetilde{f}')(\widetilde{\tau}_i)), \tag{4.2}$$

where $\{\tau_i \mid i = 1, 2, \cdots, r\}$ is the set of all n-cells of Y, for each τ_i a lifting $\widetilde{\tau}_i$ is chosen, $\theta : \pi_n(\widetilde{Y}, \widetilde{\star}) \to H_n(\widetilde{Y}, \mathbb{Z})$ is the Hurewicz homomorphism, and Φ_i is the composition of homomorphisms appearing in (4.1).

Proof By Lemma 4.2 and the homotopy invariance of Reidemeister traces, we may assume that $f'|_{Y^{(n-1)}} = f|_{Y^{(n-1)}}$.

Suppose that $\widetilde{f}(\widetilde{\tau}_i) = \sum\limits_{k=1}^{r} \widetilde{\tau}_k \beta_{i,k}$, $\widetilde{f}'(\widetilde{\tau}_i) = \sum\limits_{k=1}^{r} .\widetilde{\tau}_k \beta'_{i,k}$, where $\beta_{i,k}$ and $\beta'_{i,k}$ are elements of $\mathbb{Z}[\pi_1(Y, \star)]$. Since $f'|_{Y^{(n-1)}} = f|_{Y^{(n-1)}}$, by definition of Reidemeister trace, we see that

$$\mathrm{RT}(f) - \mathrm{RT}(f') = (-1)^n \sum_{i=1}^{r} [\beta_{i,i} - \beta'_{i,i}]_{f_\pi} = (-1)^n \sum_{i=1}^{r} [\epsilon_i \circ P_i(\widetilde{f}(\widetilde{\tau}_i) - \widetilde{f}'(\widetilde{\tau}_i))]_{f_\pi}.$$

Thus, Equation (4.2) follows immediately from Lemma 4.3.

With a given homomorphism $\varphi : \pi_1(Y, \star) \to \pi_1(Y, \star)$, we can form a set $\{f : Y \to Y \mid f_\pi = \varphi\}$/homotopy. If it is not empty, we can choose one self-map in this set and call it basic self-map, although the choice is not canonical. We would like to determine the Reidemeister trace $\mathrm{RT}(f)$ of an arbitrary self-map f. To this end, it suffices to compute the Reidemeister traces of basic maps in virtue of Theorem 4.1.

5 Spherical 3-Dimensional Manifolds

In this section, we shall apply our main theorem (Theorem 4.1) to self-maps on spherical 3-manifold X, i.e., the 3-manifold in the form of $\mathbb{S}^3/G$, where G is a finite subgroup of SU(2), acting freely on $\mathbb{S}^3$. In this situation, we shall see that Theorem 4.1 can be simplified greatly.

Since it is a smooth manifold, $X = \mathbb{S}^3/G$ has a CW cellular decomposition with unique 0-cell and unique 3-cell τ. In this section, we shall take such a decomposition of X. Hence our results in the previous section can be applied. Clearly, this decomposition yields a G-equivariant CW cellular decomposition of $\mathbb{S}^3$. For more details about equivariant CW cellular decomposition, the readers may refer to [19].

Recall that Olum [16–17] showed the following proposition.

Proposition 5.1 *Let X be an orientable* 3*-manifold with finite fundamental group and trivial $\pi_2(X)$. Every endomorphism $\psi : \pi_1(X) \to \pi_1(X)$ is induced by a* (*base-point preserving*) *continuous self-map $f : X \to X$. Furthermore, if f' is also a continuous self-map of X such that $f_\pi = f'_\pi = \psi$, then* $\deg(f) \equiv \deg(f') \mod |\pi_1(X)|$.

Clearly, each spherical 3-manifold satisfies the conditions in Proposition 5.1.

Proposition 5.2 *Let f and f' be two self-maps on a spherical 3-manifold $X = \mathbb{S}^3/G$ such that $f_\pi = f'_\pi$. Then*

$$\mathrm{RT}(f) - \mathrm{RT}(f') = \frac{\deg(f') - \deg(f)}{|G|} \cdot \sum_{\alpha \in G} [\alpha]_{f_\pi}, \tag{5.1}$$

where $|G|$ is the order of group G.

Proof By a similar argument as in Lemmas 4.1–4.2, we see that there is a self-map f'' on X such that $f'' \simeq f'$ and $f''|_{X^{(2)}} = f|_{X^{(2)}}$. Hence, $\mathrm{RT}(f') = \mathrm{RT}(f'')$ and $\deg(f') = \deg(f'')$. To prove Equation (5.1), it suffices to show that

$$\mathrm{RT}(f) - \mathrm{RT}(f'') = \frac{\deg(f'') - \deg(f)}{|G|} \sum_{\alpha \in G} [\alpha]_{f_\pi}.$$

Let $\pi : \mathbb{S}^3 \to X$ be the universal covering map. By general position, there is a ball B inside the unique 3-cell τ of X such that $f''^{-1}(B)$ is a disjoint union of balls: $B_1^+, \cdots, B_k^+$, $B_1^-, \cdots, B_l^-$. Moreover, we may assume that the restriction of f'' on each B_i^+ is orientation-preserving homeomorphism and the restriction of f'' on each B_j^+ is an orientation-reversing homeomorphism, where all orientations inherit from the universal covering space $\mathbb{S}^3$. On the other hand, $\pi^{-1}(B) = \bigcup\limits_{j=1}^{|G|} B_j$, where for each j, B_j is homeomorphic to B under $\pi|_{B_j}$. We write X' to be $\overline{X - B_1^+} \cup \overline{\mathbb{S}^3 - B_1}/\sim$, where $x \sim y$ if $f''(x) = \pi(y)$ for $x \in \partial B_1^+$ and $y \in \partial B_1$. Thus, $\deg(f'') = k - l$. Then it is clear that $X' \cong X$. Under this identification, we construct a map $f_1 : X' \to X$ as follows:

$$f_1(x) = \begin{cases} f''(x), & x \in \overline{X - B_1^X}, \\ \pi(x), & x \in \overline{\mathbb{S}^3 - B_1}. \end{cases}$$

By considering the preimage of the center point of B under f_1 and their signs, it is easy to see that $\deg(f_1) = (k-1) - l - (|G| - 1) = \deg(f'') - |G|$. By definition of f_1, we have that $f_1|_{X^{(2)}} = f''|_{X^{(2)}}$. By the definition of difference cochain $\Delta(f_1, f'')$, it follows that $\Delta(f_1, f'')(\tau)$ is equal to the homotopy class $[d(f_1, f'', \tau)]$.

Since τ preserves the orientation, it is not difficult to see that $d(f_1, f'', \tau)$ is homotopic to the composition $\pi \circ (-\mathrm{id}_{\mathbb{S}^3})$. Hence, $\Delta(f_1, f'')(\tau) = -[\pi]$. By Lemma 3.2, we see that $\Delta(\widetilde{f_1}, \widetilde{f''})(\widetilde{\tau}) = -[\mathrm{id}_{\mathbb{S}^3}]$. Then $\theta(\Delta(\widetilde{f_1}, \widetilde{f''})(\widetilde{\tau})) = -\sum\limits_{\alpha \in G} \widetilde{\tau} \cdot \alpha$. Together with Theorem 4.1, this implies that difference of the Reidemeister traces

$$\mathrm{RT}(f_1) - \mathrm{RT}(f'') = (-1)^3 \Big(- \sum_{\alpha \in G} [\alpha]_{f_\pi} \Big) = \sum_{\alpha \in G} [\alpha]_{f_\pi}.$$

Since $f|_{X^{(2)}} = f''|_{X^{(2)}}$, we have that $f_\pi = f''_\pi$. Hence, $\deg(f) - \deg(f'')$ is some multiple of $|G|$ by Proposition 5.1. Without loss of generality, we can assume that $\deg(f) = \deg(f'') - m \cdot |G|$ for some integer m. We can construct self-maps $f_1, \cdots, f_m$ such that $f_m \simeq f$ and

$\mathrm{RT}(f_i) - \mathrm{RT}(f_{i-1}) = -\sum_{\alpha\in G}[\alpha]_{f_\pi}$ for $i = 1, \cdots, m$, where f_0 is taken to be f''. Hence,

$$\begin{aligned}\mathrm{RT}(f) - \mathrm{RT}(f'') &= \mathrm{RT}(f_m) - \mathrm{RT}(f')\\ &= \sum_{i=1}^{m}(\mathrm{RT}(f_i) - \mathrm{RT}(f_{i-1}))\\ &= m\cdot\sum_{\alpha\in G}[\alpha]_{f_\pi}\\ &= \frac{\deg(f') - \deg(f)}{|G|}\sum_{\alpha\in G}[\alpha]_{f_\pi}.\end{aligned}$$

This proves the proposition.

In order to compute the Reidemeister trace $\mathrm{RT}(f)$ and hence the Nielsen number $N(f)$ for every self-map f on a spherical 3-manifold X, Equation (5.1) implies strongly for us that there may be a priori a formula for $\mathrm{RT}(f)$. Hence, the formulation of difference of Reidemeister traces in terms of difference cochain seems to be a correct way to fulfill the computations of Reidemeister traces. What is left to do is to spell out the "a priori" formula.

Note that $L(f) = 1 - \deg(f)$ for any self-map on $\mathbb{S}^3/G$. It is also true for any lifting $\widetilde{f}$ of f. From [13, Lemma 3.4] or [14, Chapter II, 5], we know that the fixed point class determined by $[\alpha]_{f_\pi}$, hence by lifting $\alpha\widetilde{f}$, has index

$$\frac{L(\widetilde{f})}{|\mathrm{Fix}(\tau_\alpha\circ f_\pi)|} = \frac{1-\deg(\widetilde{f})}{|\mathrm{Fix}(\tau_\alpha\circ f_\pi)|} = \frac{1-\deg(f)}{|\mathrm{Fix}(\tau_\alpha\circ f_\pi)|},$$

where τ_α is the inner-automorphism determined by α. We obtain the following lemma.

Lemma 5.1 *Let $f : X \to X$ be a self-map on a spherical 3-manifold X. Then*

$$\mathrm{RT}(f) = \sum_{[\alpha]\in(\pi_1(X))_{f_\pi}} \frac{1-\deg(f)}{|\mathrm{Fix}(\tau_\alpha\circ f_\pi)|}[\alpha]. \tag{5.2}$$

In what follows, we shall give a complete computation for self-map on $\mathbb{S}^3/T^\star_{24}$. The group $T^\star_{24}$ admits a presentation: $\langle a, b \mid a^2 = b^3 = (ab)^3 = -1\rangle$. There is a normal subgroup given by $\langle a, bab^{-1}\rangle$, which is isomorphic to $Q_8 = \{\pm1, \pm i, \pm j, \pm k\}$, which is a subgroup of quaternion number $\mathbb{H}$. Thus, the group $T^\star_{24}$ sits in the following short exact sequence:

$$1 \to Q_8 \to T^\star_{24} \to \mathbb{Z}_3 \to 1.$$

Since Q_8 consists of elements of order two and order four, each endomorphism ψ of $T^\star_{24}$ restricts to Q_8 as an endomorphism. In this way, by a direct computation, we know that up to an inner automorphism, all endomorphisms of $T^\star_{24}$ are

$$\mathrm{id},\ \psi^I,\ \phi,\ \psi^I\circ\phi,\ \text{trivial endomorphism},$$

where $\phi(a) = 1$, $\phi(b) = b^2$, $\psi^I(a) = a^{-1}$ and $\psi^I(b) = b^{-1}$. By (5.2) and a direct computation, we have the following proposition.

Proposition 5.3 *Let $f : \mathbb{S}^3/T^\star_{24} \to \mathbb{S}^3/T^\star_{24}$ be a self-map, having degree d. Then we have*

f_π	RT(f)	$N(f)$	$d \mod 24$
id	$\frac{1-\deg(f)}{24}([1]+[a^2]+6[a]$ $+4[b]+4[b^{-1}]+4[b^2]+4[b^{-2}])$	$\begin{cases}0, & \textit{if } d=1,\\ 7, & \textit{if } d\neq 1\end{cases}$	1
ψ^I	$\frac{1-\deg(f)}{4}(2[1]+[ab]+[ba])$	$\begin{cases}0, & \textit{if } d=1,\\ 3, & \textit{if } d\neq 1\end{cases}$	1
ϕ	$(1-\deg(f))\cdot[1]$	1	16
$\psi^I\circ\phi$	$\frac{1-\deg(f)}{3}([1]+[b]+[b^{-1}])$	3	16
trivial	$(1-\deg(f))\cdot[1]$	1	0

These results improve the computation in [11, §3.2]. The last column comes from [3]. Moreover, our arguments here can be applied to more general situation, especially for the orbit space $\mathbb{S}^n/G$.

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