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Equivalent Conditions of Complete Convergence and Complete Moment Convergence for END Random Variables*

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Abstract In this paper, the complete convergence and the complete moment convergence for extended negatively dependent (END, in short) random variables without identical distribution are investigated. Under some suitable conditions, the equivalence between the moment of random variables and the complete convergence is established. In addition, the equivalence between the moment of random variables and the complete moment convergence is also proved. As applications, the Marcinkiewicz-Zygmund-type strong law of large numbers and the Baum-Katz-type result for END random variables are established. The results obtained in this paper extend the corresponding ones for independent random variables and some dependent random variables.

Keywords Extended negatively dependent random variables, Complete convergence, Complete moment convergence
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1 Introduction

It is well known that complete convergence plays a very important role in the probability limit theory and mathematical statistics, especially in establishing the strong convergence rate for partial sums of random variables. The concept of complete convergence was introduced by Hsu and Robbins [1] as follows.

Definition 1.1 *A sequence $\{U_n, n \geq 1\}$ of random variables is said to converge completely to a constant a if for any $\varepsilon > 0$, $\sum\limits_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty$.*

In this case, we write $U_n \to a$ completely. In view of the Borel-Cantelli lemma, this implies that $U_n \to a$ almost surely (a.s., in short). The converse is true if random variables $\{U_n,\ n \geq 1\}$

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are independent. Hsu and Robbins [1] proved that the arithmetic means of independent and identically distributed (i.i.d., in short) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [2] proved the converse. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory and has been extended in several directions by many authors. One of the most important generalizations was provided by Baum and Katz [3] for the strong law of large numbers as follows.

Theorem 1.1 *Let $\frac{1}{2} < \alpha \leq 1$ and $\alpha p > 1$. Let $\{X_n, n \geq 1\}$ be independent and identically distributed random variables with zero means. Then the following statements are equivalent:*

(i) $E|X_1|^p < \infty$;

(ii) *for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} X_i\Big| > \varepsilon n^{\alpha}\Big) < \infty. \tag{1.1}$$

Up to now, there have been many versions of the Baum-Katz-type results for independent and dependent random variables, such as Gut [4], Peligrad and Gut [5], Wang et al. [6], Shen and Wu [7], and so on.

Chow [8] generalized the concept of complete convergence and introduced the concept of complete moment convergence, which is more general than complete convergence. Let $\{Z_n, n \geq 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If $\sum\limits_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty$ for all $\varepsilon > 0$, then the above result is called the complete moment convergence.

Chow [8] obtained the following result on the complete moment convergence for i.i.d. random variables.

Theorem 1.2 *Suppose that $\{X_n, n \geq 1\}$ is a sequence of* i.i.d. *random variables with $EX_1 = 0$, $\alpha > \frac{1}{2}$, $p \geq 1$ and $\alpha p > 1$. If $E[|X_1|^p + |X_1|\log(1+|X_1|)] < \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\Big\{\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} X_i\Big| - \varepsilon n^{\alpha}\Big\}_+ < \infty. \tag{1.2}$$

Since Chow [8] established the result of Theorem 1.2 for i.i.d. random variables, many authors have studied this type of complete moment convergence for dependent random variables. See, for example, Chen and Wang [9] for the φ-mixing sequence, Wu et al. [10] and Wang et al. [11] for $\widetilde{\rho}$-mixing sequence and the martingale difference sequence, respectively, and so on.

We should point out that the key techniques used in the proofs of Theorems 1.1–1.2 are the Rosenthal-type maximal moment inequality and the truncation methods. All the literatures above adopted these approaches or added extra conditions. There are many sequences of random variables satisfying the Rosenthal-type maximal moment inequality, such as independent random variables, negatively associated random variables, negatively supperadditive dependent random variables, φ-mixing random variables, $\widetilde{\rho}$-mixing random variables, asymptotically almost negatively associated random variables, and so on. But negatively orthant dependent random variables and extended negatively dependent random variables do not satisfy the Rosenthal-type maximal moment inequality. If we want to generalize the results of Theorems 1.1–1.2 for i.i.d. random variables to the case of the extended negatively dependent setting, we should use different methods. The main purpose of this paper is to generalize the results of Theorems 1.1–1.2 for i.i.d. random variables to the case of the extended negatively dependent setting without identical distribution. In addition, we will present the sufficient and necessary conditions of complete moment convergence for extended negatively dependent random variables.

Now, let us recall the definition of extended negatively dependent random variables.

Definition 1.2 *A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be extended negatively dependent (END, in short), if there exists a constant $M > 0$ such that both*

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \le M \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \le x_1, X_2 \le x_2, \cdots, X_n \le x_n) \le M \prod_{i=1}^{n} P(X_i \le x_i)$$

hold for all real numbers $x_1, x_2, \cdots, x_n$. An infinite sequence $\{X_n, n \ge 1\}$ is said to be END if every finite subcollection is END.

An array of random variables $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ is called rowwise END random variables if for every $n \ge 1$, $\{X_{ni}, 1 \le i \le n\}$ are END random variables.

The concept of the END sequence was introduced by Liu [12]. In the case $M = 1$, the notion of END random variables reduces to the well-known notion of the so-called negatively orthant dependent (NOD, in short) random variables, which was introduced by Joag-Dev and Proschan [13]. They also pointed out that the negatively associated (NA, in short) random variables are NOD and thus NA random variables are END. Hence, the class of END includes the independent sequence, the NA sequence and the NOD sequence as special cases. Studying the limiting behavior of END random variables is of great interest.

Some applications for the END sequence have been found. See, for example, Chen et al. [14] established the strong law of large numbers for extend negatively dependent random variables and showed its applications to risk theory and renewal theory; Shen [15] presented some probability inequalities for END sequences and gave some applications; Wu and Guan [16] presented some convergence properties for the partial sums of END random variables; Wang and Wang [17] investigated a more general precise large deviation result for random sums of END real-valued random variables in the presence of consistent variation; Qiu et al. [18] and Wang et al. [19–21] provided some results on complete convergence for sequences of END random variables or arrays of rowwise END random variables; Wang et al. [22] studied the complete consistency for the estimator of nonparametric regression models based on END errors, and so on forth. The main purpose of the paper is to generalize the results of Theorems 1.1–1.2 for i.i.d. random variables to the case of the END setting, and the sufficient and necessary conditions of complete moment convergence for END random variables will also be established.

This work is organized as follows: Some important lemmas are provided in Section 2. The main results and their proofs are presented in Section 3.

Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same in each appearance, and $a_n = O(b_n)$ stands for $a_n = C(b_n)$. $I(A)$ is the indicator function of an event A. Denote $\log x = \ln \max(x, e)$.

2 Preliminaries

In this section, we will provide some important lemmas, which will be applied to prove the main results of this paper. The first one is a basic property for END random variables, which was given by Liu [23].

Lemma 2.1 *Let random variables $X_1, X_2, \cdots, X_n$ be END. If $f_1, f_2, \cdots, f_n$ are all nondecreasing (or nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \cdots, f_n(X_n)$ are END.*

The next one is the Marcinkiewicz-Zygmund-type inequality and the Rosenthal-type inequality for partial sums and maximum partial sums of END random variables.

Lemma 2.2 *Let $p \geq 1$ and $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_i = 0$ and $E|X_i|^p < \infty$ for each $i \geq 1$. Then there exists a positive constant C_p depending only on p such that*

$$E\Big|\sum_{i=1}^{n} X_i\Big|^p \leq C_p \sum_{i=1}^{n} E|X_i|^p, \quad 1 \leq p \leq 2, \tag{2.1}$$

$$E\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} X_i\Big|^p\Big) \leq C_p \log^p n \sum_{i=1}^{n} E|X_i|^p, \quad 1 \leq p \leq 2, \tag{2.2}$$

$$E\Big|\sum_{i=1}^{n} X_i\Big|^p \leq C_p\Big[\sum_{i=1}^{n} E|X_i|^p + \Big(\sum_{i=1}^{n} E|X_i|^2\Big)^{\frac{p}{2}}\Big], \quad p \geq 2 \tag{2.3}$$

and

$$E\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} X_i\Big|^p\Big) \leq C_p \log^p n\Big[\sum_{i=1}^{n} E|X_i|^p + \Big(\sum_{i=1}^{n} E|X_i|^2\Big)^{\frac{p}{2}}\Big]. \tag{2.4}$$

Proof The inequality (2.3) has been established by Shen [15]. The inequality (2.1) can be obtained in a similar way as that of Corollary 2.2 in Asadian et al. [24]. The inequalities (2.2) and (2.4) can be proved by using the inequalities (2.1), (2.3) in a similar way as that of Theorem 2.3.1 in Stout [25], respectively. The details of the proof are omitted.

With Lemma 2.2 accounted for, we can get the following important property for END random variables, which will play an important role in proving the main results of this paper. The proof is similar to that of Lemma A6 in Zhang and Wen [26], so the details are omitted.

Lemma 2.3 *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables. Then there exists a positive constant C such that for any $x \geq 0$ and all $n \geq 1$,*

$$\Big[1 - P\Big(\max_{1\leq k\leq n}|X_k| > x\Big)\Big]^2 \sum_{k=1}^{n} P(|X_k| > x) \leq CP\Big(\max_{1\leq k\leq n}|X_k| > x\Big). \tag{2.5}$$

The following is a basic property for stochastic domination. For the proof, one can refer to Wu [27], or Wang et al. [6].

Lemma 2.4 *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X, i.e., there exists a positive constant C such that*

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$. Then for any $\alpha > 0$ and $b > 0$, the following two statements hold:

(i) $E|X_n|^\alpha I(|X_n| \leq b) \leq C_1[E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)]$,

(ii) $E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b)$,

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^\alpha \leq CE|X|^\alpha$, where C is a positive constant.

The last one comes from Sung [28].

Lemma 2.5 *Let $Y_n, Z_n, n \geq 1$ be random variables. Then for any $q > 1$, $\varepsilon > 0$ and $a > 0$,*

$$E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}(Y_i+Z_i)\Big|-\varepsilon a\Big)^+ \le \Big(\frac{1}{\varepsilon^q}+\frac{1}{q-1}\Big)\frac{1}{a^{q-1}}E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}Y_i\Big|^q\Big)+E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}Z_i\Big|\Big).$$

3 Main Results and Their Proofs

In this section, we will give the main results of this paper, including the sufficient and necessary conditions of complete convergence and complete moment convergence for END random variables.

3.1 Sufficient and necessary conditions for complete convergence

Theorem 3.1 *Let $\alpha > \frac{1}{2}$ and $\alpha p > 1$. Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_n = 0$ if $p \geq 1$. If there exists a random variable X and two positive constants C_1 and C_2 such that*

$$C_1P(|X|>x)\le P(|X_n|>x)\le C_2P(|X|>x) \tag{3.1}$$

for all $x \geq 0$ and $n \geq 1$, then the following statements are equivalent:

(i) $E|X|^p < \infty$;

(ii) *for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty}n^{\alpha p-2}P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}X_i\Big|>\varepsilon n^{\alpha}\Big)<\infty. \tag{3.2}$$

Proof (ii) $\Rightarrow$ (i) is trivial. So it suffices to show (i) $\Rightarrow$ (ii). We consider the following two cases.

Case 1 $0 < p < 1$.

For fixed $n \geq 1$, denote, for $1 \leq i \leq n$, that

$$Y_{ni}=X_iI(|X_i|\le n^{\alpha}),\quad Z_{ni}=X_iI(|X_i|>n^{\alpha}).$$

Noting that $X_i = Y_{ni} + Z_{ni}$, we have that for all $\varepsilon > 0$,

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{\alpha p-2}P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}X_i\Big|>\varepsilon n^{\alpha}\Big)\\ &\le\sum_{n=1}^{\infty}n^{\alpha p-2}P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}Y_{ni}\Big|>\frac{\varepsilon n^{\alpha}}{2}\Big)+\sum_{n=1}^{\infty}n^{\alpha p-2}P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}Z_{ni}\Big|>\frac{\varepsilon n^{\alpha}}{2}\Big)\\ &\doteq \mathrm{H}_1+\mathrm{H}_2.\end{aligned} \tag{3.3}$$

It follows from Markov's inequality, C_r inequality and Lemma 2.4, that

$$\begin{aligned}\mathrm{H}_1&\le C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha}\sum_{i=1}^{n}E|Y_{ni}|\\ &\le C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha}[E|X|I(|X|\le n^{\alpha})+n^{\alpha}P(|X|>n^{\alpha})]\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E|X|I(|X|\leq n^{\alpha}) + CE|X|^p \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \sum_{k=1}^{n} E|X|I((k-1)^{\alpha}<|X|\leq k^{\alpha}) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \sum_{k=1}^{n} k^{\alpha} P((k-1)^{\alpha}<|X|\leq k^{\alpha}) \\
&\leq C\sum_{k=1}^{\infty} k^{\alpha+\alpha p-\alpha} P((k-1)^{\alpha}<|X|\leq k^{\alpha}) \\
&\leq CE|X|^p<\infty
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
\mathrm{H}_2 &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\frac{\alpha p}{2}} \sum_{i=1}^{n} E|Z_{ni}|^{\frac{p}{2}} \\
&\leq C\sum_{n=1}^{\infty} n^{\frac{\alpha p}{2}-1} E|X|^{\frac{p}{2}} I(|X|>n^{\alpha}) \\
&= C\sum_{n=1}^{\infty} n^{\frac{\alpha p}{2}-1} \sum_{k=n}^{\infty} E|X|^{\frac{p}{2}} I(k^{\alpha}<|X|\leq (k+1)^{\alpha}) \\
&\leq C\sum_{n=1}^{\infty} n^{\frac{\alpha p}{2}-1} \sum_{k=n}^{\infty} k^{\frac{\alpha p}{2}} P(k^{\alpha}<|X|\leq (k+1)^{\alpha}) \\
&\leq C\sum_{k=1}^{\infty} k^{\alpha p} P(k^{\alpha}<|X|\leq (k+1)^{\alpha}) \\
&\leq CE|X|^p<\infty.
\end{aligned} \tag{3.5}
$$

Hence, the desired result (3.2) follows from (3.3)–(3.5) immediately.

Case 2 $p \geq 1$.

Noting that $\alpha p > 1$, we take a suitable q such that $\frac{1}{\alpha p} < q < 1$. For fixed $n \geq 1$, denote for $1 \leq i \leq n$ that

$$
\begin{aligned}
X_{ni}^{(1)} &= -n^{\alpha q} I(X_i < -n^{\alpha q}) + X_i I(|X_i| \leq n^{\alpha q}) + n^{\alpha q} I(X_i > n^{\alpha q}), \\
X_{ni}^{(2)} &= (X_i - n^{\alpha q}) I(X_i > n^{\alpha q}), \\
X_{ni}^{(3)} &= (X_i + n^{\alpha q}) I(X_i < -n^{\alpha q}).
\end{aligned}
$$

Noting that

$$
\sum_{i=1}^{j} X_i = \sum_{i=1}^{j} X_{ni}^{(1)} + \sum_{i=1}^{j} X_{ni}^{(2)} + \sum_{i=1}^{j} X_{ni}^{(3)}
$$

for $1 \leq j \leq n$, we have that for all $\varepsilon > 0$,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} X_i\Big| > \varepsilon n^{\alpha}\Big) \\
&\leq \sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} X_{ni}^{(1)}\Big| > \frac{\varepsilon n^{\alpha}}{3}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_{ni}^{(2)}\Big| > \frac{\varepsilon n^{\alpha}}{3}\Big)\\
&+\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_{ni}^{(3)}\Big| > \frac{\varepsilon n^{\alpha}}{3}\Big)\\
\doteq{}& \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3. \qquad (3.6)
\end{aligned}
$$

Hence, in order to prove (3.2), it suffices to show that $\mathrm{I}_1<\infty$, $\mathrm{I}_2<\infty$ and $\mathrm{I}_3<\infty$.

For I_1, we firstly show that

$$
n^{-\alpha}\max_{1\le j\le n}\Big|\sum_{i=1}^{j} EX_{ni}^{(1)}\Big| \to 0, \quad \text{as } n\to\infty. \qquad (3.7)
$$

It follows from $EX_n=0$, Markov's inequality and Lemma 2.4, that

$$
\begin{aligned}
n^{-\alpha}\max_{1\le j\le n}\Big|\sum_{i=1}^{j} EX_{ni}^{(1)}\Big| &\le n^{-\alpha}\sum_{i=1}^{n}[E|X_i|I(|X_i|>n^{\alpha q})+n^{\alpha q}P(|X_i|>n^{\alpha q})]\\
&\le Cn^{-\alpha}\sum_{i=1}^{n}[E|X|I(|X|>n^{\alpha q})+n^{\alpha q}P(|X|>n^{\alpha q})]\\
&\le Cn^{-\alpha+1+\alpha q-\alpha pq}E|X|^pI(|X|>n^{\alpha q})+Cn^{-\alpha+1+\alpha q-\alpha pq}E|X|^p\\
&\le Cn^{-\alpha+1+\alpha q-\alpha pq}E|X|^p,
\end{aligned}
$$

which together with $E|X|^p<\infty$ and $\frac{1}{\alpha p}<q<1$ yields (3.7). Hence, by (3.7), we have that

$$
\mathrm{I}_1 \le C\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}(X_{ni}^{(1)}-EX_{ni}^{(1)})\Big| > \frac{\varepsilon n^{\alpha}}{6}\Big). \qquad (3.8)
$$

For fixed $n\ge 1$, we can see that $\{X_{ni}^{(1)}-EX_{ni}^{(1)}, 1\le i\le n\}$ are still END random variables by Lemma 2.1. It follows from (3.8), Markov's inequality and Lemma 2.2, that for any $\delta\ge 2$,

$$
\begin{aligned}
\mathrm{I}_1 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\delta} E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}\Big(X_{ni}^{(1)}-EX_{ni}^{(1)}\Big)\Big|\Big)^{\delta}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\delta}\log^{\delta} n\Big[\sum_{i=1}^{n}E|X_{ni}^{(1)}|^{\delta}+\Big(\sum_{i=1}^{n}E|X_{ni}^{(1)}|^2\Big)^{\frac{\delta}{2}}\Big]\\
&\doteq C\mathrm{I}_{11}+C\mathrm{I}_{12}. \qquad (3.9)
\end{aligned}
$$

Taking $\delta>\max\big\{\frac{\alpha p-1}{\alpha-\frac{1}{2}},2,p\big\}$, we have

$$
\alpha p-1-\alpha\delta+\alpha q\delta-\alpha pq=\alpha(p-\delta)(1-q)-1<-1
$$

and

$$
\alpha p-2-\alpha\delta+\frac{\delta}{2}<-1.
$$

It follows from C_r inequality, Markov's inequality and Lemma 2.4, that

$$
\mathrm{I}_{11}\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\delta}\log^{\delta} n\sum_{i=1}^{n}[E|X_i|^{\delta}I(|X_i|\le n^{\alpha q})+n^{\alpha q\delta}P(|X_i|>n^{\alpha q})]
$$

$$\begin{aligned}
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha\delta}\log^{\delta}n[E|X|^{\delta}I(|X|\leq n^{\alpha q})+n^{\alpha q\delta}P(|X|>n^{\alpha q})]\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha\delta+\alpha q\delta-\alpha pq}E|X|^{p}\log^{\delta}n\\
&<\infty
\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}
\mathrm{I}_{12}&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha\delta}\log^{\delta}n\Big\{\sum_{i=1}^{n}[EX_i^2I(|X_i|\leq n^{\alpha q})+n^{2\alpha q}P(|X_i|>n^{\alpha q})]\Big\}^{\frac{\delta}{2}}\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha\delta+\frac{\delta}{2}}\log^{\delta}n[EX^2I(|X|\leq n^{\alpha q})+n^{2\alpha q}P(|X|>n^{\alpha q})]^{\frac{\delta}{2}}\\
&\leq\begin{cases}C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha\delta+\frac{\delta}{2}}(EX^2)^{\frac{\delta}{2}}\log^{\delta}n, & \text{if } p\geq 2,\\ C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha\delta+\frac{\delta}{2}+\alpha(2-p)\frac{\delta}{2}}(E|X|^p)^{\frac{\delta}{2}}\log^{\delta}n, & \text{if } 1\leq p<2\end{cases}\\
&=\begin{cases}C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha\delta+\frac{\delta}{2}}(EX^2)^{\frac{\delta}{2}}\log^{\delta}n, & \text{if } p\geq 2,\\ C\sum_{n=1}^{\infty}n^{(\alpha p-1)(1-\frac{\delta}{2})-1}(E|X|^p)^{\frac{\delta}{2}}\log^{\delta}n, & \text{if } 1\leq p<2\end{cases}\\
&<\infty.
\end{aligned}\tag{3.11}$$

Hence, $\mathrm{I}_1<\infty$ follows from (3.9)–(3.11) immediately.

In the following, we will show that $\mathrm{I}_2<\infty$. For fixed $n\geq 1$, denote, for $1\leq i\leq n$, that

$$X_{ni}^{(4)}=(X_i-n^{\alpha q})I(n^{\alpha q}<X_i\leq n^{\alpha}+n^{\alpha q})+n^{\alpha}I(X_i>n^{\alpha}+n^{\alpha q}).$$

It is easily checked that

$$\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j}X_{ni}^{(2)}\Big|>\frac{\varepsilon n^{\alpha}}{3}\Big)\subset\Big(\max_{1\leq i\leq n}X_i>n^{\alpha}\Big)\cup\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j}X_{ni}^{(4)}\Big|>\frac{\varepsilon n^{\alpha}}{3}\Big),$$

which implies that

$$\begin{aligned}
\mathrm{I}_2&\leq\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{i=1}^{n}P(|X_i|>n^{\alpha})+\sum_{n=1}^{\infty}n^{\alpha p-2}P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j}X_{ni}^{(4)}\Big|>\frac{\varepsilon n^{\alpha}}{3}\Big)\\
&\doteq\mathrm{I}_{21}+\mathrm{I}_{22}.
\end{aligned}\tag{3.12}$$

It follows from (3.1) and $E|X|^p<\infty$, that

$$\mathrm{I}_{21}\leq C\sum_{n=1}^{\infty}n^{\alpha p-1}P(|X|>n^{\alpha})\leq CE|X|^p<\infty.\tag{3.13}$$

Noting that $\frac{1}{\alpha p}<q<1$, we have from the definition of $X_{ni}^{(4)}$ and Lemma 2.4, that

$$n^{-\alpha}\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j}EX_{ni}^{(4)}\Big|\leq Cn^{1-\alpha}E|X|I(|X|>n^{\alpha q})$$

$$\leq Cn^{1-\alpha+\alpha q-\alpha pq}E|X|^p \to 0, \quad \text{as} \quad n \to \infty. \tag{3.14}$$

Since $X_{ni}^{(4)} > 0$, by (3.12)–(3.14), we have that

$$\mathrm{I}_2 \leq C\sum_{n=1}^{\infty} n^{\alpha p-2}P\Big(\Big|\sum_{i=1}^{n}\Big(X_{ni}^{(4)} - EX_{ni}^{(4)}\Big)\Big| > \frac{\varepsilon n^{\alpha}}{6}\Big). \tag{3.15}$$

For fixed $n \geq 1$, we can see that $\{X_{ni}^{(4)} - EX_{ni}^{(4)}, 1 \leq i \leq n\}$ are still END random variables by Lemma 2.1. It follows from Markov's inequality, C_r inequality and Lemma 2.2, that

$$\begin{aligned}
\mathrm{I}_2 &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\delta}E\Big|\sum_{i=1}^{n}\Big(X_{ni}^{(4)} - EX_{ni}^{(4)}\Big)\Big|^{\delta} \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\delta}\Big[\sum_{i=1}^{n}E|X_{ni}^{(4)}|^{\delta} + \Big(\sum_{i=1}^{n}E(X_{ni}^{(4)})^2\Big)^{\frac{\delta}{2}}\Big] \\
&\doteq \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned} \tag{3.16}$$

By C_r inequality and Lemma 2.4, we can get that

$$\begin{aligned}
\mathrm{J}_1 &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\delta}\sum_{i=1}^{n}[E|X_i - n^{\alpha q}|^{\delta}I(n^{\alpha q} < X_i \leq n^{\alpha} + n^{\alpha q}) + n^{\alpha\delta}P(X_i > n^{\alpha} + n^{\alpha q})] \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\delta}\sum_{i=1}^{n}[E|X_i|^{\delta}I(|X_i| \leq 2n^{\alpha}) + n^{\alpha\delta}P(X_i > n^{\alpha})] \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\delta}\sum_{i=1}^{n}[E|X|^{\delta}I(|X| \leq 2n^{\alpha}) + n^{\alpha\delta}P(|X| > n^{\alpha})] \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\delta}E|X|^{\delta}I(|X| \leq 2n^{\alpha}) + CE|X|^p \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\delta}\sum_{i=1}^{n}E|X|^{\delta}I(2(i-1)^{\alpha} < |X| \leq 2i^{\alpha}) + CE|X|^p \\
&\leq C\sum_{i=1}^{\infty} i^{\alpha\delta}P(2(i-1)^{\alpha} < |X| \leq 2i^{\alpha})\sum_{n=i}^{\infty} n^{\alpha p-1-\alpha\delta} + CE|X|^p \\
&\leq C\sum_{i=1}^{\infty} i^{\alpha p}P(2(i-1)^{\alpha} < |X| \leq 2i^{\alpha}) + CE|X|^p \\
&\leq CE|X|^p \;<\; \infty.
\end{aligned} \tag{3.17}$$

Similarly to the proof of (3.11) and (3.17), we can obtain that $\mathrm{J}_2 < \infty$, which, together with (3.16) and (3.17), yields that $\mathrm{I}_2 < \infty$.

Similarly to the proof of $\mathrm{I}_2 < \infty$, one can get that $\mathrm{I}_3 < \infty$. Hence, (3.2) follows from (3.6), $\mathrm{I}_1 < \infty$, $\mathrm{I}_2 < \infty$ and $\mathrm{I}_3 < \infty$ immediately. This completes the proof of the theorem.

With Theorem 3.1 accounted for, we can get the Marcinkiewicz-Zygmund-type strong law of large numbers for END random variables without identical distribution as follows.

Corollary 3.1 *Let $\alpha > \frac{1}{2}$ and $\alpha p > 1$. Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_n = 0$ if $p \geq 1$. Assume that there exists a random variable X and two* *positive constants* C_1 *and* C_2 *such that* (3.1) *holds for all* $x \geq 0$ *and* $n \geq 1$*. If* $E|X|^p < \infty$*, then*

$$\frac{1}{n^\alpha}\sum_{i=1}^n X_i \to 0 \quad \text{a.s.} \tag{3.18}$$

Proof Since $E|X|^p < \infty$, by Theorem 3.1, we have that for all $\varepsilon > 0$,

$$\sum_{n=1}^\infty n^{\alpha p-2} P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^j X_i\Big| > \varepsilon n^\alpha\Big) < \infty. \tag{3.19}$$

It follows from (3.19) that, for all $\varepsilon > 0$,

$$\begin{aligned}
\infty &> \sum_{n=1}^\infty n^{\alpha p-2} P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^j X_i\Big| > \varepsilon n^\alpha\Big) \\
&= \sum_{k=0}^\infty \sum_{n=2^k}^{2^{k+1}-1} n^{\alpha p-2} P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^j X_i\Big| > \varepsilon n^\alpha\Big) \\
&\geq \begin{cases} \displaystyle\sum_{k=0}^\infty (2^k)^{\alpha p-2} 2^k P\Big(\max_{1\leq j\leq 2^k}\Big|\sum_{i=1}^j X_i\Big| > \varepsilon 2^{(k+1)\alpha}\Big), & \text{if } \alpha p \geq 2, \\ \displaystyle\sum_{k=0}^\infty (2^{k+1})^{\alpha p-2} 2^k P\Big(\max_{1\leq j\leq 2^k}\Big|\sum_{i=1}^j X_i\Big| > \varepsilon 2^{(k+1)\alpha}\Big), & \text{if } 1 \leq \alpha p < 2 \end{cases} \\
&\geq \begin{cases} \displaystyle\sum_{k=0}^\infty P\Big(\max_{1\leq j\leq 2^k}\Big|\sum_{i=1}^j X_i\Big| > \varepsilon 2^{(k+1)\alpha}\Big), & \text{if } \alpha p \geq 2, \\ \displaystyle\frac{1}{2}\sum_{k=0}^\infty P\Big(\max_{1\leq j\leq 2^k}\Big|\sum_{i=1}^j X_i\Big| > \varepsilon 2^{(k+1)\alpha}\Big), & \text{if } 1 \leq \alpha p < 2, \end{cases}
\end{aligned}$$

which, together with the Borel-Cantelli lemma, yields that

$$\frac{\max_{1\leq j\leq 2^k}\Big|\sum_{i=1}^j X_i\Big|}{2^{(k+1)\alpha}} \to 0 \text{ a.s.} \tag{3.20}$$

For all positive integers n, there exists a positive integer k such that $2^{k-1} \leq n \leq 2^k$. We have, by (3.20), that

$$n^{-\alpha}\Big|\sum_{i=1}^n X_i\Big| \leq \max_{2^{k-1}\leq n\leq 2^k} n^{-\alpha}\Big|\sum_{i=1}^n X_i\Big| \leq \frac{2^{2\alpha}\max_{1\leq j\leq 2^k}\Big|\sum_{i=1}^j X_i\Big|}{2^{(k+1)\alpha}} \to 0 \text{ a.s.}$$

which implies (3.18). This completes the proof of the corollary.

If $\{X_n, n \geq 1\}$ is a sequence of END random variables with identical distribution, then (3.1) is obvious. By using Theorem 3.1, we can get the Baum-Katz-type result for END random variables as follows.

Corollary 3.2 *Let* $\alpha > \frac{1}{2}$ *and* $\alpha p > 1$*. Let* $\{X_n, n \geq 1\}$ *be a sequence of END random variables with identical distribution. Assume further that* $EX_1 = 0$ *if* $p \geq 1$*. Then* (i) *and* (ii) *in Theorem* 3.1 *are equivalent.*

3.2 Sufficient and necessary conditions for complete moment convergence

In this subsection, we will establish the sufficient and necessary conditions for complete moment convergence. First we present the necessary condition for complete moment convergence.

Theorem 3.2 *Suppose that the conditions of Theorem* 3.1 *hold. If for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_i\Big| - \varepsilon n^{\alpha}\Big)^{+} < \infty, \tag{3.21}$$

then $E|X|^p < \infty$.

Proof Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_i\Big| - \varepsilon n^{\alpha}\Big)^{+}\\
=&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^{\infty} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_i\Big| - \varepsilon n^{\alpha} > t\Big)dt\\
\ge&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^{\varepsilon n^{\alpha}} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_i\Big| - \varepsilon n^{\alpha} > t\Big)dt\\
\ge&\,\varepsilon\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_i\Big| > 2\varepsilon n^{\alpha}\Big).
\end{aligned} \tag{3.22}$$

Combining (3.21) and (3.22), we can get that (3.2) holds for all $\varepsilon > 0$. Hence, $E|X|^p < \infty$ follows from Theorem 3.1 immediately. This completes the proof of the theorem.

Next we present the sufficient condition for complete moment convergence. Noting that the factor $\log n$ is added to the right of the maximal moment inequality for END random variables (see Lemma 2.2), in order to establish the sufficient condition for complete moment convergence, the moment condition should be changed.

Theorem 3.3 *Suppose that the conditions of Theorem* 3.1 *hold for* $p \ge 1$. *If* $E|X|^p \log^{\theta}|X| < \infty$ *for some* $\theta > \max\{\frac{\alpha p-1}{\alpha-\frac{1}{2}}, p\}$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_i\Big| - \varepsilon n^{\alpha}\Big)^{+} < \infty. \tag{3.23}$$

Proof For fixed $n \ge 1$, denote, for $1 \le i \le n$, that

$$\begin{aligned}
Y_{ni} &= -n^{\alpha} I(X_i < -n^{\alpha}) + X_i I(|X_i| \le n^{\alpha}) + n^{\alpha} I(X_i > n^{\alpha}),\\
Z_{ni} &= X_i - Y_{ni} = (X_i - n^{\alpha}) I(X_i > n^{\alpha}) + (X_i + n^{\alpha}) I(X_i < -n^{\alpha}).
\end{aligned}$$

It follows from Lemma 2.5, that for any $\delta > 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} X_i\Big| - \varepsilon n^{\alpha}\Big)^{+}\\
\le&\, C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\delta} E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} (Y_{ni} - EY_{ni})\Big|\Big)^{\delta}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}(Z_{ni}-EZ_{ni})\Big|\Big)\\
&\doteq \mathrm{P}_1+\mathrm{P}_2.
\end{aligned}
\tag{3.24}
$$

Noting that $|Z_{ni}|=(|X_i|-n^\alpha)I(|X_i|>n^\alpha)\le |X_i|I(|X_i|>n^\alpha)$, we have, by Lemma 2.4, that

$$
\begin{aligned}
\mathrm{P}_2&=\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}(Z_{ni}-EZ_{ni})\Big|\Big)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\sum_{i=1}^{n}E|Z_{ni}|\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\sum_{i=1}^{n}E|X_i|I(|X_i|>n^\alpha)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}E|X|I(|X|>n^\alpha)\\
&= C\sum_{i=1}^{\infty} E|X|I(i^\alpha<|X|\le (i+1)^\alpha)\sum_{n=1}^{i} n^{\alpha p-1-\alpha}\\
&\le \begin{cases} C\sum_{i=1}^{\infty} E|X|I(i^\alpha<|X|\le (i+1)^\alpha)\log i, & \text{if } p=1,\\ C\sum_{i=1}^{\infty} E|X|I(i^\alpha<|X|\le (i+1)^\alpha)i^{\alpha p-\alpha}, & \text{if } p>1\end{cases}\\
&\le \begin{cases} CE|X|\log|X|, & \text{if } p=1,\\ CE|X|^p, & \text{if } p>1\end{cases}\\
&<\infty.
\end{aligned}
\tag{3.25}
$$

Next, we will show that $\mathrm{P}_1<\infty$. Noting that $\theta>p\ge 1$, we can take $\delta=\theta$. We consider the following two cases.

Case 1 $1<\theta\le 2$.

It follows from (2.2) of Lemma 2.2 and Lemma 2.4, that

$$
\begin{aligned}
\mathrm{P}_1&=C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}\Big(Y_{ni}-EY_{ni}\Big)\Big|\Big)^\theta\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}\log^\theta n\sum_{i=1}^{n}E|Y_{ni}|^\theta\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}\log^\theta n\sum_{i=1}^{n}[E|X_i|^\theta I(|X_i|\le n^\alpha)+n^{\alpha\theta}P(|X_i|>n^\alpha)]\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha\theta}\log^\theta n[E|X|^\theta I(|X|\le n^\alpha)+n^{\alpha\theta}P(|X|>n^\alpha)]\\
&= C\sum_{i=1}^{\infty} E|X|^\theta I((i-1)^\alpha<|X|\le i^\alpha)\sum_{n=i}^{\infty} n^{\alpha p-1-\alpha\theta}\log^\theta n\\
&\quad + C\sum_{i=1}^{\infty} P(i^\alpha<|X|\le (i+1)^\alpha)\sum_{n=1}^{i} n^{\alpha p-1}\log^\theta n
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{i=1}^{\infty} E|X|^{\theta} I((i-1)^{\alpha} < |X| \le i^{\alpha}) i^{\alpha p-\alpha\theta}\log^{\theta} i \\
&\quad + C\sum_{i=1}^{\infty} P(i^{\alpha} < |X| \le (i+1)^{\alpha}) i^{\alpha p}\log^{\theta} i \\
&\le CE|X|^{p}\log^{\theta}|X| \ < \ \infty.
\end{aligned} \tag{3.26}
$$

Case 2 $\theta > 2$.

Since $\theta > \frac{\alpha p-1}{\alpha-\frac{1}{2}}$, we can see that $\alpha p - 2 - \alpha\theta + \frac{\theta}{2} < -1$. Similarly to the proof of (3.26), we have, by (2.4) of Lemma 2.2, C_r inequality and Lemma 2.4, that

$$
\begin{aligned}
\mathrm{P}_1 &= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta} E\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}\Big(Y_{ni}-EY_{ni}\Big)\Big|\Big)^{\theta} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}\log^{\theta} n\Big[\sum_{i=1}^{n} E|Y_{ni}|^{\theta} + \Big(\sum_{i=1}^{n} E|Y_{ni}|^{2}\Big)^{\frac{\theta}{2}}\Big] \\
&\le C + C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}\log^{\theta} n\Big(\sum_{i=1}^{n} E|Y_{ni}|^{2}\Big)^{\frac{\theta}{2}} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta}\log^{\theta} n\Big\{\sum_{i=1}^{n}[EX_i^2 I(|X_i| \le n^{\alpha}) + n^{2\alpha}P(|X_i| > n^{\alpha})]\Big\}^{\frac{\theta}{2}} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta+\frac{\theta}{2}}\log^{\theta} n[EX^2 I(|X| \le n^{\alpha}) + n^{2\alpha}P(|X| > n^{\alpha})]^{\frac{\theta}{2}} \\
&\le \begin{cases} C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta+\frac{\theta}{2}}(EX^2)^{\frac{\theta}{2}}\log^{\theta} n < \infty, & \text{if } \ p \ge 2, \\ C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\theta+\frac{\theta}{2}+\alpha(2-p)\frac{\theta}{2}}(E|X|^{p})^{\frac{\theta}{2}}\log^{\theta} n, & \text{if } \ 1 \le p < 2 \end{cases} \\
&= \begin{cases} C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha\delta+\frac{\theta}{2}}(EX^2)^{\frac{\theta}{2}}\log^{\theta} n, & \text{if } \ p \ge 2, \\ C\sum_{n=1}^{\infty} n^{(\alpha p-1)(1-\frac{\theta}{2})-1}(E|X|^{p})^{\frac{\theta}{2}}\log^{\theta} n, & \text{if } \ 1 \le p < 2 \end{cases} \\
&< \infty.
\end{aligned} \tag{3.27}
$$

Hence, the desired result (3.23) follows from (3.24)–(3.27) immediately. This completes the proof of the theorem.

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