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Finite p-Groups with Few Non-major k-Maximal Subgroups*

Boyan WEI[1] Haipeng QU[2] Yanfeng LUO[1]

Abstract A subgroup of index p^k of a finite p-group G is called a k-maximal subgroup of G. Denote by $d(G)$ the number of elements in a minimal generator-system of G and by $\delta_k(G)$ the number of k-maximal subgroups which do not contain the Frattini subgroup of G. In this paper, the authors classify the finite p-groups with $\delta_{d(G)}(G) \leq p^2$ and $\delta_{d(G)-1}(G) = 0$, respectively.

Keywords Finite p-groups, k-Maximal subgroups, k-Major subgroups, Frattini subgroup, The number of non-major k-maximal subgroups

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1 Introduction

It is well-known that the finite p-groups are fundamental tools in understanding the structure of finite groups and in the classification of finite simple groups. After the classification of finite simple groups was completed, the study of p-groups has attracted much attention and the finite p-groups have been investigated intensively over the past decades. However, the complex behaviour of finite p-groups is well-known and it prevents any attempt at a general classification. In fact, only the p-groups of order at most p^7 have been completely classified for a general prime p (see [12] for detail). This can be extended (see [5]) to the groups of order less than or equal to p^{10} in the case $p = 2$. This situation has naturally led to restricting the study of p-groups to special families.

The enumeration problem of p-groups is important in the study of finite p-groups, which includes two aspects: One is to study the number of subgroups, elements and subsets of finite p-groups. A classical result of Kulakoff [11] asserts: For a prime $p > 2$ and a group G of order p^n, $s_k(G) \equiv 1$ or $1+p \pmod{p^2}$, where $s_k(G)$ denotes the number of subgroups of order p^k of G. Along Kulakoff's idea, many scholars investigated the possible case of $s_k(G) \pmod{p^3}$ (see [2, 7, 10, 15, 17–18]). The other is to study the structure or properties of finite p-groups by means

Manuscript received November 19, 2015. Revised August 21, 2016.
[1]School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China.
E-mail: weiboyan15@163.com luoyf@lzu.edu.cn
[2]School of Mathematics and Computer Science, Shanxi Normal University, Linfen 041004, Shanxi, China.
E-mail: orcawhale@163.com
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of the number of subgroups. Fan [8] characterized finite elementary abelian p-groups in term of $s_k(G)$. Moreover, Qu [13] characterized finite non-elementary abelian p-groups whose number of subgroups of possible order is maximal. More results about the enumeration of subgroups can be referenced to [1, 3, 14, 16, 19]. In this paper, we continue to characterize the structure of p-groups by the enumeration of subgroups.

In the enumeration of subgroups, a fundamental theorem is the enumeration principle given by P. Hall in [9]. The theorem is stated as follows.

Theorem 1.1 (P. Hall Enumeration Principle)

$$\begin{aligned} &s(G) - \sum_{M\in\mathcal{S}_1} s(M) + p\sum_{M\in\mathcal{S}_2} s(M) - p^3\sum_{M\in\mathcal{S}_3} s(M) + \cdots \\ &+ (-1)^k p^{\frac{1}{2}k(k-1)}\sum_{M\in\mathcal{S}_k} s(M) + \cdots + (-1)^d p^{\frac{1}{2}d(d-1)} s(\Phi(G)) = 0, \end{aligned}$$

where G is a finite p-group, $d = d(G)$, $s(M)$ denotes the number of subgroups contained in a subgroup M of G, S_i denotes the set consisting of all major subgroups (in the sense that subgroups containing $\Phi(G)$) of index p^i of G.

We observe that in the enumeration principle, P. Hall introduced a kind of important subgroups, major subgroups, which play important roles in the study of finite p-groups. Moreover, for a group G of order p^n, P. Hall Enumeration Principle gives the relationship between the number of major subgroups of index p^k and the number of subgroups of index p^k. For convenience, a subgroup H of index p^k is called a k-maximal subgroup of G. A k-maximal subgroup H is called a k-major subgroup of G if H is major; otherwise, H is called a non-major k-maximal subgroup of G. Clearly, if H is a k-major subgroup of G, then $1 \le k \le d = d(G)$.

Let $\gamma_k(G)$ and $\delta_k(G)$ denote the number of all k-major subgroups of G and the number of non-major k-maximal subgroups of G, respectively. Then $\delta_k(G) = s_{n-k}(G) - \gamma_k(G)$ for $k = 1, 2, \cdots, d$. Moreover, by the corresponding theorem, we have that $\gamma_k(G)$ is equal to the number of k-maximal subgroups of an elementary abelian group of order p^d. P. Hall [9] proved that

$$\delta_k(G) \equiv 0 \pmod{p^{d-k+1}}.$$

It means that $\delta_k(G) \equiv 0 \pmod p$ for $1 \le k \le d$.

A natural question is: How does $\delta_k(G)$ influence the structure of a p-group G?

Obviously, $\delta_1(G) = 0$ is true for any finite p-group G. So we get nothing in this case. It is easy to see that for a finite p-group G, if $\delta_k(G) = 0$ for some $k \in \{1, \cdots, d\}$, then $\delta_t(G) = 0$ for all $t \in \{1, \cdots, k\}$. This implies that if k is a smaller positive integer, then there are many p-groups with $\delta_k(G) = 0$. So we pay our attention to the cases of $k = d$ and $k = d - 1$. In this paper, we study the finite p-groups with few non-major k-maximal subgroups. We prove that if $0 < \delta_k(G) < p^2$ for some $k \in \{1, 2, \cdots, d\}$, then $k = d$ and $\delta_d(G) = p$. Moreover, we classify all finite p-groups with $\delta_d(G) = 0$, $\delta_{d-1}(G) = 0$, $\delta_d(G) = p$ and $\delta_d(G) = p^2$, respectively.

2 Preliminaries

In this section, we give some notation and background material of this article. The reader is referred to [3] for any undefined notation and terminology in this article.

Let G be a finite p-group. We use $\exp(G)$, $c(G)$ and $d(G)$ to denote the exponent, the nilpotency class of G and the number of elements in a minimal generator-system of G, respectively. G is said to be of maximal class if $|G| = p^n$ $(n \geq 3)$ and $c(G) = n - 1$. Denote by C_{p^n} and $C_{p^n}^m$ the cyclic group of order p^n and the direct product of m cyclic groups of order p^n, respectively. $M \lessdot G$ means that M is a maximal subgroup of G and $H \lesssim G$ means that H is isomorphic to a subgroup of G. Let H and K be two subgroups of G. We use $H * K$ to denote the central product of H and K.

For each positive integer i, we define

$$\Omega_i(G) = \langle a \in G \mid a^{p^i} = 1 \rangle \quad \text{and} \quad \mho_i(G) = \langle a^{p^i} \mid a \in G \rangle.$$

Let

$$M_p(n, m) := \langle a, b \mid a^{p^n} = b^{p^m} = 1,\ [a, b] = a^{p^{n-1}} \rangle \quad (n \geq 2,\ m \geq 1),$$
$$M_p(n, m, 1) := \langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1 \rangle \quad (n \geq m \geq 1).$$

To end this section, we list some known results which will be used in the sequel.

Lemma 2.1 (see [9]) *Let G be a finite p-group and $d = d(G)$. Then $\delta_k(G) \equiv 0 \pmod{p^{d-k+1}}$ for $1 \leq k \leq d$.*

Lemma 2.2 (see [3]) *Assume that G is a group of order p^n, $0 \leq k \leq n$. Then $s_k(G) \equiv 1 \pmod p$.*

Lemma 2.3 (see [3]) *Let G be a group of order p^n. If G is neither cyclic nor a 2-group of maximal class, then $s_k(G) \equiv 1 + p \pmod{p^2}$ for $1 \leq k < n$.*

Lemma 2.4 (see [3]) *Let G be a group of order p^n.*

(1) *If $1 < m < n$ and $s_m(G) = 1$, then G is cyclic;*

(2) *If $s_1(G) = 1$, then G is cyclic or generalized quaternion.*

Lemma 2.5 (see [3]) *Let G be a p-group with $|G'| = p$. Then*

$$G = (A_1 * A_2 * \cdots * A_s)Z(G),$$

where $A_1, A_2, \cdots, A_s$ are minimal nonabelian, so $G/Z(G)$ is elementary abelian of even rank. In particular, if G/G' is elementary abelian, then

$$|A_1| = |A_2| = \cdots = |A_s| = p^3.$$

Let $E = A_1 * A_2 * \cdots * A_s$. Then E is extraspecial and $G = EZ(G)$.

Lemma 2.6 (see [3]) *Let G be a finite p-group. If $c(G) < p$, then G is regular.*

The following lemmas are the classifications of finite p-groups with some properties, which will be used in the sequel.

Lemma 2.7 (see [4]) *Let G be a nonabelian p-group with $d(G) = 3$. Suppose that all maximal subgroups of G are generated by two elements. If $c(G) = 2$, then $|G| \leq p^6$ and one of the following holds:*

(1) $G = M_p(2,1) * C_{p^2} \cong M_p(1,1,1) * C_{p^2}$;

(2) $G = Q_8 \times C_2$;

(3) $G = \langle a, b, c \mid a^4 = b^4 = 1,\ c^2 = a^2b^2,\ [a,b] = 1,\ [a,c] = a^2,\ [b,c] = a^2b^2\rangle$;

(4) $G = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ [a,b] = 1,\ [a,c] = a^2b^2,\ [b,c] = a^2\rangle$;

(5) *If* $p > 2$, *then* $G = \langle a, b, c, x, y \mid a^p = b^{p^2} = c^{p^2} = 1,\ [b,c] = 1,\ [a,b] = x,\ [a,c] = y,\ b^p = x^\alpha y^\beta,\ c^p = x^\gamma y^\delta,\ x^p = y^p = [b,x] = [b,y] = [c,x] = [c,y] = 1\rangle$, *where* $4\beta\gamma + (\delta - \alpha)^2$ *is a quadratic non-residue mod* p;

(6) $|G| = 2^6,\ G = \langle a, b, c \mid a^4 = b^4 = c^4 = 1,\ [a,b] = c^2,\ [a,c] = b^2c^2,\ [b,c] = a^2b^2,\ [a^2,b] = [a^2,c] = [b^2,a] = [b^2,c] = [c^2,a] = [c^2,b] = 1\rangle$.

As a direct consequence of [4, Theorems 70.2, 70.4–70.5], we have the following lemma.

Lemma 2.8 *Assume that G is a nonabelian p-group with $d(G) = 3$ and all maximal subgroups of G are generated by two elements. If $c(G) > 2$, then G is one of the following nonisomorphic groups:*

(1) $\langle a, b, c, d \mid a^4 = b^4 = c^4 = d^2 = [a,d] = [b,d] = [c,d] = 1,\ [a,b] = c^2,\ [a,c] = b^2c^2,\ [b,c] = a^2b^2,\ [a^2,b] = [a^2,c] = [b^2,c] = [c^2,a] = d,\ [b^2,a] = 1,\ [c^2,b] = 1\rangle$;

(2) $H = G/G_4 = \langle a, b, c, d, e, f \mid a^4 = b^4 = c^4 = d^2 = e^2 = f^2 = def = [a,d] = [b,d] = [c,d] = [a,e] = [b,e] = [c,e] = 1,\ [a,b] = c^2d^\epsilon,\ [a,c] = b^2c^2d^\epsilon,\ [b,c] = a^2b^2,\ [a^2,b] = [b^2,c] = [c^2,a] = f,\ [a^2,c] = d,\ [b^2,a] = [c^2,b] = e\rangle$, *where* $\epsilon = 0, 1$.

Lemma 2.9 (see [6]) *Suppose that G is a group of order 2^6 and all 2-maximal subgroups of G are metacyclic. If G has a maximal subgroup which is not metacyclic, then $G > H = \langle a, b, c \mid a^4 = b^4 = 1,\ c^2 = a^2b^2,\ [a,b] = 1,\ [a,c] = a^2,\ [b,c] = a^2b^2\rangle$ and G is one of the following groups:*

(1) $\langle H, d \mid d^2 = a^2,\ [a,d] = a^2b^2,\ [b,d] = b^2,\ [c,d] = 1\rangle$;

(2) $\langle H, d \mid d^2 = b^2,\ [a,d] = a^2,\ [b,d] = a^2b^2,\ [c,d] = a^{-1}\rangle$;

(3) $\langle H, d \mid d^2 = b,\ [a,d] = b^2,\ [b,d] = 1,\ [c,d] = a^{-1}\rangle$.

Lemma 2.10 (see [3]) *Let G be a nonabelian group of order p^n with cyclic maximal subgroup. Then G is isomorphic to one of the following groups:*

(1) $M_p(n-1, 1)$, *where* $n \geq 4$ *if* $p = 2$;

(2) $D_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1,\ b^2 = 1,\ [a,b] = a^{-2}\rangle$ $(n \geq 3)$, *the dihedral group;*

(3) $Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1,\ b^2 = a^{2^{n-2}},\ [a, b] = a^{-2}\rangle$ $(n \geq 3)$, *the generalized quaternion group;*

(4) $SD_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1,\ b^2 = 1,\ [a, b] = a^{-2+2^{n-2}}\rangle$ $(n \geq 4)$, *the semidihedral group.*

3 The Classification for the Cases $\delta_d(G) = 0$ and $\delta_{d-1}(G) = 0$

In this section, we classify all finite p-groups G with $\delta_d(G) = 0$ and $\delta_{d-1}(G) = 0$, respectively. Throughout this section, we always assume that $d = d(G) > 1$.

We first consider the case $\delta_d(G) = 0$. The next theorem gives the classification of all finite p-groups G with $\delta_d(G) = 0$.

Theorem 3.1 *Let G be a group of order p^n and $d = d(G) > 1$. Then $\delta_d(G) = 0$ if and only if either $G \cong Q_8$ or $G \cong C_p^n$.*

Proof If $G \cong Q_8$ or $G \cong C_p^n$, then it is easy to verify that $\delta_d(G) = 0$.

Conversely, suppose that $\delta_d(G) = 0$. Then G contains only one d-maximal subgroup $\Phi(G)$ and $s_{n-d}(G) = 1 + \delta_d(G) = 1$. If $n - d > 1$, then G is cyclic by Lemma 2.4. This contradicts the assumption $d > 1$. Thus $n - d \leq 1$. If $n - d = 0$, then $G \cong C_p^n$. If $n - d = 1$, then $G \cong Q_{2^n}$ by Lemma 2.4. Thus $d = 2$ and $n = 3$. That is, $G \cong Q_8$.

To classify all finite p-groups with $\delta_{d-1}(G) = 0$, we need the following lemmas which give some properties of a finite p-group G with $\delta_k(G) = 0$ for some $k \in \{2, \cdots, d\}$.

Lemma 3.1 *Let G be a finite p-group and k a positive integer such that $2 \leq k \leq d$. Then the following conditions are equivalent:*

(1) $\delta_k(G) = 0$.

(2) $\Phi(H) = \Phi(G)$ *for every $(k-1)$-maximal subgroup H of G.*

(3) $d(H) = d - (k-1)$ *for every $(k-1)$-maximal subgroup H of G.*

Proof (1)$\Rightarrow$(2) Suppose that (1) holds. Let H be any $(k-1)$-maximal subgroup of G. Then every maximal subgroup $\widetilde{H}$ of H is a k-maximal subgroup of G. Since $\delta_k(G) = 0$, we have $\Phi(G) \leq \widetilde{H}$ for all maximal subgroups $\widetilde{H}$ of H. It follows that

$$\Phi(G) \leq \bigcap_{\widetilde{H} \lessdot H} \widetilde{H} = \Phi(H).$$

Since every subgroup of a finite p-group is subnormal, we have $\Phi(H) \leq \Phi(G)$. Therefore $\Phi(H) = \Phi(G)$ and (2) holds.

(2)$\Rightarrow$(3) It is obvious.

(3)$\Rightarrow$(1) Suppose that (3) holds. To show that $\delta_k(G) = 0$, it suffices to prove that $\Phi(G) \leq \widetilde{H}$ for every k-maximal subgroup $\widetilde{H}$ of G. Let H be a $(k-1)$-maximal subgroup of G such that $\widetilde{H} \lessdot H$. Then $\Phi(H) \leq \widetilde{H}$. Now $d(H) = d - (k-1)$ implies that $|\Phi(H)| = \frac{|H|}{p^{d(H)}} = \frac{|H|p^{k-1}}{p^d} = \frac{|G|}{p^d} = |\Phi(G)|$. It follows from $\Phi(H) \leq \Phi(G)$ that $\Phi(G) = \Phi(H)$. Hence $\Phi(G) \leq \widetilde{H}$ and (1) holds.

Lemma 3.2 *Let G be a finite p-group. If $\delta_k(G) = 0$ for some k with $2 \le k \le d$, then $G' = \Phi(G)$. In particular, if G is abelian, then G is elementary abelian.*

Proof Let $\overline{G} = G/G'$. It suffices to prove $\Phi(\overline{G}) = 1$. Assume that $\overline{G} = \langle x_1\rangle \times \langle x_2\rangle \times \cdots \times \langle x_d\rangle$ with $o(x_1) \ge o(x_2) \ge \cdots \ge o(x_d)$. Then $M = \langle x_1^p\rangle \times \langle x_2\rangle \times \cdots \times \langle x_d\rangle$ is a maximal subgroup of $\overline{G}$. Since $G' \le \Phi(G)$, we have $\delta_k(\overline{G}) = 0$ by the corresponding theorem. Whence $\delta_2(\overline{G}) = 0$. It follows from Lemma 3.1(3) that $d(M) = d - 1$. We thus have $x_1^p = 1$. Therefore $\Phi(\overline{G}) = 1$.

Lemma 3.3 *Let G be a finite p-group. If $\delta_k(G) = 0$ for some positive integer $k \in \{2, 3, \cdots, d\}$, then $\delta_{k-1}(M) = 0$ for every maximal subgroup M of G.*

Proof Observe that $\delta_2(G) = 0$. Then, by Lemma 3.1(2), $\Phi(M) = \Phi(G)$ for every maximal subgroup M of G. Notice that every $(k-1)$-maximal subgroup of M is a k-maximal subgroup of G. It follows from the fact that $\delta_k(G) = 0$ that all $(k-1)$-maximal subgroups of M contain $\Phi(G) = \Phi(M)$. Therefore $\delta_{k-1}(M) = 0$.

Now we are ready for the classification of all finite p-groups G with $\delta_{d-1}(G) = 0$.

Theorem 3.2 *Let G be a finite p-group with $d > 2$. Then $\delta_{d-1}(G) = 0$ if and only if one of the following holds:*

(1) *G is elementary abelian;*

(2) *$d = 3$, G is one of the groups listed in Lemmas 2.7–2.8;*

(3) *$d = 4$, $G \cong D_8 * Q_8$ or $G \cong P$, where $P = \langle x_1, x_2, x_3, x_4 \mid x_1^4 = x_2^4 = 1,\ x_3^2 = x_1^2x_2^2,\ x_4^2 = x_1^2,\ [x_1, x_2] = 1,\ [x_1, x_3] = x_1^2,\ [x_1, x_4] = [x_2, x_3] = x_1^2x_2^2,\ [x_2, x_4] = x_2^2,\ [x_3, x_4] = 1\rangle$.*

Proof It is routine matter to verify that each group listed in the theorem satisfies $\delta_{d-1}(G) = 0$.

Conversely, suppose that G is a finite p-group with $d > 2$ such that $\delta_{d-1}(G) = 0$. Then $G' = \Phi(G)$ by Lemma 3.2. Since $d \ge 3$ and $\delta_{d-1}(G) = 0$, we have $\delta_2(G) = 0$. If G is abelian, then G is elementary abelian by Lemma 3.2 and (1) holds. Suppose that G is nonabelian. There are three cases to be considered.

Case 1 $d = 3$. Then $\delta_2(G) = 0$. It follows from Lemma 3.1(3) that all maximal subgroups of G are generated by two elements. It follows from Lemmas 2.7–2.8 that G is one of the groups listed in Lemmas 2.7–2.8. Hence (2) holds.

Case 2 $d = 4$. Then $\delta_3(G) = 0$ and $\delta_2(G) = 0$. Let M be any maximal subgroup of G. Then $d(M) = 3$ by Lemma 3.1(3). In addition, $\delta_2(M) = 0$ by Lemma 3.3. Hence M is one of the groups listed in Lemmas 2.7–2.8 by Case 1.

We claim that $p = 2$.

Otherwise, let G be a counterexample of minimal order. Then $G' = \Phi(G)$ by Lemma 3.2. Let N be a normal subgroup of G such that $|G' : N| = p$. Then G/N is also a counterexample. By the minimality of G, we must have $N = 1$. Hence $|G| = p^5$, $|G'| = |\Phi(G)| = p$ and

$M \cong M_p(1,1,1) * C_{p^2}$. Notice that $1 \neq \mho_1(M) \leq \mho_1(G) \leq \Phi(G)$. We have $|\mho_1(G)| = p$. Since $|G'| = p$ and $p > 2$, we have that G is regular by Lemma 2.6. Hence $|G : \Omega_1(G)| = |\mho_1(G)| = p$. Thus $\Omega_1(G)$ is a maximal subgroup of G. But $\exp(\Omega_1(G)) = p$ since G is regular and $|G'| = p$. This contradicts the fact that $\mho_1(M) \neq 1$.

Suppose that $|G| = 2^n$. Then $n \geq 5$.

(i) Assume $|G| = 2^5$. Then $|G'| = |\Phi(G)| = 2$ since $G' = \Phi(G)$. It follows from Lemma 2.5 that G is isomorphic to one of the following groups:

$$D_8 * Q_8, \quad D_8 * D_8 = Q_8 * Q_8, \quad D_8 * C_4 \times C_2 = Q_8 * C_4 \times C_2, \quad D_8 \times C_2^2, \quad Q_8 \times C_2^2.$$

If $G \ncong D_8 * Q_8$, then G has an elementary abelian subgroup H of order 8. By Lemma 3.1(2), $\Phi(G) = \Phi(H) = 1$, a contradiction. Therefore $G \cong D_8 * Q_8$.

(ii) Assume $|G| = 2^6$. Then $|G'| = |\Phi(G)| = 4$. It follows that M is the group listed in Lemma 2.7(3) or (4).

We claim that every maximal subgroup of G is isomorphic to the group listed in Lemma 2.7(3).

Suppose to contrary that there exists a maximal subgroup M of G which is isomorphic to the group listed in Lemma 2.7(4). Let K be a subgroup of M such that $K \cong C_2^3$ and $M' \leq K$. Notice that $|M'| = 4$ and $M' = G'$. Then $K \trianglelefteq G$. If $K < C_G(K)$, then there exists $L \leq C_G(K)$ such that $K \lessdot L$. So L is abelian, $|L| = 2^4$ and $d(L) \geq d(K) = 3$. Since $\delta_3(G) = 0$, we have $d(L) = 2$ by Lemma 3.1(3), a contradiction. So $C_G(K) = K$ and hence $G/K \lesssim \mathrm{Aut}(K) \cong \mathrm{GL}(3,2)$. Notice that the Sylow 2-subgroup of $\mathrm{GL}(3,2)$ is isomorphic to D_8. It follows that $G/K \cong D_8$. However, since $\Phi(G) = G' \leq K$, we have G/K is elementary abelian, a contradiction. Therefore every maximal subgroup of G is isomorphic to the group listed in Lemma 2.7(3).

By a direct calculation, we may show that every maximal subgroup of the group listed in Lemma 2.7(3) is isomorphic to C_4^2 or $M_2(2,2)$. So all 2-maximal subgroups of G are metacyclic. Hence G is isomorphic to one of the groups listed in Lemma 2.9. Notice that $d = 4$. Hence $G \cong P$.

(iii) We claim that $|G| = 2^5$ or 2^6.

Otherwise, suppose $n \geq 7$ and let G be a counterexample of minimal order. Then $G' = \Phi(G)$ by Lemma 3.2. Let N be a normal subgroup of G such that $|G : N| = 2^7$ and $N \leq \Phi(G)$. Then G/N is also a counterexample. By the minimality of G, we must have $N = 1$. Hence $|G| = 2^7$, $|G'| = |\Phi(G)| = 8$ and M is isomorphic to the group listed in Lemma 2.7(6).

Now assume

$$M = \left\langle a, b, c \;\middle|\; \begin{array}{l} a^4 = b^4 = c^4 = 1,\ [a,b] = c^2,\ [a,c] = b^2c^2,\ [b,c] = a^2b^2, \\ [a^2,b] = [a^2,c] = [b^2,a] = [b^2,c] = [c^2,a] = [c^2,b] = 1 \end{array} \right\rangle.$$

Let $\overline{G} = G/\langle a^2 \rangle$. Then $|\overline{G}| = 2^6$ and $\delta_3(\overline{G}) = 0$. So every maximal subgroup H of $\overline{G}$ should be isomorphic to the group listed in Lemma 2.7(3). In particular, each element in $H \setminus \Phi(H)$ is of order 4. But $\overline{a} \in \overline{M} \setminus \Phi(\overline{M})$ is an involution, a contradiction.

Case 3 $d \geq 5$. In this case, we show that there is no finite p-group G such that $\delta_{d-1}(G) = 0$.

Otherwise, let G be a counterexample of minimal order. Let H be a $(d-5)$-maximal subgroup of G. Then $d(H) = 5$. Take $N \lhd H$ such that $|\Phi(H) : N| = p$. Then H/N is also a counterexample. By the minimality of G, we have $N = 1$ and $H = G$. So $|G| = p^6$ and $|G'| = |\Phi(G)| = p$. Assume that M is an arbitrary maximal subgroup of G. Then $d(M) = 4$ and $\delta_3(M) = 0$. Hence $p = 2$ and $M \cong D_8 * Q_8$ by Case 2. It follows from Lemma 2.5 that

$$G \cong D_8 * Q_8 \times C_2 \quad \text{or} \quad D_8 * Q_8 * C_4 = D_8 * D_8 * C_4.$$

Thus G has an elementary abelian subgroup H of order 8. By Lemma 3.1(2), $\Phi(G) = \Phi(H) = 1$. This is a contradiction.

4 The Classification for the Cases $\delta_d(G) = p$ and $\delta_d(G) = p^2$

In this section, we classify all finite p-groups G with $\delta_d(G) = p$ and $\delta_d(G) = p^2$, where $d = d(G) > 1$.

We first consider the case $\delta_d(G) = p^2$. The next theorem gives the classification of all finite p-groups G with $\delta_d(G) = p^2$.

Theorem 4.1 *Let G be a noncyclic group of order p^n. Then $\delta_d(G) = p^2$ if and only if G is a 2-group of maximal class and $G \ncong Q_8$.*

Proof If G is a 2-group of maximal class and $G \ncong Q_8$, then $d = 2$, and $\delta_2(G) = 2^2$ by counting the 2-maximal subgroups of G. Conversely, assume that $\delta_d(G) = p^2$. Then G is not elementary abelian and so $1 \leq n - d < n$. If G is not a 2-group of maximal class, then $s_{n-d}(G) = 1 + \delta_d(G) \equiv 1 + p \pmod{p^2}$ by Lemma 2.3. This contradicts the fact that $\delta_d(G) = p^2$. Hence G is a 2-group of maximal class. Moreover, $G \ncong Q_8$ by Theorem 3.1.

To classify all finite p-groups with $\delta_d(G) = p$, we need the following results.

Proposition 4.1 *Let G be a finite p-group with $|G| = p^n$. Then $0 < \delta_k(G) < p^2$ for some $k \in \{2, \cdots, d\}$ if and only if $k = d$ and $\delta_d(G) = p$.*

Proof The necessity is trivial.

Conversely, suppose that $0 < \delta_k(G) < p^2$ for some $k \in \{2, \cdots, d\}$. It follows from Lemma 2.1 that $k = d$. Notice that $0 < \delta_d(G) < p^2$. Then G is not elementary abelian and so $1 < d < n$. If G is a 2-group of maximal class, then $\delta_d(G) = 0$ or 2^2 by Theorem 3.1 and Theorem 4.1. This contradicts the fact that $0 < \delta_d(G) < p^2$. Hence G is neither cyclic nor a 2-group of maximal class. It follows from Lemma 2.3 that $s_{n-d}(G) = \delta_d(G) + 1 \equiv 1 + p \pmod{p^2}$. This together with $0 < \delta_d(G) < p^2$ implies that $\delta_d(G) = p$.

Lemma 4.1 *Let G be a finite p-group with $|G| = p^n$ $(n \geq 3)$ and $d \geq 2$. If $\delta_d(G) = p$, then $\delta_{d-1}(G) = 0$.*

Proof It suffices to prove that each $(d-1)$-maximal subgroup H contains $\Phi(G)$.

If H is cyclic, then $\Phi(H)$ is a d-maximal subgroup of G. Hence $|\Phi(H)| = |\Phi(G)|$. Clearly, $\Phi(H) \leq \Phi(G)$. Therefore $\Phi(G) = \Phi(H)$ is a subgroup of H.

If H is not cyclic, then H has at least $1+p$ maximal subgroups by Lemma 2.2. But G has only $1+p$ d-maximal subgroups. Notice that every maximal subgroup of H is a d-maximal subgroup of G. Hence H has just $1+p$ maximal subgroups which are d-maximal subgroups of G. Thus the d-maximal subgroup $\Phi(G)$ of G must be a maximal subgroup of H.

Now we give the classification of finite p-groups G with $\delta_d(G) = p$.

Theorem 4.2 *Let G be a finite p-group of order p^n with $d \geq 2$. Then $\delta_d(G) = p$ if and only if G is one of the following non-isomorphic groups:*

(1) $C_{p^{n-1}} \times C_p$ $(n \geq 3)$;

(2) $M_p(n-1, 1)$, *where* $n \geq 4$ *if* $p = 2$;

(3) $Q_8 \times C_2$.

Proof If G is one of the groups listed in the theorem, then it is easy to verify that $\delta_d(G) = p$.

Conversely, suppose that G is a finite p-group of order p^n such that $\delta_d(G) = p$. Then $\delta_{d-1}(G) = 0$ by Lemma 4.1. Notice that G has exactly $1+p$ d-maximal subgroups. If there is no cyclic $(d-1)$-maximal subgroup of G, then each $(d-1)$-maximal subgroup can be generated by the $1+p$ d-maximal subgroups by Lemma 2.2. Thus G has only one $(d-1)$-maximal subgroup. This contradicts $\delta_{d-1}(G) = 0$. Therefore G has a cyclic $(d-1)$-maximal subgroup. Moreover, the number of noncyclic $(d-1)$-maximal subgroups of G is at most 1.

If $d \geq 3$, then G is isomorphic to one of the groups listed in Theorem 3.2. By checking the groups listed in Theorem 3.2, it follows that $G \cong Q_8 \times C_2$. Assume that $d = 2$. Then G has a cyclic maximal subgroup. If G is abelian, then $G \cong C_{p^{n-1}} \times C_p$ $(n \geq 3)$. If G is nonabelian, then G is isomorphic to one of the groups listed in Lemma 2.10. By checking the groups listed in Lemma 2.10, it follows that $G \cong M_p(n-1, 1)$, where $n \geq 4$ if $p = 2$.

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References

[1] An, L. J. and Liu, Y. D., Characterization of finite p-group with the number of subgroups, *Adv. Math.* (China), **40**(3), 2011, 285–292.

[2] Berkovich, Y., On the number of subgroups of given order in a finite p-group of expotent p, *Proc. Amer. Math. Soc.*, **109**, 1990, 875–879.

[3] Berkovich, Y., Groups of Prime Power Order I, Walter de Gruyter, Berlin, New York, 2008.

[4] Berkovich, Y. and Janko, Z., Groups of Prime Power Order II, Walter de Gruyter, Berlin, New York, 2008.

[5] Besche, H. U., Eick, B. and O'Brien, E. A., A millennium project: Constructing small groups, *International J. of Algebra and Computation*, **12**(5), 2002, 623–644.

[6] Ćepulić, V., Ivanković, M. and Kovač Striko, E., Second-metacyclic finite 2-groups, *Glas. Mat. Ser.*, **40**(60), 2005, 59–69.

[7] Dyubyuk, P. E., On the number of subgroups of certain categories of finite p-groups (in Russian), *Mat. Sbornik N. S.,* **30**(72), 1952, 575–580.

[8] Fan, Y., A characterization of elementary abelian p-group by counting subgroups (in Chinese), *Math. Pract. Theory*, **1**, 1988, 63–64.

[9] Hall, P., A contribution to the theory of groups of prime power order, *Proc. London Math. Soc.*, **36**, 1933, 29–95.

[10] Hua, L. K., Some "Anzahl" theorems for groups of prime power orders, *Sci. Rep. Nat. Tsing-Hua Univ. Ser. A*, **4**, 1947, 313–327.

[11] Kulakoff, A., Über die Anzahl der eigentlichen Untergruppen und der Elemente von gegebener Ordnung in p-Gruppen, *Math. Ann.*, **104**, 1931, 778–793.

[12] O'Brien, E. A. and Vaughan-Lee, M., The groups with order p^7 for odd prime p, *J. Algebra,* **292**, 2005, 243–258.

[13] Qu, H. P., Finite non-elementary abelian p-groups whose number of subgroups is maximal, *Israel J. Math.*, **195**(2), 2013, 773–781.

[14] Qu, H. P., Sun, Y. and Zhang, Q. H., Finite p-groups in which the number of subgroups of possible order is less than or equal to p^3, *Chin. Ann. Math. Ser. B*, **31**(4), 2010, 497–506.

[15] Tuan, H. F., An "Anzahl" theorem of Kulakoff's type for p-groups, *Sci. Rep. Nat. Tsing-Hua Univ. Ser. A*, **5**, 1948, 182–189.

[16] Zhang, Q. H., A characterization of metacyclic p-groups by counting subgroups, Proc. of International Conference on Algebra 2010, World Sci. Publ., Hackensack, NJ, 2012, 713–720.

[17] Zhang, Q. H. and Qu, H. P., On Hua-Tuan's conjecture, *Sci. China Math.*, **52**(2), 2009, 389–393.

[18] Zhang, Q. H. and Qu, H. P., On Hua-Tuan's conjecture II, *Sci. China Math.*, **54**(1), 2011, 65–74.

[19] Zhang, Q. H., Zhao, L. B., Li, M. M. and Shen, Y. Q., Finite p-groups all of whose subgroups of index p^3 are abelian, *Commun. Math. Stat.*, **3**(1), 2015, 69–162.