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# On 3-Submanifolds of $S^3$ Which Admit Complete Spanning Curve Systems*

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**Abstract** Let $M$ be a compact connected 3-submanifold of the 3-sphere $S^3$ with one boundary component $F$ such that there exists a collection of $n$ pairwise disjoint connected orientable surfaces $\mathcal{S} = \{S_1, \cdots, S_n\}$ properly embedded in $M$, $\partial\mathcal{S} = \{\partial S_1, \cdots, \partial S_n\}$ is a complete curve system on $F$. We call $\mathcal{S}$ a complete surface system for $M$, and $\partial\mathcal{S}$ a complete spanning curve system for $M$. In the present paper, the authors show that the equivalent classes of complete spanning curve systems for $M$ are unique, that is, any complete spanning curve system for $M$ is equivalent to $\partial\mathcal{S}$. As an application of the result, it is shown that the image of the natural homomorphism from the mapping class group $\mathcal{M}(M)$ to $\mathcal{M}(F)$ is a subgroup of the handlebody subgroup $\mathcal{H}_n$.

**Keywords** Complete surface system, Complete spanning curve system, Heegaard diagram, Handlebody addition
**2000 MR Subject Classification** 57M99

## 1 Introduction

A complete curve system (CCS, simply) $\mathcal{J} = \{J_1, \cdots, J_n\}$ on a connected orientable closed surface $F$ of genus $n$ is a collection of pairwise disjoint simple closed curves such that the surface obtained by cutting $F$ open along $\mathcal{J}$ is a $2n$-punctured 2-sphere. Two CCSs on $F$ are equivalent if one can be obtained from another via finite number of band moves (defined in Section 2) and isotopies.

Let $M$ be a 3-manifold with one boundary component $F$. A complete surface system (CSS, simply) $\mathcal{S} = \{S_1, \cdots, S_n\}$ for $M$ is a collection of $n$ pairwise disjoint connected orientable surfaces properly embedded in $M$ such that $\partial\mathcal{S} = \{\partial S_1, \cdots, \partial S_n\}$ is a CCS on $F$. We also call $\partial\mathcal{S}$ a complete spanning curve system (CSCS, simply) for $M$ on $F$. Two CSSs $\mathcal{S}_1$ and $\mathcal{S}_2$ for $M$ are equivalent if the corresponding CSCSs $\partial\mathcal{S}_1$ and $\partial\mathcal{S}_2$ are equivalent on $F$.

By [8, Corollary 1.4], a CSS in a handlebody $H_n$ is just a complete disk system. It is a well-known fact that any two complete disk systems in a handlebody are equivalent, that is, the equivalent classes of complete disk systems for a handlebody are unique.

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Let $M$ be a compact 3-submanifold of the 3-sphere $S^3$ with one boundary component which admits a CSCS $\mathcal{J}$. Our first main result states that the equivalent classes of CSCSs for $M$ are unique, that is, any complete spanning curve system for $M$ is equivalent to $\mathcal{J}$.

Let $K$ be a knot in $S^3$. The exterior $\overline{S^3\backslash N(K)}$ of $K$ is denoted by $E(K)$. The slopes (isotopy classes of essential simple closed curves on $\partial E(K) = \partial N(K)$) can be parameterized by $\mathbb{Q} \cup \{\frac{1}{0}\}$. Let $\gamma$ be a slope. By doing a Dehn surgery on $K$ along $\gamma$, we obtain a 3-manifold $S^3(K;\gamma)$. Dehn surgery on a link in $S^3$ can be similarly defined.

Gordon-Luecke [3] showed that no non-trivial Dehn surgery on non-trivial knots in $S^3$ yields the 3-sphere $S^3$. That is, $S^3(K;\gamma)$ is not homeomorphic to $S^3$ for any non-trivial knot $K$ and $\gamma \neq \frac{1}{0}$. This implies that two knots are equivalent if and only if they have homeomorphic complements. On the other hand, there are non-trivial Dehn surgeries on links in $S^3$ producing $S^3$ (see for example, [10]).

Let $M$ be a compact 3-manifold with one boundary component $S$ of genus $n$, and $H_n$ a handlebody of genus $n$. Let $h : \partial H_n \to S$ be a homeomorphism. By gluing $M$ and $H_n$ via $h$, we obtain a closed 3-manifold $\widetilde{M} = M \cup_h H_n$. For a CCS $\mathcal{J}$ on $S = h(\partial H_n)$, if $\mathcal{J}$ occurs to be the boundary of a complete disk system for $H_n$ in $\widetilde{M}$, we say that $\widetilde{M}$ is obtained from $M$ by adding a handlebody along $\mathcal{J}$.

Let $M$ be a compact 3-submanifold of $S^3$ with one boundary component $S$ which admits a CSCS $\mathcal{K}$. Let $\mathcal{J}$ be a CCS on $S$, and $\widetilde{M}$ the manifold obtained from $M$ by adding a handlebody along $\mathcal{J}$. We know that if $\widetilde{M}$ is homeomorphic to $S^3$, then $(F;\mathcal{J},\mathcal{K})$ is a Heegaard diagram of $S^3$. From the uniqueness of Heegaard splittings of $S^3$, there exists a CSCS $\{\alpha_1,\cdots,\alpha_n\}$ for $M$ on $S$ which is equivalent to $\mathcal{K}$ and a complete disk system $\mathcal{E}$ for $H_n$ with $\partial\mathcal{E} = \{\beta_1,\cdots,\beta_n\}$ on $S$ which is equivalent to $\mathcal{J}$ such that $|\alpha_i \cap \beta_i| = 1$ for $1 \leq i \leq n$, and $|\alpha_i \cap \beta_j| = 0$ for $1 \leq i \neq j \leq n$. Our second result says that the same phenomenon happens when the surface $S$ splits $S^3$ into two general 3-submanifolds each of which admits a CSCS on $S$.

The paper is organized as follows. Section 2 contains some necessary preliminaries. In Section 3, we prove the main results and obtain some other properties of such submanifolds in $S^3$, including that the image of the natural homomorphism from the mapping class group $\mathcal{M}(M)$ to $\mathcal{M}(F)$ is a subgroup of the handlebody subgroup $\mathcal{H}_n$ of $\mathcal{M}(F)$.

## 2 Preliminaries

The terminologies and definitions used in the paper are all standard (see for example, refer to [6–7, 16]).

### 2.1 Equivalent complete spanning curve systems for 3-manifolds

In the subsection, we will introduce some definitions and facts on complete spanning curve systems for 3-manifolds.

**Definition 2.1** (1) *Let $F_n$ be a closed orientable surface of genus $n$. A CCS on $F_n$ is a collection $\mathcal{J}$ of $n$ pairwise disjoint simple closed curves on $F_n$ such that the surface obtained by cutting $F_n$ along $\mathcal{J}$ is a $2n$-punctured sphere.*

(2) *Let $M$ be a compact 3-manifold and $S$ a boundary component of $M$ with $g(S) = n \geq 1$. Let $\mathcal{J} = \{J_1, \cdots, J_n\}$ be a CCS on $S$. If there exists a collection of pairwise disjoint compact orientable surfaces $S_1, \cdots, S_n$ properly embedded in $M$ such that $\partial S_i = J_i$ for all $1 \leq i \leq n$, we call $\mathcal{S} = \{S_1, \cdots, S_n\}$ a CSS in $M$ (respect to $S$), and call $\mathcal{J}$ a CSCS for $M$ on $S$.*

A simple closed curve $J$ on a surface $S$ is essential if $J$ does not bound a disk on $S$.

**Definition 2.2** (1) *Let $J_1, J_2$ be two disjoint essential simple closed curves on a surface $S$, and $\gamma$ a simple arc on $S$ such that $\gamma \cap J_1$ is an end point of $\gamma$, $\gamma \cap J_2$ is another end point of $\gamma$, and the interior of $\gamma$ is disjoint from $J_1 \cup J_2$. Let $P = N(J_1 \cup \gamma \cup J_2)$ be a compact regular neighborhood of $J_1 \cup \gamma \cup J_2$ on $S$. Denote by $J_1 \#_\gamma J_2$ the boundary component of $P$ which is not isotopic to $J_1$ or $J_2$ on $P$, and call it the band sum of $J_1$ and $J_2$ along $\gamma$.*

(2) *Let $\mathcal{J} = \{J_1, \cdots, J_n\}$ be a CCS on a closed surface $S$ of genus $n > 0$. For $1 \leq i \neq j \leq n$, let $\gamma$ be a simple arc on $S$ such that $\gamma \cap J_i$ is an end point of $\gamma$, $\gamma \cap J_j$ is another end point of $\gamma$, and the interior of $\gamma$ is disjoint from $\bigcup\limits_{1 \leq i \leq n} J_i$. Let $J_{ij} = J_i \#_\gamma J_j$ be the band sum of $J_i$ and $J_j$ along $\gamma$. See Figure 1. By isotopy, we may assume that $J_{ij}$ is disjoint from the curves in $\mathcal{J}$. Replace $J_i$ or $J_j$ with $J_{ij}$ in $\mathcal{J}$ to get a CCS $\mathcal{J}'$ on $S$. We call $\mathcal{J}'$ a band move of $\mathcal{J}$.*

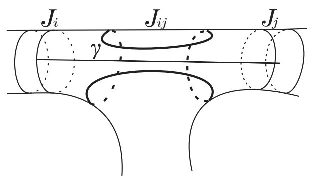


Figure 1 The band sum of $J_i$ and $J_j$.

It is clear that if $\mathcal{J}'$ is a band move of $\mathcal{J}$, then $\mathcal{J}$ is also a band move of $\mathcal{J}'$.

**Definition 2.3** (1) *Two CCSs $\mathcal{C}_1$ and $\mathcal{C}_2$ on a closed surface $S$ of genus $n > 0$ are called equivalent if one can be obtained from another by a finite number of band moves and isotopies.*

(2) *Let $M$ be a compact 3-manifold and $S$ a boundary component of $M$ with $g(S) = n \geq 1$. Let $\mathcal{S} = \{S_1, \cdots, S_n\}$ and $\mathcal{S}' = \{S'_1, \cdots, S'_n\}$ be two CSSs for $M$ with respect to $S$, and $\mathcal{J}$, $\mathcal{J}'$, the corresponding CSCSs on $S$. We say that $\mathcal{S}$ and $\mathcal{S}'$ are equivalent if $\mathcal{J}$ and $\mathcal{J}'$ are equivalent on $S$.*

A handlebody $H$ of genus $n$ is a 3-manifold which admits a complete disk system $\mathcal{D}$ such that the manifold obtained by cutting $H$ open along $\mathcal{D}$ is a 3-ball.

The following facts are well known, and proofs can be found in [8].

**Proposition 2.1** *Let $H$ be a handlebody of genus $n \geq 1$.*

(1) *The only complete surface system in $H$ is the complete disk system.*

(2) *Let $\mathcal{D} = \{D_1, \cdots, D_n\}$ be a complete disk system for $H$, and $\mathcal{J} = \partial\mathcal{D} = \{\partial D_1, \cdots, \partial D_n\}$. Then any CCS $\mathcal{K}$ on $\partial H$ which is equivalent to $\mathcal{J}$ is the boundary of a complete disk system for $H$. Moreover, any two CSCSs (therefore the boundaries of two complete disk systems) for $H$ are equivalent.*

**Proposition 2.2** *Let $M$ be a compact 3-manifold, and $S$ a boundary component of $M$ with genus $n > 0$. Let $\mathcal{J}$ be a CSCS for $M$ with respect to $S$. Then any CCS $\mathcal{J}'$ on $S$ which is equivalent to $\mathcal{J}$ is also a CSCS for $M$.*

### 2.2 Heegaard splittings

In the subsection, we will review some fundamental facts on Heegaard splittings of 3-manifolds.

A Heegaard splitting for a compact closed orientable 3-manifold $M$ is a decomposition of $M$ into two handlebodies $H$ and $H'$ of the same genus such that $H \cap H' = \partial H = \partial H' = S$ and $M = H \cup_S H'$. $S$ is called a Heegaard surface of $M$, and $g(S)$, the genus of $S$, is called the genus of the splitting. The genus of $M$, denoted by $g(M)$, is defined to be the minimal genus over all Heegaard splittings for $M$. It is a well-known fact that any compact closed orientable 3-manifold admits a Heegaard splitting (see [5]).

Let $V \cup_S W$ and $V' \cup_{S'} W'$ be two Heegaard splittings for a 3-manifold $M$. $V \cup_S W$ and $V' \cup_{S'} W'$ are called isotopic if their splitting surfaces $S$ and $S'$ are isotopic in $M$, and equivalent if, there exists a homeomorphism $h : M \to M$ which takes $S$ to $S'$ (that is, $h$ is a homeomorphism of triples from $(M, V, W)$ to $(M, V', W')$ or from $(M, V, W)$ to $(M, W', V')$). $V \cup_S W$ and $V' \cup_{S'} W'$ are called stably equivalent if, after a finite number of elementary stabilizations, they become isotopic.

**Lemma 2.1** *Let $(S; \mathcal{J}, \mathcal{J}')$ be a Heegaard diagram associated to a Heegaard splitting $H \cup_S H'$, and $h : S \to S$ a self-homeomorphism of $S$. Then the Heegaard splitting determined by the Heegaard diagram $(S; h(\mathcal{J}), h(\mathcal{J}'))$ is equivalent to $H \cup_S H'$.*

**Proof** Let $V \cup_S V'$ be the Heegaard splitting for $M'$ determined by the Heegaard diagram $(S; h(\mathcal{J}), h(\mathcal{J}'))$. From the definition of Heegaard diagram and [1, Theorem 3.8], it is clear that $h : S \to S$ can be extended to a homeomorphism $\overline{h} : (M, H, H') \to (M', V, V')$ of triples. Thus $V \cup_S V'$ is equivalent to $H \cup_S H'$.

It is a classical theorem (Reidemeister-Singer theorem, refer to [7] or [9]) on the stabilizations of Heegaard splittings that any two Heegaard splittings $V \cup_S W$ and $V' \cup_{S'} W'$ for a 3-manifold $M$ are stably equivalent.

Waldhausen [18] proved the following uniqueness theorem of the Heegaard splittings for $S^3$.

**Theorem 2.1** (see [18]) *Let $V \cup_S W$ be a Heegaard splitting of genus $n \geq 1$ for $S^3$. Then $V \cup_S W$ is a stabilization of the Heegaard splitting of genus 0 for $S^3$, i.e., for each genus, the Heegaard splitting for $S^3$ is unique.*

Let $V \cup_S W$ be a Heegaard splitting of genus $n \geq 1$ for $S^3$. From Theorem 2.1, it is a stabilization of the genus 0 splitting for $S^3$, there exists a Heegaard diagram $(S; \{\alpha_1, \cdots, \alpha_n\}, \{\beta_1, \cdots, \beta_n\})$ for $S^3$ associated to the splitting such that $|\alpha_i \cap \beta_i| = 1$ for $1 \leq i \leq n$, and $|\alpha_i \cap \beta_j| = 0$ for $1 \leq i \neq j \leq n$. We call $(S; \{\alpha_1, \cdots, \alpha_n\}, \{\beta_1, \cdots, \beta_n\})$ the canonical Heegaard diagram for $S^3$. See Figure 2 below.

There is a very elegant characterization of the 3-sphere in terms of any corresponding Heegaard diagram.

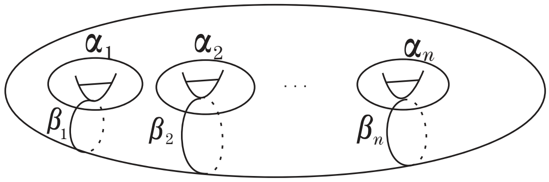


Figure 2 The canonical Heegaard diagram of $S^3$.

**Theorem 2.2** *Let $M$ be a closed orientable 3-manifold, and $V \cup_F W$ a Heegaard splitting of genus $n$ for $M$ with an associated Heegaard diagram $(V; J_1, \cdots, J_n)$. Then $M$ is homeomorphic to $S^3$ if and only if there exists an embedding $i : V \hookrightarrow S^3$ such that $K = \{i(J_1), \cdots, i(J_n)\}$ is a CSCS for $W' = \overline{S^3 \backslash i(V)}$.*

**Remark 2.1** (1) Theorem 2.2 was first stated in Haken's paper [4] for the homotopy 3-sphere and attributed to Moise and others (refer to [15] for a proof). By Perelman's work (see [14]) on Thurston's geometrization conjecture (which implies that Poincaré conjecture hold), the homotopy 3-sphere is the 3-sphere.

(2) The embedding $i : V \hookrightarrow S^3$ might be complicated, even for a Heegaard diagram $(V; J)$ associated to the genus 1 Heegaard splitting $V \cup_S W$ of $S^3$, where $i(J)$ could be any knot in $S^3$.

### 2.3 Fox's re-embedding theorem of 3-submanifolds of $S^3$

The following is a classical re-embedding theorem of Fox [2] for a compact connected 3-submanifold of $S^3$. See [11, 13, 17] for reproofs of the theorem.

**Theorem 2.3** *Let $X$ be a compact connected 3-submanifold of $S^3$. Then $X$ can be re-embedded in $S^3$ so that the complement of the image of $X$ is a union of handlebodies.*

## 3 Some Properties of 3-Submanifolds in $S^3$ Which Admit CSCSs

In the section, we will always assume that $M$ is a compact 3-submanifold of $S^3$ with one boundary component $F$ which admits CSCSs. We say that there always exist such 3-submanifolds in $S^3$, for example, let $M$ be the complement of a non-trivial knot in $S^3$, then the preferred longitude for the knot is a CSCS for $M$ on the $\partial M$.

The following theorem shows that the equivalent classes of CSCSs for such 3-submanifolds of $S^3$ are unique.

**Theorem 3.1** *Let $\mathcal{J}$, $\mathcal{K}$ be two CSCSs for $M$. Then $\mathcal{J}$ and $\mathcal{K}$ are equivalent.*

**Proof** Assume $g(F) = n \geq 1$, and $\mathcal{J} = \{J_1, \cdots, J_n\}$, $\mathcal{K} = \{K_1, \cdots, K_n\}$. By Theorem 2.3, there exists an embedding $i : M \hookrightarrow S^3$ such that $V = \overline{S^3 \backslash i(M)}$ is a handlebody of genus $n$. Set $S = i(F)$. By Theorem 2.2, both $(V; i(\mathcal{J}))$ and $(V; i(\mathcal{K}))$ are Heegaard diagrams of $S^3$. By Theorem 2.1, the Heegaard splittings of $S^3$ are unique for each genus, so $(V; i(\mathcal{J}))$ and

$(V; i(\mathcal{K}))$ are Heegaard diagrams associated to the unique Heegaard splitting $V \cup_S V'$ of $S^3$ with genus $n$. Thus, $i(\mathcal{J})$ and $i(\mathcal{K})$ are boundaries of complete disk systems for the handlebody $V'$. By Proposition 2.1, $i(\mathcal{J})$ and $i(\mathcal{K})$ are equivalent on $S$. Hence, $\mathcal{J}$ and $\mathcal{K}$ are equivalent on $F$. This completes the proof.

Let $X$ be a compact 3-manifold with one boundary component $S$ of genus $n$, and $H_n$ a handlebody of genus $n$. Let $h : \partial H_n \to S$ be a homeomorphism. By gluing $X$ and $H_n$ via $h$, we obtain a closed 3-manifold $\widetilde{M} = X \cup_h H_n$. For a CCS $\mathcal{J}$ on $S = h(\partial H_n)$, if $\mathcal{J}$ occurs to be a complete disk system for $H_n$ in $\widetilde{M}$, we say that $\widetilde{M}$ is obtained from $X$ by adding a handlebody along $\mathcal{J}$. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two CCSs on $\partial X$. We say that the two handlebody additions to $X$ along $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively, are equivalent, if $\mathcal{J}_1$ and $\mathcal{J}_2$ are equivalent CCSs on $\partial X$. Clearly, equivalent handlebody additions to $X$ yield homeomorphic 3-manifolds.

For a 3-submanifold $M$ in $S^3$ with one boundary component $F$, Fox's re-embedding theorem (see Theorem 2.3) implies that there is always a handlebody addition to $M$ which yields $S^3$. Suppose that $M$ admits a CSCS $\mathcal{K}$, $\mathcal{J}$ is a CCS on $F$. Let $\widetilde{M}$ be the manifold obtained by adding a handlebody to $M$ along $\mathcal{J}$. If $\widetilde{M}$ is homeomorphic to $S^3$, then by Theorem 2.2, $(H_n; \mathcal{K})$ is a Heegaard diagram of $S^3$ associated to the unique Heegaard splitting of genus $n$. $\mathcal{J}$ bounds a complete disk system of handlebody $H_n$. Therefore, $(F; \mathcal{J}, \mathcal{K})$ is a Heegaard diagram of $S^3$.

The following are examples of handlebody additions which have some special property.

**Example 3.1** Let $F$ be a closed surface of genus $n \geq 1$ in $S^3$ which splits $S^3$ into $M_1$ and $M_2$. Suppose that at least one of $M_1$ and $M_2$ is a handlebody. Assume that $M_i$ admits a CSCS on $F$, $i = 1, 2$.

(1) When $g(F) = 1$, one of $M_1$ and $M_2$, say, $M_1$, is a solid torus (a tubular neighborhood of a knot $K$ in $S^3$), and $M_2$ is the knot exterior. It is clear that there exists a meridian disk $D$ in $M_1$ and a Seifert surface $S$ in $M_2$ such that $\partial D$ and $\partial S$ meet in one point on $F$.

(2) $F$ is a Heegaard surface of genus $n \geq 1$ in $S^3$. There exists a canonical Heegaard diagram $(F; \{\alpha_1, \cdots, \alpha_n\}, \{\beta_1, \cdots, \beta_n\})$ associated to the splitting such that $|\alpha_i \cap \beta_i| = 1$ for $1 \leq i \leq n$, and $|\alpha_i \cap \beta_j| = 0$ for $1 \leq i \neq j \leq n$.

(3) Suppose that $M_1$ admits a CSCS $\mathcal{J}$, and $M_2$ is a handlebody. Choose a complete disk system $\mathcal{D}$ for $M_2$ and set $\mathcal{K} = \partial\mathcal{D}$. Then we can consider $S^3$ as a handlebody addition to $M_1$, and $(F; \mathcal{J}, \mathcal{K})$ is a Heegaard diagram of $S^3$. Theorem 2.1 implies that there exists a CSCS $\{\alpha_1, \cdots, \alpha_n\}$ for $M_1$ on $\partial M_1$ which is equivalent to $\mathcal{J}$ and a complete disk system $\mathcal{E}$ for $M_2$ with $\partial\mathcal{E} = \{\beta_1, \cdots, \beta_n\}$ on $\partial M_2$ which is equivalent to $\mathcal{K}$ such that $|\alpha_i \cap \beta_i| = 1$ for $1 \leq i \leq n$, and $|\alpha_i \cap \beta_j| = 0$ for $1 \leq i \neq j \leq n$.

The following theorem shows that the same phenomenon happens for a general surface $F$.

**Theorem 3.2** *Let $F$ be a closed surface of genus $n \geq 1$ in $S^3$ which splits $S^3$ into two 3-manifolds $M_1$ and $M_2$. Assume that $M_i$ admits a CSCS $\mathcal{J}_i$ on $F$, $i = 1, 2$. Then there exists a CSCS $\mathcal{L}_1 = \{\alpha_1, \cdots, \alpha_n\}$ for $M_1$ which is equivalent to $\mathcal{J}_1$ on $F$ and a CSCS $\mathcal{L}_2 = \{\beta_1, \cdots, \beta_n\}$ for $M_2$ which is equivalent to $\mathcal{J}_2$ on $F$, such that $|\alpha_i \cap \beta_i| = 1$ for $1 \leq i \leq n$, and $|\alpha_i \cap \beta_j| = 0$ for $1 \leq i \neq j \leq n$.*

**Proof** By assumption and Theorem 2.3, there exists an embedding $i_j : M_j \hookrightarrow S^3$ such that, $U_j = \overline{S^3 \backslash i_j(M)}$ is a handlebody of genus $n$, $j = 1, 2$, and both $(U_1; i_1(\mathcal{J}_1))$ and $(U_2; i_2(\mathcal{J}_2)$ are Heegaard diagrams of $S^3$. Choose a CCS $\mathcal{K}_j$ on $\partial U_j$ which bounds a complete disk system for $U_j$, $j = 1, 2$. Then $(i_1(F); \mathcal{K}_1, i_1(\mathcal{J}_1), )$, $(i_2(F); i_2(\mathcal{J}_2), \mathcal{K}_2)$ are Heegaard diagram for $S^3$. Thus, by Lemma 2.1, $(F; i_1^{-1}(\mathcal{K}_1), \mathcal{J}_1)$ and $(F; \mathcal{J}_2, i_2^{-1}(\mathcal{K}_2))$ are both Heegaard diagrams for $S^3$. By Theorem 2.1, the Heegaard splittings of $S^3$ are unique for each genus, so $i_1^{-1}(\mathcal{K}_1)$ and $\mathcal{J}_2$ are equivalent CCSs on $F$, and $i_2^{-1}(\mathcal{K}_2)$ and $\mathcal{J}_1$ are equivalent CCSs on $F$. Therefore, $(F; \mathcal{J}_2, \mathcal{J}_1)$ (as well as, $(F; \mathcal{J}_1, \mathcal{J}_2)$) is again a Heegaard diagram for $S^3$.

Let $(F; \mathcal{L}_1 = \{\alpha_1, \cdots, \alpha_n\}, \mathcal{L}_2 = \{\beta_1, \cdots, \beta_n\})$ be the canonical Heegaard diagram for $S^3$, i.e., $|\alpha_i \cap \beta_i| = 1$ for $1 \le i \le n$, and $|\alpha_i \cap \beta_j| = 0$ for $1 \le i \ne j \le n$. Again by Theorem 2.1, $\mathcal{J}_1$ is equivalent to $\mathcal{L}_1$ on $F$, and $\mathcal{J}_2$ is equivalent to $\mathcal{L}_2$ on $F$.

Since $\mathcal{J}_j$ is a CSCS on $F$ for $M_j$, by Proposition 2.2, $\mathcal{L}_j$ is also a CSCS on $F$ for $M_j$, $j = 1, 2$. The conclusion follows.

**Remark 3.1** One way to obtain the 3-submanifolds in $S^3$ which admit CSCSs is as follows. Let $L = \{l_1, \cdots, l_n\}$ be a boundary link in $S^3$. $L$ bounds a disjoint union of $n$ Seifert surfaces $S_1 \cdots, S_n$ in $S^3$ such that $l_i$ bounds $S_i$ for $i = 1, \cdots, n$. Choose a point $P$ in $S^3$ so that $P$ is not contained in any $S_i$, $1 \le i \le n$. For each $i$, $1 \le i \le n$, choose a simple arc $\alpha_i$ in $S^3$ connecting $P$ and a point $P_i \in l_i$, such that $\alpha_i \cap S_i = \alpha_i \cap l_i = P_i$, and for $i \ne j$, $\alpha_i \cap \alpha_j = \{P\}$. Set $\Gamma = \bigcup\limits_{i=1}^{n} \alpha_i \cup l_i$. Then $\Gamma$ is a connected graph with $\chi(\Gamma) = 1 - n$. Let $H$ be a regular neighborhood of $\Gamma$ in $S^3$. $H$ is a handlebody of genus $n$. Clearly, $M = \overline{S^3 \backslash H}$ admits a CSCS on $\partial M$.

As before, let $M$ be a compact 3-submanifold of $S^3$ with one boundary component $F$ which admits CSCSs. We use $\mathcal{M}(F)$ (resp. $\mathcal{M}(M)$) to denote the mapping class group of $F$ (resp. $M$), the group of isotopy classes of orientation preserving self-homeomorphisms of $F$ (resp. $M$). For $X = F$ or $M$, we call each element of $\mathcal{M}(X)$ an automorphism of $X$ and when there is no confusion we feel free to go back and forward between this element and a representative of this isotopy class.

For each $\varphi \in \mathcal{M}(M)$, $\varphi|_F \in \mathcal{M}(F)$. So there is a natural homomorphism $j : \mathcal{M}(M) \to \mathcal{M}(F)$. When $M$ is a handlebody $H_n$ of genus $n$ and $F = \partial H_n$, we call $j(\mathcal{M}(H_n))$ the handlebody subgroup of $\mathcal{M}(F)$, and denote it by $\mathcal{H}_n$. $\mathcal{H}_n$ consists of the elements in $\mathcal{M}(F)$ which can extend to automorphisms of $H_n$. A classification of $\mathcal{H}_n$ is given in [12].

As an application of Theorem 3.1, we have the following theorem.

**Theorem 3.3** *Let $M$ be a compact 3-submanifold of $S^3$ with one boundary component $F$ of genus $n$ which admit a CSCS $\mathcal{K}$, and $j : \mathcal{M}(M) \to \mathcal{M}(F)$ the natural homomorphism. Let $H$ be a handlebody of genus $n$ with $\partial H = F$ and $\mathcal{K}$ is the boundary of a complete disk system for $H$. Then the image $j(\mathcal{M}(M))$ is a subgroup of $\mathcal{H}_n$.*

**Proof** Take a $\varphi \in \mathcal{M}(M)$, $\varphi' = \varphi|_F = j(\varphi) \in \mathcal{M}(F)$. We only need to show $\varphi' \in \mathcal{H}_n$. Set $\varphi'(\mathcal{K}) = \mathcal{K}'$. Since $\mathcal{K}$ is a CSCS for $M$, $\mathcal{K}'$ is also a CSCS for $M$. By Theorem 3.1, $\mathcal{K}$ and $\mathcal{K}'$ are equivalent CCSs on $F$. From Proposition 2.1, $\mathcal{K}'$ also bounds a complete disk system for $H$. It is clear that $\varphi'$ extends an automorphism of $H$.

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