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Exponential Convergence to Time-Periodic Viscosity Solutions in Time-Periodic Hamilton-Jacobi Equations*

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Abstract Consider the Cauchy problem of a time-periodic Hamilton-Jacobi equation on a closed manifold, where the Hamiltonian satisfies the condition: The Aubry set of the corresponding Hamiltonian system consists of one hyperbolic 1-periodic orbit. It is proved that the unique viscosity solution of Cauchy problem converges exponentially fast to a 1-periodic viscosity solution of the Hamilton-Jacobi equation as the time tends to infinity.

Keywords Hamilton-Jacobi equations, Viscosity solutions, Weak KAM theory
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1 Introduction

Consider the time-periodic Hamilton-Jacobi equation

$$u_t + H(x, u_x, t) = 0, \quad t \in [0, +\infty),\ x \in M, \tag{1.1}$$

where M is a closed (i.e., compact without boundary) and connected smooth manifold of dimension m. We choose, once and for all, a C^∞ Riemannian metric on M. It is classical that there is a canonical way to associate to it a Riemannian metric on TM. The Hamiltonian $H(x, p, t) : T^*M \times \mathbb{R} \to \mathbb{R}$, defined by $H(x, p, t) = \sup\limits_{v \in T_x M} \{\langle p, v\rangle_x - L(x, v, t)\}$, is 1-periodic in t, where $\langle \cdot, \cdot\rangle_x$ represents the canonical pairing between the tangent and cotangent space, and $L(x, v, t) : TM \times \mathbb{R} \to \mathbb{R}$ is a C^2 Lagrangian and satisfies the following conditions:

(H1) Periodicity. L is 1-periodic in the $\mathbb{R}$ factor.

(H2) Positive Definiteness. For each $x \in M$ and each $t \in \mathbb{R}$, the restriction of L to $T_x M \times \{t\}$ is strictly convex in the sense that its Hessian second derivative is everywhere positive definite.

(H3) Superlinear Growth. $\lim\limits_{\|v\|_x \to +\infty} \frac{L(x,v,t)}{\|v\|_x} = +\infty$ uniformly on $x \in M$, $t \in \mathbb{R}$, where $\|\cdot\|_x$ denotes the norm on $T_x M$ induced by the Riemannian metric on M.

(H4) Completeness of the Euler-Lagrange Flow. The maximal solutions of the Euler-Lagrange equation, which in local coordinates is

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial v}(x, \dot{x}, t) = \frac{\partial L}{\partial x}(x, \dot{x}, t),$$

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are defined on all of $\mathbb{R}$.

Such a Lagrangian L is usually called a time-periodic Tonelli Lagrangian in the literature. Without loss of generality, we will from now on always assume that the Mañé critical value (see [12]) of L is 0.

For a given time-periodic Tonelli Lagrangian L, it is well known that the function $U : M \times [0, +\infty) \to \mathbb{R}$ defined by $U(x,t) = T_t u_0(x)$ is the unique viscosity solution of the Cauchy problem

$$\begin{cases} u_t + H(x, u_x, t) = 0 & \text{in } M \times (0, +\infty), \\ u|_{t=0} = u_0 & \text{on } M, \end{cases} \tag{1.2}$$

where $u_0 : M \to \mathbb{R}$ is a continuous function and $T_t : C(M, \mathbb{R}) \to C(M, \mathbb{R})$, $t \geq 0$ is the Lax-Oleinik operator (see Section 2 for a definition) associated with the Lagrangian L (see [9] for instance).

(H5) The Aubry set of L consists of one hyperbolic 1-periodic orbit.

For any given time-periodic Tonelli Lagrangian L satisfying (H5), we show that for each $u_0 \in C(M, \mathbb{R})$, the unique viscosity solution $U(x,t)$ of the Cauchy problem (1.2) converges exponentially fast to a 1-periodic viscosity solution of (1.1) as $t \to +\infty$.

The main result of this paper is as follows.

Theorem 1.1 *If a time-periodic Tonelli Lagrangian $L : TM \times \mathbb{R} \to \mathbb{R}$ satisfies* (H5)*, then there exists $\rho > 0$ such that for each $u_0 \in C(M, \mathbb{R})$, there exists a constant $K > 0$ and a* 1*-periodic viscosity solution $\overline{u}$ of* (1.1) *such that*

$$\|U(x, n + \tau) - \overline{u}(x, \langle \tau \rangle)\|_\infty \leq K e^{-\rho n}, \quad \forall n \in \mathbb{N}, \tag{1.3}$$

where $\tau \in [0, 1]$, $\langle \tau \rangle = \tau \bmod 1$, and $\| \cdot \|_\infty$ denotes the supremum norm in the space $C(M \times [0, 1], \mathbb{R})$.

Remark 1.1 In fact, $\overline{u}(x, s) = \inf_{y \in M} (u_0(y) + h_{0,s}(y, x))$ for all $(x, s) \in M \times \mathbf{S}$, where $\mathbf{S}$ is the unit circle and h denotes the (extended) Peierls barrier (see Section 2 for a definition).

Remark 1.2 Inequality (1.3) implies that $\|U(x,t) - \overline{u}(x, \langle t \rangle)\|_0 \leq K_1 e^{-\rho t}$, $\forall t > 0$, where $K_1 > 0$ is a constant and $\| \cdot \|_0$ denotes the supremum norm in the space $C(M, \mathbb{R})$.

Remark 1.3 The essence of Theorem 1.1 is that the Lax-Oleinik operators possess an exponential convergence rate under the assumptions (H1)–(H5). See [8, 16–18] for various results on the rate of convergence of the Lax-Oleinik operators for the autonomous case.

Remark 1.4 In [15], Sánchez-Morgado provides a similar result to Theorem 1.1 for $M = \mathbf{T}^m$, where $\mathbf{T}^m$ denotes the flat m-torus. Our method here is totally different from that used in [15].

2 Preliminaries

The methods here are inspired from Mather-Mañé-Fathi theory (see [4–7, 10–14]) on Tonelli Lagrangian systems. We introduce the notations used in the sequel and review some definitions and results of Mather-Mañé-Fathi theory in this section.

We view $\mathbf{S}$ as a fundamental domain in $\mathbb{R}: \overline{I} = [0,1]$ with the two endpoints identified. The unique coordinate s of a point in $\mathbf{S}$ will belong to $I = [0,1)$. The standard universal covering projection $\pi : \mathbb{R} \to \mathbf{S}$ takes the form $\pi(\widetilde{s}) = \langle \widetilde{s} \rangle$, where $\langle \widetilde{s} \rangle = \widetilde{s} \mod 1$ denotes the fractional part of $\widetilde{s}$ ($\widetilde{s} = [\widetilde{s}] + \langle \widetilde{s} \rangle$, where $[\widetilde{s}]$ is the greatest integer not greater than $\widetilde{s}$). $\| \cdot \|$ denotes the usual Euclidean norm.

The Euler-Lagrange equation generates a flow of diffeomorphisms $\phi_t^L : TM \times \mathbf{S} \to TM \times \mathbf{S}$, $t \in \mathbb{R}$, defined by

$$\phi_t^L(x_0, v_0, t_0) = (x(t+t_0), \dot{x}(t+t_0), \langle t+t_0 \rangle),$$

where $x : \mathbb{R} \to M$ is the maximal solution of the Euler-Lagrange equation with initial conditions $x(t_0) = x_0$, $\dot{x}(t_0) = v_0$. The completeness and periodicity conditions grant that this correctly defines a flow on $TM \times \mathbf{S}$.

Consider the action functional A_L from the space of continuous and piecewise C^1 curves $\gamma : [a,b] \to M$, defined by

$$A_L(\gamma) = \int_a^b L(\mathrm{d}\gamma(\sigma), \sigma) \mathrm{d}\sigma,$$

where $\mathrm{d}\gamma : [a,b] \to TM$ denotes the differential of γ.

Recall the definition of the Lax-Oleinik operators T_t associated with L. For each $t \geq 0$ and each $u_0 \in C(M, \mathbb{R})$, let

$$T_t u_0(x) = \inf_{\gamma} \{u_0(\gamma(0)) + A_L(\gamma)\}$$

for all $x \in M$, where the infimum is taken among the continuous and piecewise C^1 paths $\gamma : [0,t] \to M$ with $\gamma(t) = x$. For each $t \geq 0$, T_t is an operator from $C(M, \mathbb{R})$ to itself.

As done by Mather in [14], it is convenient to introduce, for all $t < t' \in \mathbb{R}$ and $x, x' \in M$, the following quantity:

$$F_{t,t'}(x, x') = \inf_{\gamma} A_L(\gamma),$$

where the infimum is taken over the continuous and piecewise C^1 paths $\gamma : [t, t'] \to M$ such that $\gamma(t) = x$ and $\gamma(t') = x'$. For all $t < t' \in \mathbb{R}$ and all $x, x' \in M$, there exists a continuous and piecewise C^1 path $\overline{\gamma} : [t, t'] \to M$ with $\overline{\gamma}(t) = x$ and $\gamma(t') = x'$ such that $F_{t,t'}(x, x') = A_L(\overline{\gamma})$ (see [13, Tonelli's Theorem]). Such a curve is called a Tonelli minimizer (for the fixed endpoint problem). The function $F : \mathbb{R} \times \mathbb{R} \times M \times M \to \mathbb{R}$, $(t, t', x, x') \mapsto F_{t,t'}(x, x')$ is Lipschitz and bounded on $\{t' \geq t+1\}$ (see for example [2, Lemma 3.3]).

Following Mañé [12] and Mather [14], define the action potential and the extended Peierls barrier as follows.

Action Potential. For each $(s, s') \in \mathbf{S} \times \mathbf{S}$, let

$$\Phi_{s,s'}(x, x') = \inf F_{t,t'}(x, x')$$

for all $(x, x') \in M \times M$, where the infimum is taken on the set of $(t, t') \in \mathbb{R}^2$ such that $s = \langle t \rangle$, $s' = \langle t' \rangle$ and $t' \geq t+1$.

Extended Peierls Barrier. For each $(s, s') \in \mathbf{S} \times \mathbf{S}$, let

$$h_{s,s'}(x, x') = \liminf_{t'-t \to +\infty} F_{t,t'}(x, x')$$

for all $(x,x') \in M \times M$, where the lim inf is restricted to the set of $(t,t') \in \mathbb{R}^2$ such that $s = \langle t\rangle$, $s' = \langle t'\rangle$. The function $h : \mathbf{S} \times \mathbf{S} \times M \times M \to \mathbb{R}$, $(s,s',x,x') \mapsto h_{s,s'}(x,x')$ is Lipschitz (see [3, Proposition 2] for details).

A continuous and piecewise C^1 curve $\gamma : \mathbb{R} \to M$ is called global semi-static if

$$A_L(\gamma|_{[t,t']}) = \Phi_{\langle t\rangle,\langle t'\rangle}(\gamma(t),\gamma(t'))$$

for all $[t,t'] \subset \mathbb{R}$. An orbit $(\mathrm{d}\gamma(\sigma),\langle\sigma\rangle)$ is called global semi-static if γ is a global semi-static curve. The Mañé set $\widetilde{\mathcal{N}}_0$ is the union in $TM \times \mathbf{S}$ of the images of global semi-static orbits. A continuous and piecewise C^1 curve $\gamma : \mathbb{R} \to M$ is called global static if

$$A_L(\gamma|_{[t,t']}) = -\Phi_{\langle t'\rangle,\langle t\rangle}(\gamma(t'),\gamma(t))$$

for all $[t,t'] \subset \mathbb{R}$. An orbit $(\mathrm{d}\gamma(\sigma),\langle\sigma\rangle)$ is called global static if γ is a global static curve. The Aubry set $\widetilde{\mathcal{A}}_0$ is the union in $TM \times \mathbf{S}$ of the images of global static orbits. For a time-periodic Tonelli Lagrangian satisfying (H5), we have $\widetilde{\mathcal{A}}_0 = \widetilde{\mathcal{N}}_0$.

A time-periodic Tonelli Lagrangian L is called regular, if the lim inf in the definition of the functions $h_{s,s'}$ is a limit for all s, s', x, x'. According to [2, Lemma 5.4], a time-periodic Tonelli Lagrangian L satisfying (H5) is regular. Thus, under the assumptions of Theorem 1.1, we have

$$\lim_{n\to+\infty} F_{0,n+\tau}(x,y) = h_{0,\langle\tau\rangle}(x,y), \quad \forall(\tau,x,y) \in [0,1] \times M \times M.$$

Since the family of functions $\{F_{0,n+\cdot}(\cdot,\cdot)\}_n$ is equi-Lipschitzian, we have

$$\lim_{n\to+\infty} F_{0,n+\tau}(x,y) = h_{0,\langle\tau\rangle}(x,y) \tag{2.1}$$

uniformly on $(\tau,x,y) \in [0,1] \times M \times M$. Note that for each $u_0 \in C(M,\mathbb{R})$, each $\tau \in [0,1]$, each $n \in \mathbb{N}$ and each $x \in M$, we have

$$T_{n+\tau}u_0(x) = \inf_{y\in M}(u_0(y) + F_{0,n+\tau}(y,x)). \tag{2.2}$$

From (2.1)–(2.2), it is easy to see that

$$\lim_{n\to+\infty} \|T_{n+\tau}u_0(x) - \inf_{y\in M}(u_0(y) + h_{0,\langle\tau\rangle}(y,x))\|_\infty = 0. \tag{2.3}$$

In view of (2.3), the function $\overline{u}$ in Theorem 1.1 has the form

$$\overline{u}(x,s) = \inf_{y\in M}(u_0(y) + h_{0,s}(y,x))$$

for all $(x,s) \in M \times \mathbf{S}$. Furthermore, from [17, Propositions 3.12–3.13], $\{\overline{u}\}_{u_0\in C(M,\mathbb{R})}$ is exactly the set of 1-periodic viscosity solutions or backward weak KAM solutions of (1.1). Now we recall the definition of the weak KAM solution of (1.1).

A backward weak KAM solution of the Hamilton-Jacobi equation (1.1) is a function $w : M \times \mathbf{S} \to \mathbb{R}$ such that w is dominated by L, i.e.,

$$w(x_1,s_1) - w(x_2,s_2) \le \Phi_{s_2,s_1}(x_2,x_1), \quad \forall(x_1,s_1),\ (x_2,s_2) \in M \times \mathbf{S},$$

and for every $(x,s) \in M \times \mathbf{S}$, there exists a curve $\gamma : (-\infty,\widetilde{s}] \to M$ with $\gamma(\widetilde{s}) = x$ and $\langle\widetilde{s}\rangle = s$ such that

$$w(x,s) - w(\gamma(t),\langle t\rangle) = A_L(\gamma_{[t,\widetilde{s}]}), \quad \forall t \in (-\infty,\widetilde{s}].$$

Similarly, we say that $w : M\times\mathbf{S} \to \mathbb{R}$ is a forward weak KAM solution of (1.1) if w is dominated by L, and for every $(x, s) \in M \times \mathbf{S}$, there exists a curve $\gamma : [\widetilde{s}, +\infty) \to M$ with $\gamma(\widetilde{s}) = x$ and $\langle\widetilde{s}\rangle = s$ such that $w(\gamma(t), \langle t\rangle) - w(x, s) = A_L(\gamma_{[\widetilde{s},t]})$, $\forall t \in [\widetilde{s}, +\infty)$.

We denote by $\mathcal{S}_-$ ($\mathcal{S}_+$) the set of backward (forward) weak KAM solutions. Given $(x_0, s_0) \in M \times \mathbf{S}$, define $w^*(x, s) := h_{s_0,s}(x_0, x)$, $w_*(x, s) := -h_{s,s_0}(x, x_0)$ for $(x, s) \in M \times \mathbf{S}$. Then $w^* \in \mathcal{S}_-$, $w_* \in \mathcal{S}_+$ (see [3, Lemma 9]).

Define the projected Aubry set $\mathcal{A}_0$ as

$$\mathcal{A}_0 := \{(x, s) \in M \times \mathbf{S} \mid h_{s,s}(x, x) = 0\}.$$

Note that $\mathcal{A}_0 = \Pi\widetilde{\mathcal{A}_0}$, where $\Pi : TM\times\mathbf{S} \to M\times\mathbf{S}$ denotes the projection. Define an equivalence relation on $\mathcal{A}_0$ by saying that (x_1, s_1) and (x_2, s_2) are equivalent if and only if

$$\Phi_{s_1,s_2}(x_1, x_2) + \Phi_{s_2,s_1}(x_2, x_1) = 0.$$

The equivalent classes of this relation are called static classes. Let $\mathbf{A}$ be the set of static classes. For each static class $\Gamma \in \mathbf{A}$, choose a point $(x, 0) \in \Gamma$ and let $\mathbb{A}_0$ be the set of such points. Under the assumptions of Theorem 1.1, $\mathbb{A}_0$ consists of only one point, denoted by $(p, 0) \in \mathcal{A}_0$. Thus, for each backward weak KAM solution w of (1.1), we have

$$w(x, s) = \min_{(q,0)\in\mathbb{A}_0} (w(q, 0) + h_{0,s}(q, x)) = w(p, 0) + h_{0,s}(p, x) \tag{2.4}$$

for all $(x, s) \in M \times \mathbf{S}$ (see [3, Theorem 7]).

Proposition 2.1 *Under the assumptions of Theorem* 1.1*, let V be a neighborhood of the Aubry set $\widetilde{\mathcal{A}_0}$ in $TM\times\mathbf{S}$. Given $0 < a_1 < a_2 < 1$, there exists $T > 0$ such that if $n \geq T$, $n \in \mathbb{N}$, $\tau \in [0, 1]$, and $\gamma : [0, n + \tau] \to M$ is a Tonelli minimizer for the fixed point problem, then*

$$(\mathrm{d}\gamma(\sigma), \langle\sigma\rangle)|_{[a_1 n,a_2 n]} \subset V.$$

Proof Suppose by contradiction that there exist $\{n_i\}_{i=1}^{+\infty} \subset \mathbb{N}$ with $n_i \to +\infty$ as $i \to +\infty$, $\{\tau_{n_i}\}_{i=1}^{+\infty} \subset [0, 1]$, a sequence $\{\gamma_{n_i}\}_{i=1}^{+\infty} : [0, n_i + \tau_{n_i}] \to M$ of Tonelli minimizers, and $\{\sigma_{n_i}\}_{i=1}^{+\infty}$ with $a_1 n_i \leq \sigma_{n_i} \leq a_2 n_i$ such that

$$(\mathrm{d}\gamma_{n_i}(\sigma_{n_i}), \langle\sigma_{n_i}\rangle) \notin V, \quad i = 1, 2, \cdots . \tag{2.5}$$

For each i, we set $x_{n_i} = \gamma_{n_i}(n_i + \tau_{n_i})$, $y_{n_i} = \gamma_{n_i}(0)$. Passing as necessary to a subsequence, we may suppose that $x_{n_i} \to x_0$, $y_{n_i} \to y_0$ and $\tau_{n_i} \to \tau_0$ as $i \to +\infty$, where x_0, $y_0 \in M$ and $\tau_0 \in [0, 1]$.

Since

$$\begin{aligned}|F_{0,n_i+\tau_{n_i}}(y_{n_i}, x_{n_i}) - h_{0,\langle\tau_0\rangle}(y_0, x_0)| \leq{}& |F_{0,n_i+\tau_{n_i}}(y_{n_i}, x_{n_i}) - h_{0,\langle\tau_{n_i}\rangle}(y_{n_i}, x_{n_i})|\\ &+ |h_{0,\langle\tau_{n_i}\rangle}(y_{n_i}, x_{n_i}) - h_{0,\langle\tau_0\rangle}(y_{n_i}, x_{n_i})|\\ &+ |h_{0,\langle\tau_0\rangle}(y_{n_i}, x_{n_i}) - h_{0,\langle\tau_0\rangle}(y_0, x_0)|,\end{aligned}$$

from (2.1) and the Lipschitz property of h, we have

$$\lim_{i\to+\infty} A_L(\gamma_{n_i}) = \lim_{i\to+\infty} F_{0,n_i+\tau_{n_i}}(y_{n_i}, x_{n_i}) = h_{0,\langle\tau_0\rangle}(y_0, x_0). \tag{2.6}$$

For each i, we set

$$(\widetilde{x}_{n_i}, \dot{\widetilde{x}}_{n_i}, s_{n_i}) = (\gamma_{n_i}(\sigma_{n_i}), \dot{\gamma}_{n_i}(\sigma_{n_i}), \langle \sigma_{n_i} \rangle).$$

By (2.5), $(\widetilde{x}_{n_i}, \dot{\widetilde{x}}_{n_i}, s_{n_i}) \notin V$, $\forall i$. Since γ_{n_i} are minimizing extremal curves, using the a priori compactness Lemma 3.4 in [17], we conclude that $\{(\widetilde{x}_{n_i}, \dot{\widetilde{x}}_{n_i}, s_{n_i})\}_{i=1}^{+\infty}$ are contained in a compact subset of $TM \times \mathbf{S}$. So we may assume upon passing if necessary to a subsequence that $(\widetilde{x}_{n_i}, \dot{\widetilde{x}}_{n_i}, s_{n_i}) \to (\widetilde{x}, \dot{\widetilde{x}}, s) \in TM \times \mathbf{S}$ as $i \to +\infty$. Since $(\widetilde{x}_{n_i}, \dot{\widetilde{x}}_{n_i}, s_{n_i}) \notin V$, $\forall i$, one has $(\widetilde{x}, \dot{\widetilde{x}}, s) \notin \widetilde{\mathcal{A}}_0$.

Let $(\mathrm{d}\gamma(\sigma), \langle \sigma \rangle) = \phi^L_{\sigma - s}(\widetilde{x}, \dot{\widetilde{x}}, s)$, $\sigma \in \mathbb{R}$. We assert that the orbit $(\mathrm{d}\gamma(\sigma), \langle \sigma \rangle)$ is global semi-static, i.e., γ is a global semi-static curve. If this assertion is true, then $(\widetilde{x}, \dot{\widetilde{x}}, s) \in \widetilde{\mathcal{N}}_0 = \widetilde{\mathcal{A}}_0$, which is impossible since $(\widetilde{x}, \dot{\widetilde{x}}, s) \notin \widetilde{\mathcal{A}}_0$. This contradiction proves the proposition.

Based on the above arguments, it is sufficient to show that γ is a global semi-static curve. We prove it by contradiction. Otherwise, there would be j_1, $j_2 \in \mathbb{N}$ such that

$$A_L(\gamma|_{[s-j_1, s+j_2]}) > \Phi_{s,s}(\gamma(s-j_1), \gamma(s+j_2)).$$

It implies that there exist j_1', $j_2' \in \mathbb{N}$ with $s - j_1' + 1 \le s + j_2'$ and a minimizing curve $\widetilde{\gamma} : [s - j_1', s + j_2'] \to M$ satisfying $\widetilde{\gamma}(s - j_1') = \gamma(s - j_1)$ and $\widetilde{\gamma}(s + j_2') = \gamma(s + j_2)$ such that $A_L(\gamma|_{[s-j_1, s+j_2]}) > A_L(\widetilde{\gamma}|_{[s-j_1', s+j_2']})$. Thus, there exists $\delta > 0$ such that

$$A_L(\widetilde{\gamma}|_{[s-j_1', s+j_2']}) \le A_L(\gamma|_{[s-j_1, s+j_2]}) - \delta. \tag{2.7}$$

Since $(\widetilde{x}_{n_i}, \dot{\widetilde{x}}_{n_i}, s_{n_i}) \to (\widetilde{x}, \dot{\widetilde{x}}, s) \in TM \times \mathbf{S}$ as $i \to +\infty$, for every $\varepsilon > 0$, by the differentiability of the solutions of the Euler-Lagrange equation with respect to initial values, we have

$$d((\mathrm{d}\gamma(\sigma), \langle \sigma \rangle), (\mathrm{d}\gamma_{n_i}(\sigma + \sigma_{n_i} - s), \langle \sigma + \sigma_{n_i} - s \rangle)) < \varepsilon \tag{2.8}$$

for all $\sigma \in [s - j_1, s + j_2]$ and i large enough. Using the periodicity of L, we have

$$A_L(\gamma_{n_i}|_{[\sigma_{n_i} - j_1, \sigma_{n_i} + j_2]}) = \int_{s-j_1}^{s+j_2} L(\mathrm{d}\gamma_{n_i}(\sigma + \sigma_{n_i} - s), \langle \sigma + \sigma_{n_i} - s \rangle) \mathrm{d}\sigma, \tag{2.9}$$

In view of (2.8)–(2.9), we have

$$|A_L(\gamma_{n_i}|_{[\sigma_{n_i} - j_1, \sigma_{n_i} + j_2]}) - A_L(\gamma|_{[s-j_1, s+j_2]})| \le C\varepsilon \tag{2.10}$$

for some constant $C > 0$ independent of ε and sufficiently large i. Since ε may be taken arbitrary small, from (2.7) and (2.10) we obtain

$$\begin{aligned} A_L(\gamma_{n_i}|_{[\sigma_{n_i} - j_1, \sigma_{n_i} + j_2]}) &\ge A_L(\gamma|_{[s-j_1, s+j_2]}) - C\varepsilon \\ &\ge A_L(\widetilde{\gamma}|_{[s-j_1', s+j_2']}) + \frac{3\delta}{4}, \end{aligned} \tag{2.11}$$

provided that i is large enough.

We set

$$\overline{x} = \widetilde{\gamma}(s - j_1') = \gamma(s - j_1) \quad \text{and} \quad \underline{x} = \widetilde{\gamma}(s + j_2') = \gamma(s + j_2).$$

For i large enough, consider the following curves on M. Let $\alpha_i^1 : [0, \sigma_{n_i} - j_1] \to M$ with $\alpha_i^1(0) = y_{n_i}$, $\alpha_i^1(\sigma_{n_i} - j_1) = \overline{x}$ and $\alpha_i^2 : [\sigma_{n_i} - j_1 + j_1' + j_2', \tau_{n_i} + n_i - j_1 - j_2 + j_1' + j_2'] \to M$ with

$\alpha_i^2(\sigma_{n_i} - j_1 + j_1' + j_2') = \underline{x}$ and $\alpha_i^2(\tau_{n_i} + n_i - j_1 - j_2 + j_1' + j_2') = x_{n_i}$ be Tonelli minimizers. Set

$$\widetilde{\gamma}_{n_i}(\sigma) = \begin{cases} \alpha_i^1(\sigma), & \sigma \in [0, \sigma_{n_i} - j_1], \\ \widetilde{\gamma}(\sigma - \sigma_{n_i} + j_1 + s - j_1'), & \sigma \in [\sigma_{n_i} - j_1, \sigma_{n_i} - j_1 + j_1' + j_2'], \\ \alpha_i^2(\sigma), & \sigma \in [\sigma_{n_i} - j_1 + j_1' + j_2', \tau_{n_i} + n_i - j_1 - j_2 + j_1' + j_2']. \end{cases}$$

It is clear that $\widetilde{\gamma}_{n_i} : [0, \tau_{n_i} + n_i - j_1 - j_2 + j_1' + j_2'] \to M$ is a continuous and piecewise C^1 curve connecting y_{n_i} and x_{n_i}.

We set $\overline{x}_{n_i} = \gamma_{n_i}(\sigma_{n_i} - j_1)$ and $\underline{x}_{n_i} = \gamma_{n_i}(\sigma_{n_i} + j_2)$. For i large enough, compare $A_L(\widetilde{\gamma}_{n_i})$ with $A_L(\gamma_{n_i})$ as follows. In view of (2.8), we have

$$\begin{aligned} &|A_L(\widetilde{\gamma}_{n_i}|_{[0,\sigma_{n_i}-j_1]}) - A_L(\gamma_{n_i}|_{[0,\sigma_{n_i}-j_1]})| \\ &= |F_{0,\sigma_{n_i}-j_1}(y_{n_i}, \overline{x}) - F_{0,\sigma_{n_i}-j_1}(y_{n_i}, \overline{x}_{n_i})| \\ &\le D_{\mathrm{Lip}}\varepsilon, \end{aligned} \tag{2.12}$$

where $D_{\mathrm{Lip}} > 0$ is a Lipschitz constant of $F_{t,t'}$ which is independent of t, t' with $t+1 \le t'$ (see [2, Lemma 3.3]).

Note that

$$\begin{aligned} &A_L(\widetilde{\gamma}_{n_i}|_{[\sigma_{n_i}-j_1,\sigma_{n_i}-j_1+j_1'+j_2']}) - A_L(\gamma_{n_i}|_{[\sigma_{n_i}-j_1,\sigma_{n_i}+j_2]}) \\ &= \int_{s-j_1'}^{s+j_2'} L(\mathrm{d}\widetilde{\gamma}(\sigma), \sigma + s_{n_i} - s)\mathrm{d}\sigma - A_L(\gamma_{n_i}|_{[\sigma_{n_i}-j_1,\sigma_{n_i}+j_2]}). \end{aligned}$$

Since $s_{n_i} \to s$ as $i \to +\infty$,

$$\left| A_L(\widetilde{\gamma}|_{[s-j_1',s+j_2']}) - \int_{s-j_1'}^{s+j_2'} L(\mathrm{d}\widetilde{\gamma}(\sigma), \sigma + s_{n_i} - s)\mathrm{d}\sigma \right| \le \frac{\delta}{4}$$

for i large enough. Hence,

$$A_L(\widetilde{\gamma}_{n_i}|_{[\sigma_{n_i}-j_1,\sigma_{n_i}-j_1+j_1'+j_2']}) - A_L(\gamma_{n_i}|_{[\sigma_{n_i}-j_1,\sigma_{n_i}+j_2]}) \le -\frac{\delta}{2}. \tag{2.13}$$

From the Lipschitz property of $F_{t,t'}$ and (2.8), we find

$$\begin{aligned} &|A_L(\widetilde{\gamma}_{n_i}|_{[\sigma_{n_i}-j_1+j_1'+j_2',\tau_{n_i}+n_i-j_1-j_2+j_1'+j_2']}) - A_L(\gamma_{n_i}|_{[\sigma_{n_i}+j_2,\tau_{n_i}+n_i]})| \\ &= |F_{\sigma_{n_i}-j_1+j_1'+j_2',\tau_{n_i}+n_i-j_1-j_2+j_1'+j_2'}(\underline{x}, x_{n_i}) - F_{\sigma_{n_i}+j_2,\tau_{n_i}+n_i}(\underline{x}_{n_i}, x_{n_i})| \\ &\le D_{\mathrm{Lip}}\varepsilon. \end{aligned} \tag{2.14}$$

Since ε may be taken arbitrary small, from (2.12)–(2.14), we have

$$A_L(\widetilde{\gamma}_{n_i}) \le A_L(\gamma_{n_i}) - \frac{\delta}{4} \tag{2.15}$$

for i large enough.

Since

$$\begin{aligned} &|F_{0,\tau_{n_i}+n_i-j_1-j_2+j_1'+j_2'}(y_{n_i}, x_{n_i}) - h_{0,\langle\tau_0\rangle}(y_0, x_0)| \\ &\le |F_{0,\tau_{n_i}+n_i-j_1-j_2+j_1'+j_2'}(y_{n_i}, x_{n_i}) - h_{0,\langle\tau_{n_i}\rangle}(y_{n_i}, x_{n_i})| + |h_{0,\langle\tau_{n_i}\rangle}(y_{n_i}, x_{n_i}) - h_{0,\langle\tau_0\rangle}(y_{n_i}, x_{n_i})| \\ &\quad + |h_{0,\langle\tau_0\rangle}(y_{n_i}, x_{n_i}) - h_{0,\langle\tau_0\rangle}(y_0, x_0)|, \end{aligned}$$

from (2.1) and the Lipschitz property of h, we have

$$\lim_{i\to+\infty} F_{0,\tau_{n_i}+n_i-j_1-j_2+j_1'+j_2'}(y_{n_i},x_{n_i}) = h_{0,\langle\tau_0\rangle}(y_0,x_0). \tag{2.16}$$

Combining (2.6) and (2.15)–(2.16), we have

$$\begin{aligned} h_{0,\langle\tau_0\rangle}(y_0,x_0) - \frac{\delta}{4} &= \lim_{i\to+\infty} A_L(\gamma_{n_i}) - \frac{\delta}{4} \\ &\geq \liminf_{i\to+\infty} A_L(\widetilde{\gamma}_{n_i}) \\ &\geq \lim_{i\to+\infty} F_{0,\tau_{n_i}+n_i-j_1-j_2+j_1'+j_2'}(y_{n_i},x_{n_i}) \\ &= h_{0,\langle\tau_0\rangle}(y_0,x_0), \end{aligned}$$

a contradiction. This contradiction shows that γ is global semi-static, which completes the proof of the proposition.

3 Proof of the Main Result

Let $(p,v_p,0)$ be the unique point in $\widetilde{\mathcal{A}}_0$ with $\Pi(p,v_p,0)=(p,0)\in\mathbb{A}_0$, where $\Pi: TM\times\mathbf{S}^1\to M\times\mathbf{S}$ denotes the projection. By (H5) the Aubry set $\widetilde{\mathcal{A}}_0$ consists of one hyperbolic 1-periodic orbit, denoted by $\Gamma:\phi_\sigma^L(p,v_p,0)=(\mathrm{d}\gamma_p(\sigma),\langle\sigma\rangle)$, $\sigma\in\mathbb{R}$.

Proof of Theorem 1.1 Our purpose is to show that there exists $\rho>0$ such that for each $u_0\in C(M,\mathbb{R})$, there exists $K>0$ such that the following two inequalities hold:

$$\overline{u}(x,\langle\tau\rangle) - T_{n+\tau}u_0(x) \leq K\mathrm{e}^{-\rho n}, \quad \forall n\in\mathbb{N},\ \forall (x,\tau)\in M\times[0,1]; \tag{I1}$$

$$T_{n+\tau}u_0(x) - \overline{u}(x,\langle\tau\rangle) \leq K\mathrm{e}^{-\rho n}, \quad \forall n\in\mathbb{N},\ \forall (x,\tau)\in M\times[0,1]. \tag{I2}$$

Step 1 We first prove inequality (I1). For any given $y\in M$, $h_{0,\cdot}(y,\cdot)$ is a backward weak KAM solution of (1.1). In view of (2.4), we have

$$h_{0,\langle\tau\rangle}(y,x) = h_{0,0}(y,p) + h_{0,\langle\tau\rangle}(p,x) \tag{3.1}$$

for all $(x,\tau)\in M\times[0,1]$. Given $u_0\in C(M,\mathbb{R})$ and $(x,\tau)\in M\times[0,1]$, it is easy to see that for each $n\in\mathbb{N}$, there exists a minimizing extremal curve $\gamma_n:[0,\tau+n]\to M$ such that $\gamma_n(\tau+n)=x$ and

$$T_{n+\tau}u_0(x) = u_0(\gamma_n(0)) + A_L(\gamma_n). \tag{3.2}$$

In view of (3.1), we have

$$\begin{aligned} \overline{u}(x,\langle\tau\rangle) &= \inf_{y\in M}(u_0(y) + h_{0,\langle\tau\rangle}(y,x)) \\ &= \inf_{y\in M}(u_0(y) + h_{0,0}(y,p) + h_{0,\langle\tau\rangle}(p,x)). \end{aligned}$$

Thus, we have

$$\begin{aligned} \overline{u}(x,\langle\tau\rangle) &\leq u_0(\gamma_n(0)) + h_{0,0}(\gamma_n(0),p) + h_{0,\langle\tau\rangle}(p,x) \\ &\leq u_0(\gamma_n(0)) + F_{0,n_1}(\gamma_n(0),\gamma_n(\sigma)) + h_{0,0}(\gamma_n(\sigma),p) \\ &\quad + h_{0,0}(p,\gamma_n(\sigma)) + F_{0,n_2+\tau}(\gamma_n(\sigma),x) \end{aligned} \tag{3.3}$$

for all $\sigma \in [0, \tau+n]$ and all $n_1, n_2 \in \mathbb{N}$. For $n \in \mathbb{N}$ large enough, let $j_n = \left[\frac{2n}{3}\right] - \left[\frac{n}{3}\right] - 1$. Taking $n_1 = \left[\frac{j_n}{2}\right] + \left[\frac{n}{3}\right] + 1$, $\sigma = n_1$ and $n_2 = n - n_1$, by (3.3), we obtain

$$\overline{u}(x, \langle\tau\rangle) \leq u_0(\gamma_n(0)) + A_L(\gamma_n) + 2C_{\text{Lip}} d\Big(\gamma_n\Big(\Big[\frac{j_n}{2}\Big] + \Big[\frac{n}{3}\Big] + 1\Big), p\Big), \tag{3.4}$$

where $C_{\text{Lip}} > 0$ is a Lipschitz constant of h. From (3.2) and (3.4), we have

$$\overline{u}(x, \langle\tau\rangle) - T_{n+\tau} u_0(x) \leq 2C_{\text{Lip}} d\Big(\gamma_n\Big(\Big[\frac{j_n}{2}\Big] + \Big[\frac{n}{3}\Big] + 1\Big), p\Big). \tag{3.5}$$

We now estimate the term in the right-hand side of (3.5). Consider the Poincaré map for the time-periodic Lagrangian system L,

$$\varphi_{1,0} : TM \to TM, \quad (x_0, v_0) \mapsto \varphi_{1,0}(x_0, v_0),$$

where $\varphi_{t,0}(x_0, v_0) = (x(t), \dot{x}(t))$ and $x(t)$ denotes the solution to the Euler-Lagrange equation with initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$. Obviously, $\phi_t^L(x_0, v_0, 0) = (\varphi_{t,0}(x_0, v_0), \langle t\rangle)$. It is easy to see that (p, v_p) is a hyperbolic fixed point of $\varphi_{1,0}$. According to the Hartman-Grobman theorem, the Poincaré map $\varphi_{1,0}$ is locally conjugate to its linear part at the hyperbolic fixed point (p, v_p). More precisely, there exist a neighborhood $V(p, v_p)$ of (p, v_p) in TM as well as a neighborhood $U(0)$ of 0 in $T_{(p,v_p)}(TM)$ and a homeomorphism $f : V(p, v_p) \to U(0)$, such that

$$D\varphi_{1,0}(p, v_p) \circ f = f \circ \varphi_{1,0}. \tag{3.6}$$

Furthermore, there exists $0 < \alpha < 1$ such that f and f^{-1} are α-Hölder continuous (see [1]). Denote for brevity $P = (p, v_p)$. As the problem here is a local one, we can, using a local chart, suppose that $\varphi_{1,0}$ is a map from $\mathbb{R}^{2m}$ to itself with P as a hyperbolic fixed point.

Let $B(P)$ be a sufficiently small neighborhood of P in $\mathbb{R}^{2m}$ such that $B(P) \subset V(P) = V(p, v_p)$. We choose a tubular neighborhood W_Γ of Γ such that for each $(q, v, \langle\sigma\rangle) \in \Gamma$, $d((q, v, \langle\sigma\rangle), \partial W_\Gamma) = \kappa$, where ∂W_Γ denotes the boundary of W_Γ and κ is a positive constant small enough such that for each $(q, v, 0) \in W_\Gamma$, $(q, v) \in B(P)$. For the tubular neighborhood W_Γ, applying Proposition 2.1, there exists $T > 0$ such that for $n \in \mathbb{N}$ with $n \geq T$, we have

$$(\mathrm{d}\gamma_n(\sigma), \langle\sigma\rangle)|_{[\frac{n}{3}, \frac{2n}{3}]} \subset W_\Gamma.$$

It follows that

$$\Big(\mathrm{d}\gamma_n\Big(\Big[\frac{n}{3}\Big] + 1\Big), 0\Big), \cdots, \Big(\mathrm{d}\gamma_n\Big(\Big[\frac{2n}{3}\Big]\Big), 0\Big) \in W_\Gamma.$$

Thus, we have

$$\Big(\mathrm{d}\gamma_n\Big(\Big[\frac{n}{3}\Big] + 1\Big), \cdots, \mathrm{d}\gamma_n\Big(\Big[\frac{2n}{3}\Big]\Big)\Big) \in B(P),$$

i.e.,

$$\varphi_{1,0}^{[\frac{n}{3}]+1}(P_0^n), \cdots, \varphi_{1,0}^{[\frac{2n}{3}]}(P_0^n) \in B(P), \tag{3.7}$$

where $P_0^n = (\gamma_n(0), \dot{\gamma}_n(0))$. Set $A = D\varphi_{1,0}(P)$ and $P_1^n = \varphi_{1,0}^{[\frac{n}{3}]+1}(P_0^n)$. By (3.6)–(3.7), we have

$$Af(P_1^n) = f \circ \varphi_{1,0}^{[\frac{n}{3}]+2}(P_0^n), \quad \cdots, \quad A^{j_n} f(P_1^n) = f \circ \varphi_{1,0}^{[\frac{2n}{3}]}(P_0^n).$$

Thus $A^i f(P_1^n) \in U(0),\ i = 0, 1, \cdots, j_n$. Hence, there exists $\Delta > 0$ such that

$$\|A^i f(P_1^n)\| \leq \Delta, \quad i = 0, 1, \cdots, j_n. \tag{3.8}$$

As $A : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is hyperbolic, there exists an invariant splitting $\mathbb{R}^{2m} = E^s \oplus E^u$. For each $z \in \mathbb{R}^{2m}$, we have $z = z_s + z_u$, $z_s \in E^s$, $z_u \in E^u$ and $Az = A_s z_s + A_u z_u$, where $A_s = A|_{E^s}$ and $A_u = A|_{E^u}$. Let $f(P_1^n) = y_s^n + y_u^n$, $y_s^n \in E^s$, $y_u^n \in E^u$ and $A^{j_n} f(P_1^n) = z_s^n + z_u^n$, $z_s^n \in E^s$, $z_u^n \in E^u$. Let $\lambda_1, \cdots, \lambda_m$ be the eigenvalues of A_s. Then $|\lambda_i| < 1$ for $i = 1, \cdots, m$. Since A is similar to a symplectic matrix, $\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_m}$ are the eigenvalues of A_u. Set $\lambda_{\max} = \max\limits_{1 \leq i \leq m} |\lambda_i|$. It is a standard result that for arbitrary $\varepsilon > 0$, we have

$$\|A_s^i z_s\| \leq (\lambda_{\max} + \varepsilon)^i \|z_s\|, \quad \forall z_s \in E^s \tag{3.9}$$

for $i \in \mathbb{N}$ large enough. We choose $\varepsilon_0 > 0$ small enough such that $\lambda_{\max} + \varepsilon_0 < 1$. Then from (3.9) we have $\|A_s^{[\frac{j_n}{2}]} y_s^n\| \leq (\lambda_{\max} + \varepsilon_0)^{[\frac{j_n}{2}]} \|y_s^n\| \leq (\lambda_{\max} + \varepsilon_0)^{[\frac{j_n}{2}]} \Delta$ for n large enough. Similarly, we have $\|A_u^{[\frac{j_n}{2}]} y_u^n\| = \|A_u^{-(j_n - [\frac{j_n}{2}])} z_u^n\| \leq (\lambda_{\max} + \varepsilon_0)^{j_n - [\frac{j_n}{2}]} \|z_u^n\| \leq (\lambda_{\max} + \varepsilon_0)^{[\frac{j_n}{2}]} \Delta$ for n large enough. Thus, we have

$$\|A^{[\frac{j_n}{2}]} f(P_1^n)\| \leq \|A_s^{[\frac{j_n}{2}]} y_s^n\| + \|A_u^{[\frac{j_n}{2}]} y_u^n\| \leq 2\Delta (\lambda_{\max} + \varepsilon_0)^{[\frac{j_n}{2}]} \tag{3.10}$$

for n large enough. Since $j_n = \left[\frac{2n}{3}\right] - \left[\frac{n}{3}\right] - 1$, from (3.10) we have

$$\|A^{[\frac{j_n}{2}]} f(P_1^n)\| \leq 2\Delta (\lambda_{\max} + \varepsilon_0)^{\frac{n}{12}} \tag{3.11}$$

for n large enough. Note that $A^{[\frac{j_n}{2}]} f(P_1^n) = f \circ \varphi_{1,0}^{[\frac{j_n}{2}] + [\frac{n}{3}] + 1}(P_0^n)$ and $f(P) = 0$. Since f^{-1} is α-Hölder continuous, from (3.11) we have

$$\begin{aligned} \|\varphi_{1,0}^{[\frac{j_n}{2}] + [\frac{n}{3}] + 1}(P_0^n) - P\| &= \|f^{-1} \circ A^{[\frac{j_n}{2}]} f(P_1^n) - f^{-1}(0)\| \\ &\leq C_1 \|A^{[\frac{j_n}{2}]} f(P_1^n) - 0\|^\alpha \\ &\leq C_1 2^\alpha \Delta^\alpha (\lambda_{\max} + \varepsilon_0)^{\frac{\alpha n}{12}} \end{aligned} \tag{3.12}$$

for n large enough, where $C_1 > 0$ is a constant. Therefore, there exists a constant $C_2 > 0$ independent of $u_0 \in C(M, \mathbb{R})$ and $(x, \tau) \in M \times [0, 1]$ such that

$$d\Big(\gamma_n\Big(\Big[\frac{j_n}{2}\Big] + \Big[\frac{n}{3}\Big] + 1\Big), p\Big) \leq C_2 (\lambda_{\max} + \varepsilon_0)^{\frac{\alpha n}{12}} \tag{3.13}$$

for n large enough. Note that the above estimate is independent of (x, τ). By (3.5) and (3.13), for sufficiently large n, we have

$$\overline{u}(x, \langle \tau \rangle) - T_{n+\tau} u_0(x) \leq 2C_{\mathrm{Lip}} C_2 (\lambda_{\max} + \varepsilon_0)^{\frac{\alpha n}{12}}, \quad \forall (x, \tau) \in M \times [0, 1].$$

Hence, there exists a constant $C_3 > 0$ such that

$$\overline{u}(x, \langle \tau \rangle) - T_{n+\tau} u_0(x) \leq C_3 (\lambda_{\max} + \varepsilon_0)^{\frac{\alpha n}{12}}, \quad \forall n \in \mathbb{N},\ \forall (x, \tau) \in M \times [0, 1],$$

where the constant C_3 depends on u_0. Since $0 < \lambda_{\max} + \varepsilon_0 < 1$, there exists $\rho_1 > 0$ such that $(\lambda_{\max} + \varepsilon_0)^{\frac{\alpha}{12}} = \mathrm{e}^{-\rho_1}$. Thus, we have

$$\overline{u}(x, \langle \tau \rangle) - T_{n+\tau} u_0(x) \leq C_3 \mathrm{e}^{-\rho_1 n}, \quad \forall n \in \mathbb{N},\ \forall (x, \tau) \in M \times [0, 1]. \tag{3.14}$$

Step 2 We now prove inequality (I2). Given $u_0 \in C(M,\mathbb{R})$ and $(x,\tau) \in M \times [0,1]$, there exists $y \in M$ such that

$$\overline{u}(x,\langle\tau\rangle) = u_0(y) + h_{0,0}(y,p) + h_{0,\langle\tau\rangle}(p,x). \tag{3.15}$$

To prove (I2), it suffices to show that for $n \in \mathbb{N}$ large enough, we can find a curve $\eta : [0,\tau+n] \to M$ with $\eta(0) = y$ and $\eta(\tau+n) = x$, such that

$$u_0(\eta(0)) + A_L(\eta) - \overline{u}(x,\langle\tau\rangle) \le C\mathrm{e}^{-\theta n} \tag{3.16}$$

for some constants C, $\theta > 0$ independent of $u_0 \in C(M,\mathbb{R})$, $(x,\tau) \in M \times [0,1]$ and $n \in \mathbb{N}$. In fact, for $n \in \mathbb{N}$ large enough, if such a curve exists, then we have

$$T_{n+\tau}u_0(x) - \overline{u}(x,\langle\tau\rangle) \le u_0(\eta(0)) + A_L(\eta) - \overline{u}(x,\langle\tau\rangle) \le C\mathrm{e}^{-\theta n},$$

which immediately implies the desired inequality (I2).

Our task now is to construct the curve mentioned above. Since $h_{0,\cdot}(p,\cdot)$ is a backward weak KAM solution of (1.1), there is a curve $\beta_{x,\langle\tau\rangle} : (-\infty,\widetilde{\tau}] \to M$ with $\beta_{x,\langle\tau\rangle}(\widetilde{\tau}) = x$ and $\langle\widetilde{\tau}\rangle = \langle\tau\rangle$ such that

$$h_{0,\langle\tau\rangle}(p,x) - h_{0,\langle t\rangle}(p,\beta_{x,\langle\tau\rangle}(t)) = A_L(\beta_{x,\langle\tau\rangle}|_{[t,\widetilde{\tau}]}), \quad \forall t \in (-\infty,\widetilde{\tau}]. \tag{3.17}$$

It is clear that $\beta_{x,\langle\tau\rangle}$ is a minimizing curve. From [2, Lemma 3.9], the α-limit set for any minimizing orbit is contained in the Aubry set $\widetilde{\mathcal{A}}_0$. Since $\widetilde{\mathcal{A}}_0$ consists of one hyperbolic 1-periodic orbit Γ, the α-limit set for $(\mathrm{d}\beta_{x,\langle\tau\rangle}(\sigma),\langle\sigma\rangle)$ is exactly Γ. Similarly, since $-h_{\cdot,0}(\cdot,p)$ is a forward weak KAM solution of (1.1), there exists a curve $\omega_{y,0} : [\widetilde{o},+\infty) \to M$ with $\omega_{y,0}(\widetilde{o}) = y$ and $\langle\widetilde{o}\rangle = 0$ such that

$$h_{0,0}(y,p) - h_{\langle t\rangle,0}(\omega_{y,0}(t),p) = A_L(\omega_{y,0}|_{[\widetilde{o},t]}), \quad \forall t \in [\widetilde{o},+\infty). \tag{3.18}$$

Moreover, $\omega_{y,0}$ is a minimizing curve and the ω-limit set for $(\mathrm{d}\omega_{y,0}(\sigma),\langle\sigma\rangle)$ is also the hyperbolic 1-periodic orbit Γ (see [2, Lemma 3.9]).

Since Γ is a hyperbolic 1-periodic orbit, for the tubular neighborhood W_Γ there exist constants $T_1 > 0$ and $C_4 > 0$, such that

$$d((\mathrm{d}\omega_{y,0}(\sigma+\widetilde{o}),\langle\sigma+\widetilde{o}\rangle),(\mathrm{d}\gamma_p(\sigma),\langle\sigma\rangle)) \le C_4\mathrm{e}^{-\mu\sigma} \tag{3.19}$$

for all $\sigma > T_1$, and

$$d((\mathrm{d}\beta_{x,\langle\tau\rangle}(\sigma+\widetilde{\tau}),\langle\sigma+\widetilde{\tau}\rangle),\big(\mathrm{d}\gamma_p(\sigma+\langle\tau\rangle),\langle\sigma+\langle\tau\rangle\rangle)) \le C_4\mathrm{e}^{\mu\sigma} \tag{3.20}$$

for all $\sigma < -T_1$, where T_1 and C_4 depend only on W_Γ, and μ denotes the smallest positive Lyapunov exponent of Γ.

We are now in a position to construct the curve η. For $n \in \mathbb{N}$ large enough such that $\frac{n}{3} > \max\{T_1,2\}$, choose $0 \le d_1 < 1$ so that $\big(\mathrm{d}\gamma_p\big(\frac{n}{3}+d_1\big),\big\langle\frac{n}{3}+d_1\big\rangle\big) = (p,v_p,0)$. Then from (3.19) we obtain

$$d\Big(\Big(\mathrm{d}\omega_{y,0}\Big(\frac{n}{3}+\widetilde{o}+d_1\Big),\Big\langle\frac{n}{3}+\widetilde{o}+d_1\Big\rangle\Big),(p,v_p,0)\Big) \le C_4\mathrm{e}^{-\mu\frac{n}{3}}. \tag{3.21}$$

From $\langle\widetilde{o}\rangle = 0$ and the property of $F_{t,t'}$, we have

$$\begin{aligned}F_{0,\frac{n}{3}+d_1}\Big(y,\omega_{y,0}\Big(\frac{n}{3}+\widetilde{o}+d_1\Big)\Big) &= F_{\widetilde{o},\frac{n}{3}+\widetilde{o}+d_1}\Big(y,\omega_{y,0}\Big(\frac{n}{3}+\widetilde{o}+d_1\Big)\Big)\\ &= A_L(\omega_{y,0}|_{[\widetilde{o},\frac{n}{3}+\widetilde{o}+d_1]}),\end{aligned} \tag{3.22}$$

where the last equality holds since $\omega_{y,0}$ is a minimizing curve. Let $\eta_1 : [0, \frac{n}{3}+d_1] \to M$ with $\eta_1(0) = y$ and $\eta_1(\frac{n}{3}+d_1) = p$ be a Tonelli minimizer. Then, in view of (3.21)–(3.22), we have

$$\begin{aligned}&|A_L(\eta_1) - A_L(\omega_{y,0}|_{[\widetilde{o},\frac{n}{3}+\widetilde{o}+d_1]})|\\ &= \Big|F_{0,\frac{n}{3}+d_1}(y,p) - F_{0,\frac{n}{3}+d_1}\Big(y,\omega_{y,0}\Big(\frac{n}{3}+\widetilde{o}+d_1\Big)\Big)\Big|\\ &\le D_{\mathrm{Lip}}C_4\mathrm{e}^{-\mu\frac{n}{3}},\end{aligned} \tag{3.23}$$

where $D_{\mathrm{Lip}} > 0$ is a Lipschitz constant of $F_{t,t'}$ which is independent of t, t' with $t+1 \le t'$.

For the above sufficiently large $n \in \mathbb{N}$ with $\frac{n}{3} > \max\{T_1, 2\}$, let $a(n) = \frac{2n}{3} - d_1 + \tau$. It is clear that $a(n) \ge \frac{n}{3}$ and $(\mathrm{d}\gamma_p(-a(n)+\langle\tau\rangle), \langle -a(n)+\langle\tau\rangle\rangle) = (p, v_p, 0)$. From (3.20) we have

$$d((\mathrm{d}\beta_{x,\langle\tau\rangle}(-a(n)+\widetilde{\tau}), \langle -a(n)+\widetilde{\tau}\rangle), (p, v_p, 0)) \le C_4\mathrm{e}^{-\mu\frac{n}{3}}. \tag{3.24}$$

Since $\beta_{x,\langle\tau\rangle}$ is a minimizing curve,

$$F_{-a(n)+\widetilde{\tau},\widetilde{\tau}}\big(\beta_{x,\langle\tau\rangle}\big(-a(n)+\widetilde{\tau}\big), x\big) = A_L(\beta_{x,\langle\tau\rangle}|_{[-a(n)+\widetilde{\tau},\widetilde{\tau}]}). \tag{3.25}$$

Let $\widetilde{\eta}_2 : [-a(n)+\widetilde{\tau}, \widetilde{\tau}] \to M$ with $\widetilde{\eta}_2(-a(n)+\widetilde{\tau}) = p$ and $\widetilde{\eta}_2(\widetilde{\tau}) = x$ be a Tonelli minimizer. Then, by (3.24)–(3.25), we obtain

$$\begin{aligned}&|A_L(\widetilde{\eta}_2) - A_L(\beta_{x,\langle\tau\rangle}|_{[-a(n)+\widetilde{\tau},\widetilde{\tau}]})|\\ &= |F_{-a(n)+\widetilde{\tau},\widetilde{\tau}}(p,x) - F_{-a(n)+\widetilde{\tau},\widetilde{\tau}}(\beta_{x,\langle\tau\rangle}(-a(n)+\widetilde{\tau}),x)|\\ &\le D_{\mathrm{Lip}}C_4\mathrm{e}^{-\mu\frac{n}{3}}.\end{aligned} \tag{3.26}$$

Define a curve $\eta_2 : [\frac{n}{3}+d_1, \frac{n}{3}+d_1+a(n)] \to M$ by $\eta_2(\varsigma) = \widetilde{\eta}_2(\varsigma - \frac{n}{3} - a(n) - d_1 + \widetilde{\tau})$. Then $A_L(\eta_2) = A_L(\widetilde{\eta}_2)$.

Consider the curve $\eta : [0, \tau+n] \to M$ connecting y and x defined by

$$\eta(\sigma) = \begin{cases}\eta_1(\sigma), & \sigma \in \Big[0, \dfrac{n}{3}+d_1\Big],\\ \eta_2(\sigma), & \sigma \in \Big[\dfrac{n}{3}+d_1, \tau+n\Big].\end{cases} \tag{3.27}$$

Now it remains to show that the curve defined by (3.27) is just the one we need. For $n \in \mathbb{N}$ large enough, from (3.15) we get

$$\begin{aligned}u_0(\eta(0)) + A_L(\eta) - \overline{u}(x,\langle\tau\rangle) &= u_0(\eta(0)) + A_L(\eta) - u_0(y) - h_{0,0}(y,p) - h_{0,\langle\tau\rangle}(p,x)\\ &= A_L(\eta_1) + A_L(\eta_2) - h_{0,0}(y,p) - h_{0,\langle\tau\rangle}(p,x).\end{aligned} \tag{3.28}$$

In view of (3.28), (3.23) and (3.26), we have

$$\begin{aligned}u_0(\eta(0)) + A_L(\eta) - \overline{u}(x,\langle\tau\rangle) &\le A_L(\omega_{y,0}|_{[\widetilde{o},\frac{n}{3}+\widetilde{o}+d_1]}) + A_L(\beta_{x,\langle\tau\rangle}|_{[-a(n)+\widetilde{\tau},\widetilde{\tau}]})\\ &\quad + 2D_{\mathrm{Lip}}C_4\mathrm{e}^{-\mu\frac{n}{3}} - h_{0,0}(y,p) - h_{0,\langle\tau\rangle}(p,x).\end{aligned} \tag{3.29}$$

From (3.29) and (3.17)–(3.18), we have

$$\begin{aligned}
&u_0(\eta(0)) + A_L(\eta) - \overline{u}(x, \langle\tau\rangle) \\
&\le -h_{0,0}\Big(\omega_{y,0}\Big(\frac{n}{3} + \widetilde{o} + d_1\Big), p\Big) - h_{0,0}(p, \beta_{x,\langle\tau\rangle}(-a(n) + \widetilde{\tau})) + 2D_{\mathrm{Lip}}C_4\mathrm{e}^{-\mu\frac{n}{3}} \\
&\le 2(C_{\mathrm{Lip}} + D_{\mathrm{Lip}})C_4\mathrm{e}^{-\mu\frac{n}{3}},
\end{aligned}$$

where the last inequality follows from $h_{0,0}(p,p) = 0$, (3.21) and (3.24). Let

$$C_5 = 2(C_{\mathrm{Lip}} + D_{\mathrm{Lip}})C_4.$$

Note that C_5 and μ are independent of $(x,\tau) \in M \times [0,1]$, $u_0 \in C(M,\mathbb{R})$ and $n \in \mathbb{N}$, which means that (3.16) holds.

Thus, for $n \in \mathbb{N}$ large enough, we have

$$T_{n+\tau}u_0(x) - \overline{u}(x,\langle\tau\rangle) \le C_5\mathrm{e}^{-\mu\frac{n}{3}}, \quad \forall (x,\tau) \in M \times [0,1].$$

Hence, there exists a constant $C_6 > 0$ such that

$$T_{n+\tau}u_0(x) - \overline{u}(x,\langle\tau\rangle) \le C_6\mathrm{e}^{-\mu\frac{n}{3}}, \quad \forall n \in \mathbb{N},\ \forall (x,\tau) \in M \times [0,1], \tag{3.30}$$

where the constant C_6 depends on u_0.

Let $\rho_2 = \frac{1}{3}\mu$, $K = \max\{C_3, C_6\}$ and $\rho = \min\{\rho_1, \rho_2\}$. Then from (3.14) and (3.30), we have

$$\|T_{n+\tau}u_0(x) - \overline{u}(x,\langle\tau\rangle)\|_\infty \le K\mathrm{e}^{-\rho n}, \quad \forall n \in \mathbb{N}.$$

The proof is now complete.

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