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Nongeneric Bifurcations Near a Nontransversal Heterodimensional Cycle*

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Abstract In this paper bifurcations of heterodimensional cycles with highly degenerate conditions are studied in three dimensional vector fields, where a nontransversal intersection between the two-dimensional manifolds of the saddle equilibria occurs. By setting up local moving frame systems in some tubular neighborhood of unperturbed heterodimensional cycles, the authors construct a Poincaré return map under the nongeneric conditions and further obtain the bifurcation equations. By means of the bifurcation equations, the authors show that different bifurcation surfaces exhibit variety and complexity of the bifurcation of degenerate heterodimensional cycles. Moreover, an example is given to show the existence of a nontransversal heterodimensional cycle with one orbit flip in three dimensional system.

Keywords Local moving frame, Nontransversal heterodimensional cycle, Orbit flip, Poincaré return map
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1 Introduction and Hypotheses

In recent years, bifurcation theory has attracted lots of attention due to its important role in applications (see [1–3]). Especially, different kinds of high co-dimensional homoclinic or heteroclinic bifurcations have been studied in detail. [4] studied the inclination-flip homoclinic orbit together with two other codimension 2 homoclinic bifurcations, which are cases of resonant bifurcation and orbit-flip bifurcation. [5] investigated codimension-two bifurcations of homoclinic orbits with an orbit flip. For other references, see [6–8] and the references cited therein.

[9] considered the bifurcation of heterodimensional cycles in dynamical systems. A heteroclinic cycle is said to be equidimensional if all the equilibrium points in the cycle have the same index (dimension of the stable manifold). Otherwise, such a cycle is called heterodimensional. Heterodimensional cycles, as a special kind of heteroclinic cycle, were found in many practical problems (see [10–11]). Bykov made an essential contribution to the topic of the paper under consideration (see [12], where the unfolding of codim-0/codim-2 cycles was studied). [13] analyzed homoclinic orbits near heterodimensional cycles between an equilibrium and a periodic orbit in three dimensions. For other references about heterodimensional cycles, see [14–19] and the references cited therein.

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Usually, a generic heterodimensional cycle is composed by a codim-0/codim-2 heteroclinic orbit between two real saddle equilibrium. However a heterodimensional cycle may exhibit different degeneracies for some reasons (see [18–20]). The study in [20] revealed another degeneracy that the two heteroclinic orbits of the heterodimensional cycle are both nontransversal, that was found in Chua's equation. Notice that there are few papers on nontransversal heterodimensional cycle problems concerning orbit flips. Motivated by this fact, in this paper, we confine ourselves to study the bifurcation of the nongeneric heterodimensional cycle with orbit flip if the nontransversal intersection of the two-dimensional manifolds occurs at the same time.

We will present the bifurcation results on different parameter regions, and we will show that under the stronger degeneracy conditions-nontransversality and orbit flip, the problem under consideration in our paper has the richer dynamics than the problem discussed in the literature [17], where they discussed the nontransversal heterodimensional cycle with no orbit flip. For example, the heterodimensional cycle can coexist with periodic orbit, but this can not happen in the case in [17]. In addition, we also give an example to demonstrate the existence of the system which has a nontransversal heterodimensional with one orbit flip.

The difficulty for us is how to show the different degeneracy (including the nontransversality and the orbit flip) in the return map. The technique we have used here is the Shilnikov coordinates and the local moving frame, the latter is introduced in [21], and then improved in [22–23] etc. By establishing the local coordinates and Poincaré maps in a sufficiently small neighborhood of the primary cycle, we theoretically show that the different bifurcation surfaces exhibits variety and complexity of the bifurcations of degenerate heterodimensional cycles.

Consider the following C^r system:

$$\dot{z} = f(z) + g(z, \mu), \tag{1.1}$$

and its unperturbed system

$$\dot{z} = f(z), \tag{1.2}$$

where $r \geqslant 4,\ z \in \mathbb{R}^3,\ \mu \in \mathbb{R}^l,\ l \geq 3,\ 0 \leqslant |\mu| \ll 1,\ g(z,0) = 0,\ \ f(z)$ is C^r with respect to the phase variable z, $g(z,\mu)$ is C^r with respect to the phase variable z and the parameter μ. We also assume that:

(H_1) System (1.2) has two hyperbolic equilibria p_i, $i = 1, 2$. $W^s_{p_i}$ and $W^u_{p_i}$ are the C^r stable and unstable manifolds of p_i, respectively. In addition, the linearization matrix $Df(p_1)$ has three simple real eigenvalues: $-\rho_1^1,\ \lambda_1^1,\ \lambda_1^2$ satisfying

$$-\rho_1^1 < 0 < \lambda_1^1 < \lambda_1^2, \quad \lambda_1^2 \geq 3\lambda_1^1, \tag{1.3}$$

and $Df(p_2)$ has three simple real eigenvalues: $-\rho_2^1, -\rho_2^2, \lambda_2^1$ satisfying

$$-\rho_2^2 < -\rho_2^1 < 0 < \lambda_2^1, \quad \rho_2^2 \geq 3\rho_2^1. \tag{1.4}$$

(H_2) There is a heteroclinic cycle $\Gamma = \Gamma_1 \cup \Gamma_2$ connecting p_1 and p_2, where $\Gamma_i = \{z = r_i(t) :\ t \in R\},\ r_1(+\infty) = r_2(-\infty) = p_2,\ r_1(-\infty) = r_2(+\infty) = p_1$.

(H_3) Let $e_i^{\pm} = \lim\limits_{t \to \mp\infty} \frac{\dot{r}_i(t)}{|\dot{r}_i(t)|}$, then $e_1^+ \in T_{p_1}W^u_{p_1},\ e_2^+ \in T_{p_2}W^u_{p_2},\ e_1^- \in T_{p_2}W^{ss}_{p_2},\ e_2^- \in T_{p_1}W^s_{p_1}$ be unit eigenvectors corresponding to $\lambda_1^1, \lambda_2^2, -\rho_2^1, -\rho_1^1$, respectively, where $W^{ss}_{p_2}$ is the strong stable manifold of p_2. By T_qM, we denote the tangent space of the manifold M at q.

Remark 1.1 Under the assumption (H$_1$), we know that Γ is a heterodimensional cycle. By (H$_3$), e_1^+ and e_1^- are the eigenvalues corresponding to λ_1^1 and $-\rho_2^2$, respectively, which means that Γ_1 enters p_1 along the leading unstable direction of $W_{p_1}^u$, and enters p_2 along the strong stable direction of $W_{p_2}^{ss}$. From [24], we know that Γ_1 takes orbit-flip when $t \to +\infty$ (see Figure 1).

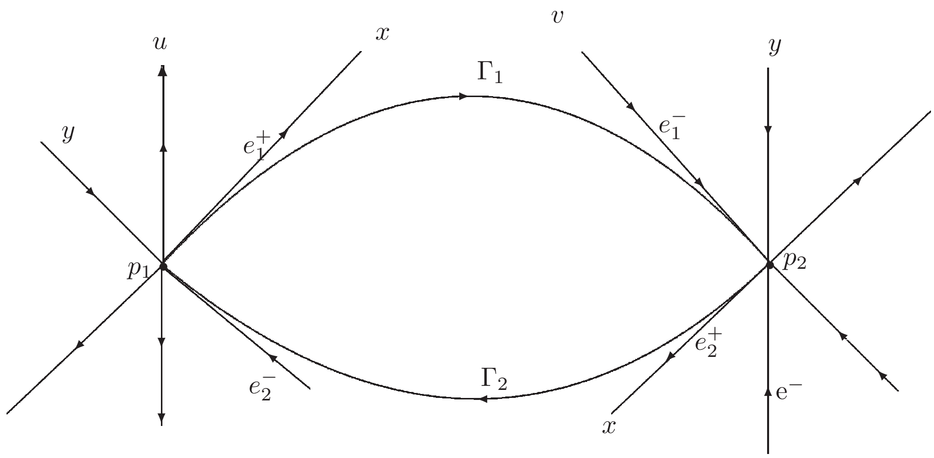

Figure 1 Heterodimensional cycle $\Gamma = \Gamma_1 \cup \Gamma_2$.

(H$_4$) (Nontransversal condition) There is a nontransversal intersection between the two-dimensional manifolds of p_i along the heteroclinic orbit Γ_1, that is, $W_{p_1}^u$ is coincident with $W_{p_2}^s$ along Γ_1.

As we will see, the bifurcations under consideration heavily depend on the relations between the eigenvalues of p_i, $i = 1, 2$. Without loss of generality, we may assume

(H$_5$)

$$\frac{\lambda_2^1}{\rho_2^1} > \frac{\rho_1^1}{\lambda_1^1} > 1.$$

The rest of the paper is organized as follows. In Section 2, the Poincaré map and the successor function are obtained by the establishment of a local moving frame system near the unperturbed heterodimensional cycle. Then, bifurcation equations are derived by using the implicit function theorem. Section 3 presents the bifurcation results on different parameter regions and the sufficient conditions for the persistence of heterodimensional cycle, the existence of homoclinic orbit and periodic orbit, the noncoexistence and coexistence of heterodimensional cycle, periodic orbit and homoclinic orbit. An analytical example is demonstrated to illustrate our main results in the last section.

2 Local Coordinates and Poincaré Maps

Following [25], as a direct application of the stable (unstable) manifold theorem and the strong stable (unstable) manifold theorem, we take two successive C^r and C^{r-1} transformations to straighten the local stable manifold, unstable manifold, strong unstable manifold in the region of U_i such that the system (1.1) has the following form in the small neighborhood U_1 of p_1:

$$\begin{cases} \dot{x} = [\lambda_1^1(\mu) + \cdots]x + O(u)O(y), \\ \dot{y} = [-\rho_1^1(\mu) + \cdots]y, \\ \dot{u} = [\lambda_1^2(\mu) + \cdots]u + O(x)O(y), \end{cases} \tag{2.1}$$

and has the following form in the neighborhood U_2 of p_2:

$$\begin{cases} \dot{x} = [\lambda_2^1(\mu) + \cdots]x, \\ \dot{y} = [-\rho_2^1(\mu) + \cdots]y + O(v)O(x), \\ \dot{v} = [-\rho_2^2(\mu) + \cdots]v + O(y)[O(x) + O(y)]. \end{cases} \tag{2.2}$$

Systems (2.1)–(2.2) are at least C^k, where $k = \min\{r-3, [\frac{\lambda_1^2}{\lambda_1^1}]-1, [\frac{\rho_2^2}{\rho_2^1}]-1\} \geq 2$, which is owing to that the weak unstable manifold of p_1 and the weak stable manifold of P_2 are approximately $C^{[\frac{\lambda_1^2}{\lambda_1^1}]}$, $C^{[\frac{\rho_2^2}{\rho_2^1}]}$, respectively (see [24, p. 56]). Of course, the same kind of change of variable can be achieved by using the theory of exponential dichotomies and weighted exponential dichotomies. However, by [24], we know that the extra conditions $\lambda_1^2 \geq 3\lambda_1^1$ and $\rho_2^2 \geq 3\rho_2^1$ are needed to ensure such change of coordinates are possible, so that the systems (2.1)–(2.2) are smooth enough. For notational convenience, we use $\lambda_1^i(\mu)$, $-\rho_1^1(\mu)$, $i = 1, 2$ and $-\rho_2^j(\mu)$, $j = 1, 2$ $\lambda_2^1(\mu)$ as the corresponding eigenvalues of the linearization matrix of perturbed system (1.1), which indicate dependence on μ, where $\lambda_1^i(0) = \lambda_1^i$, $\rho_1^1(0) = \rho_1^1$, $i = 1, 2$, $\rho_2^j(0) = \rho_2^j$, $\lambda_2^1(0) = \lambda_2^1$, $j = 1, 2$.

Take the coordinate expression of $r_i(t)$ as $r_i(t) = (r_i^x(t), r_i^y(t), r_i^u(t))^*$ in the small neighborhood U_1, and $r_i(t) = (r_i^x(t), r_i^y(t), r_i^v(t))^*$, $i = 1, 2$, in the small neighborhood U_2, $i = 1, 2$. Take the time T_i large enough such that $r_1(-T_1) = (\delta, 0, 0)^*$, $r_1(T_1) = (0, 0, \delta)^*$, $r_2(-T_2) = (\delta, 0, 0)^*$, $r_2(T_2) = (0, \delta, 0)^*$, where the sign "*" means the transposition, and $\delta > 0$ is small enough such that

$$\{(x, y, u)^* \mid |x|, |y|, |u| < 2\delta\} \subset U_1, \quad \{(x, y, v)^* \mid |x|, |y|, |v| < 2\delta\} \subset U_2.$$

Consider the linear variational system of (1.2)

$$\dot{Z} = Df(r_i(t))Z \tag{2.3}$$

and its adjoint system

$$\dot{\Phi} = -(Df(r_i(t)))^*\Phi. \tag{2.4}$$

Note that these two systems are adjoint in the sense that if $Z(t)$ is the solution matrix of (2.3), then $(Z^{-1}(t))^*$ is the solution matrix of (2.4).

In the following, we will choose suitable solutions of the corresponding linear variational equation as a local coordinate system along Γ_i.

Following the idea in [17], we know that there exists a fundamental solution matrix $Z_1(t) = (z_1^1(t), z_1^2(t), z_1^3(t))$ for the system (2.3) satisfying

$$\begin{aligned} &z_1^1(t) \in (T_{r_1(t)}W_{p_1}^u)^c, \\ &z_1^2(t) = \frac{\dot{r_1}(t)}{|\dot{r_1}(-T_1)|} \in T_{r_1(t)}W_{p_1}^u \cap T_{r_1(t)}W_{p_2}^s, \end{aligned}$$

$$z_1^3(t) \in T_{r_1(t)} W_{p_1}^u \cap (T_{r_1(t)}\Gamma_1)^c,$$

such that

$$Z_1(-T_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \omega_1^{13} & 0 & 1 \end{pmatrix}, \quad Z_1(T_1) = \begin{pmatrix} \omega_1^{11} & 0 & 0 \\ 0 & 0 & \omega_1^{32} \\ \overline{\omega}_1^{13} & \omega_1^{23} & \omega_1^{33} \end{pmatrix},$$

where $\omega_1^{32} \neq 0,\ \omega_1^{11} \neq 0,\ \omega_1^{23} < 0,\ |\overline{\omega}_1^{13} \cdot (\omega_1^{11})^{-1}| \ll 1,\ |\omega_1^{33} \cdot (\omega_1^{32})^{-1}| \ll 1$. The notation $(M)^c$ means subspace complementary to M.

Also, there exists a fundamental solution matrix $Z_2(t) = (z_2^1(t), z_2^2(t), z_2^3(t))$ for the system (2.3) satisfying

$$z_2^1(t), z_2^3(t) \in (T_{r_2(t)}\Gamma_2)^c,$$
$$z_2^2(t) = \frac{\dot{r_2}(t)}{|\dot{r_2}(-T_2)|} \in T_{r_2(t)} W_{p_2}^u \cap T_{r_2(t)} W_{p_1}^s,$$

such that

$$Z_2(-T_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Z_2(T_2) = \begin{pmatrix} \omega_2^{11} & 0 & \omega_2^{31} \\ \omega_2^{12} & \omega_2^{22} & \omega_2^{32} \\ \omega_2^{13} & 0 & \omega_2^{33} \end{pmatrix},$$

where $\omega_2^{22} \neq 0,\ \ \omega = \begin{vmatrix} \omega_2^{11} & \omega_2^{31} \\ \omega_2^{13} & \omega_2^{33} \end{vmatrix} \neq 0,\ |\omega_2^{i2} \cdot \omega^{-1}| \ll 1,\ i = 1, 3.$

In what follows, we choose $(z_i^1(t), z_i^2(t), z_i^3(t))$ as a new local coordinate system along Γ_i. Let $\Phi_i(t) = (\phi_i^1(t), \phi_i^2(t), \phi_i^3(t)) = (Z_i^{-1}(t))^*$, then $\Phi_i(t)$ is a fundamental solution matrix of (2.4), $i = 1, 2$. Take a coordinate transformation near the orbits Γ_i as

$$z = r_i(t) + Z_i(t)N_i(t) \triangleq h_i(t), \quad i = 1, 2,$$

where $N_i(t) = (n_i^1(t), 0, n_i^3(t))^*,\ i = 1, 2$ are the coordinate decomposition of system (1.1) in the new local coordinate system corresponding to $z_i^1(t), z_i^3(t)$.

Let

$$S_1^0 = \{z = h_1(-T_1) : |x|, |y|, |u| < 2\delta\}, \quad S_2^0 = \{z = h_2(-T_2) : |x|, |y|, |v| < 2\delta\},$$
$$S_1^1 = \{z = h_1(T_1) : |x|, |y|, |v| < 2\delta\}, \qquad S_2^1 = \{z = h_2(T_2) : |x|, |y|, |u| < 2\delta\}$$

be cross-sections of Γ_i at $t = -T_i$ and $t = T_i$, respectively, which intersect Γ_i transversally.

Now we start to construct the Poincaré map step by step. Consider the map $F_i^0 : q_{i-1}^1 \in S_{i-1}^1 \to q_i^0 \in S_i^0$ and $F_i^1 : q_i^0 \in S_i^0 \to q_i^1 \in S_i^1$, where $S_0^1 = S_2^1,\ q_0^1 = q_2^1$ (see Figure 2).

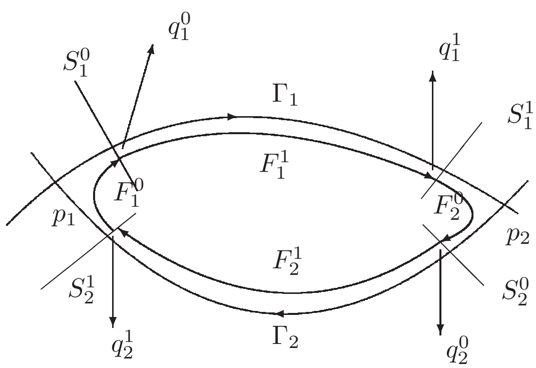

Figure 2 The cross-sections and Poincaré map.

In order to obtain the Poincaré map, first we should establish the relationship between the old coordinates

$$q_1^0(x_1^0, y_1^0, u_1^0)^*,\ q_1^1(x_1^1, y_1^1, v_1^1)^*,\ q_2^0(x_2^0, y_2^0, v_2^0)^*,\ q_2^1(x_2^1, y_2^1, u_2^1)^*$$

and their new coordinates

$$q_1^0(n_1^{0,1}, 0, n_1^{0,3})^*,\ q_1^1(n_1^{1,1}, 0, n_1^{1,3})^*,\ q_2^0(n_2^{0,1}, 0, n_2^{0,3})^*,\ q_2^1(n_2^{1,1}, 0, n_2^{1,3})^*.$$

By the coordinate transformation $h_i(t) = r_i(t) + Z_i(t)N_i(t)$, we have

$$\begin{aligned}
q_1^0 &= (x_1^0, y_1^0, u_1^0)^* = r_1(-T_1) + Z_1(-T_1)N_1(-T_1), \quad && N_1(-T_1) = (n_1^{0,1}, 0, n_1^{0,3})^*,\\
q_1^1 &= (x_1^1, y_1^1, v_1^1)^* = r_1(T_1) + Z_1(T_1)N_1(T_1), && N_1(T_1) = (n_1^{1,1}, 0, n_1^{1,3})^*.
\end{aligned}$$

Then combining with the expressions of $Z_i(-T_i)$, $Z_i(T_i)$ $(i = 1, 2)$, we obtain

$$\begin{cases} n_1^{0,1} = y_1^0,\\ n_1^{0,3} = u_1^0 - \omega_1^{13} y_1^0,\\ x_1^0 = \delta \end{cases} \tag{2.5}$$

and

$$\begin{cases} n_1^{1,1} = (\omega_1^{11})^{-1} x_1^1,\\ n_1^{1,3} = (\omega_1^{32})^{-1} y_1^1,\\ v_1^1 = \delta + \overline{\omega}_1^{13}(\omega_1^{11})^{-1} x_1^1 + \omega_1^{33}(\omega_1^{32})^{-1} y_1^1 \approx \delta. \end{cases} \tag{2.6}$$

For

$$\begin{aligned}
q_2^0 &= (x_2^0, y_2^0, v_2^0)^* = r_2(-T_2) + Z_2(-T_2)N_2(-T_2), \quad && N_2(-T_2) = (n_2^{0,1}, 0, n_2^{0,3})^*,\\
q_2^1 &= (x_2^1, y_2^1, u_2^1)^* = r_2(T_2) + Z_2(T_2)N_2(T_2), && N_2(T_2) = (n_2^{1,1}, 0, n_2^{1,3}),
\end{aligned}$$

a similar calculation shows that

$$\begin{cases} n_2^{0,1} = v_2^0,\\ n_2^{0,3} = y_2^0,\\ x_2^0 = \delta \end{cases} \tag{2.7}$$

and

$$\begin{cases} n_2^{1,1} = \omega^{-1}(\omega_2^{33} x_2^1 - \omega_2^{31} u_2^1),\\ n_2^{1,3} = \omega^{-1}(\omega_2^{11} u_2^1 - \omega_2^{13} x_2^1),\\ y_2^1 \approx \delta. \end{cases} \tag{2.8}$$

On the other hand, suppose that $h_i(t) = r_i(t) + Z_i(t)N_i(t)$ is the solution of (1.1) in the small tube neighborhood of Γ_i. Then substitute it into (1.1), and we have

$$\begin{aligned}
&\dot r_i(t) + \dot Z_i(t)N_i(t) + Z_i(t)\dot N_i(t)\\
&= f(r_i(t) + Z_i(t)N_i(t)) + g(r_i(t) + Z_i(t)N_i(t), \mu)\\
&= f(r_i(t)) + Df(r_i(t))Z_i(t)N_i(t) + g(r_i(t), 0)
\end{aligned}$$

$$+ g_z(r_i(t),0)\cdot Z_i(t)N_i(t) + g_\mu(r_i(t),0)\cdot\mu + \text{h.o.t.}$$

By $\dot r_i(t) = f(r_i(t))$, $\dot Z_i(t) = Df(r_i(t))\cdot Z_i(t)$ and $g(z,0)=0$, we obtain

$$\dot N_i(t) = Z_i^{-1}(t)\cdot g_\mu(r_i(t),0)\mu + \text{h.o.t.}$$

Integrating both sides of this equation from $-T_i$ to T_i, we have

$$N_i(T_i) - N_i(-T_i) = \int_{-T_i}^{T_i} Z_i^{-1}(t) g_\mu(r_i(t),0)\mu\,\mathrm{d}t + \text{h.o.t.},$$

which produce the global map $F_1^1 : S_1^0 \longrightarrow S_1^1$ and $F_2^1 : S_2^0 \longrightarrow S_2^1$, as follows

$$\begin{aligned} F_1^1(n_1^{0,1},0,n_1^{0,3})^* &= (\widetilde n_1^{1,1},0,\widetilde n_1^{1,3})^*,\\ F_2^1(n_2^{0,1},0,n_2^{0,3})^* &= (\widetilde n_2^{1,1},0,\widetilde n_2^{1,3})^* \end{aligned}$$

with the expression given by

$$\begin{aligned} \widetilde n_1^{1,j} &= n_1^{0,j} + M_1^j\mu + \text{h.o.t.}, \quad j=1,3,\\ \widetilde n_2^{1,k} &= n_2^{0,k} + M_2^k\mu + \text{h.o.t.}, \quad k=1,3, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} M_1^j &= \int_{-T_1}^{T_1} \phi_1^{j*}(t) g_\mu(r_1(t),0)\,\mathrm{d}t, \quad j=1,3,\\ M_2^k &= \int_{-T_2}^{T_2} \phi_2^{k*}(t) g_\mu(r_2(t),0)\,\mathrm{d}t, \quad k=1,3. \end{aligned}$$

Next we consider the local maps $F_1^0 : q_2^1 \in S_2^1 \longrightarrow q_1^0 \in S_1^0$ and $F_2^0 : q_1^1 \in S_1^1 \longrightarrow q_2^0 \in S_2^0$ induced by flows confined in the neighborhood U_i.

Let τ_i $(i=1,2)$ be the time spent from q_{i-1}^1 to q_i^0, $q_0^1 = q_2^1$. Suppose $\rho_1^1 > \lambda_1^1$, $\lambda_2^1 > \rho_2^1$, then we select $s_1 = \mathrm{e}^{-\lambda_1^1(\mu)\tau_1}$, $s_2 = \mathrm{e}^{-\rho_2^1(\mu)\tau_2}$ (if $\rho_1^1 < \lambda_1^1$, $\lambda_2^1 < \rho_2^1$, then we select $s_1 = \mathrm{e}^{-\rho_1^1(\mu)\tau_1}$, $s_2 = \mathrm{e}^{-\lambda_2^1(\mu)\tau_2}$). Define $\beta_1(\mu) = \frac{\rho_1^1(\mu)}{\lambda_1^1(\mu)}$, $\beta_2(\mu) = \frac{\rho_2^1(\mu)}{\lambda_2^1(\mu)}$, then by (H_5), $1 < \beta_1(\mu) < \frac{1}{\beta_2(\mu)}$ holds for $|\mu| \ll 1$ on the basis of the continuity.

Then under the assumption (H_5) of the non-resonance conditions among the eigenvalues, by the normal forms (2.1)–(2.2), and the formula of variation of constants, we obtain the local map $F_1^0 : q_2^1(x_2^1,y_2^1,u_2^1) \in S_2^1 \to q_1^0(x_1^0,y_1^0,u_1^0) \in S_1^0$ as follows:

$$x_2^1 = x(T_2) \approx \delta s_1, \quad y_1^0 = y(T_2+\tau_1) \approx \delta s_1^{\frac{\rho_1^1(\mu)}{\lambda_1^1(\mu)}}, \quad u_2^1 = u(T_2) \approx s_1^{\frac{\lambda_1^2(\mu)}{\lambda_1^1(\mu)}} u_1^0, \tag{2.10}$$

and the local map $F_2^0 : q_1^1(x_1^1,y_1^1,v_1^1) \in S_1^1 \to q_2^0(x_2^0,y_2^0,v_2^0) \in S_2^0$ as follows:

$$x_1^1 = x(T_1) \approx \delta s_2^{\frac{\lambda_2^1(\mu)}{\rho_2^1(\mu)}}, \quad y_2^0 = y(T_1+\tau_2) \approx s_2 y_1^1, \quad v_2^0 = v(T_1+\tau_2) \approx \delta s_2^{\frac{\rho_2^2(\mu)}{\rho_2^1(\mu)}}, \tag{2.11}$$

where (s_1, s_2, u_1^0, y_1^1) are called Shilnikov variables.

Remark 2.1 Shilnikov variables were introduced by Shilnikov in 1968 to compute the local transition map near equilibria to leading order. Instead of solving an initial-value problem, solutions near the equilibrium are found using an appropriate boundary-value problem. Further information on Shilnikov variables can be found in [24, p. 62] and [26].

In the following, for convenience, we may denote $\lambda_1^i=\lambda_1^i(\mu),\ i=1,2;\ \rho_1^1=\rho_1^1(\mu),\ \beta_1=\frac{\rho_1^1(\mu)}{\lambda_1^1(\mu)},\ \beta_2=\frac{\rho_2^1(\mu)}{\lambda_2^1(\mu)};\ \rho_2^j(\mu)=\rho_2^j,\ \lambda_2^j=\lambda_2^j(\mu),\ j=1,2.$

Thus, by (2.5), (2.9)–(2.10), we obtain the Poincaré map $F_1=F_1^1\circ F_1^0$: $S_2^1\to S_1^1$ as follows:

$$\begin{cases}\widetilde{n}_1^{1,1}=s_1^{\beta_1}\delta+M_1^1\mu+\text{h.o.t.},\\ \widetilde{n}_1^{1,3}=u_1^0-\omega_1^{13}s_1^{\beta_1}\delta+M_1^3\mu+\text{h.o.t.},\end{cases}\tag{2.12}$$

and by (2.7), (2.9), (2.11), we obtain the Poincaré map $F_2=F_2^1\circ F_2^0$: $2S_1^1\to S_2^1$ as follows:

$$\begin{cases}\widetilde{n}_2^{1,1}=s_2^{\frac{\rho_2^2}{\rho_2^1}}\delta+M_2^1\mu+\text{h.o.t.},\\ \widetilde{n}_2^{1,3}=s_2y_1^1+M_2^3\mu+\text{h.o.t.}\end{cases}\tag{2.13}$$

Then, by (2.6), (2.8), (2.12)–(2.13), we obtain the successor functions

$$(G_1,G_2)\triangleq G(s_1,s_2,u_1^0,y_1^1)=(G_1^1,G_1^3,G_2^1,G_2^3)=(F_1(q_2^1)-q_1^1,\ F_2(q_1^1)-q_2^1)$$

as follows:

$$\begin{aligned}G_1^1&=s_1^{\beta_1}\delta-(\omega_1^{11})^{-1}s_2^{\frac{1}{\beta_2}}\delta+M_1^1\mu+\text{h.o.t.},\\ G_1^3&=u_1^0-\omega_1^{13}s_1^{\beta_1}\delta-(\omega_1^{32})^{-1}y_1^1+M_1^3\mu+\text{h.o.t.},\\ G_2^1&=s_2^{\frac{\rho_2^2}{\rho_2^1}}\delta-\omega^{-1}\omega_2^{33}s_1\delta+\omega^{-1}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}u_1^0+M_2^1\mu+\text{h.o.t.},\\ G_2^3&=s_2y_1^1+\omega^{-1}\omega_2^{13}s_1\delta-\omega^{-1}\omega_2^{11}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}u_1^0+M_2^3\mu+\text{h.o.t.}\end{aligned}$$

By the implicit function theorem, solving the equation $G_1^3=0$, we have

$$u_1^0=\omega_1^{13}s_1^{\beta_1}\delta+(\omega_1^{32})^{-1}y_1^1-M_1^3\mu+\text{h.o.t.}$$

Substituting it into $(G_1^1,G_2^1,G_2^3)=0$, we obtain the bifurcation equations, which have the following three different expressions:

(I) $\omega_2^{13}\neq 0,\ \omega_2^{33}\neq 0$

$$\begin{cases}s_1^{\beta_1}\delta-(\omega_1^{11})^{-1}s_2^{\frac{1}{\beta_2}}\delta+M_1^1\mu+\text{h.o.t.}=0,\\ s_2^{\frac{\rho_2^2}{\rho_2^1}}\delta-\omega^{-1}\omega_2^{33}s_1\delta+M_2^1\mu+\text{h.o.t.}=0,\\ s_2y_1^1+\omega^{-1}\omega_2^{13}s_1\delta+M_2^3\mu+\text{h.o.t.}=0.\end{cases}\tag{2.14}$$

(II) $\omega_2^{13}=0,\ \omega_2^{33}\neq 0$

$$\begin{cases}s_1^{\beta_1}\delta-(\omega_1^{11})^{-1}s_2^{\frac{1}{\beta_2}}\delta+M_1^1\mu+\text{h.o.t.}=0,\\ s_2^{\frac{\rho_2^2}{\rho_2^1}}\delta-\omega^{-1}\omega_2^{33}s_1\delta+M_2^1\mu+\text{h.o.t.}=0,\\ s_2y_1^1-\omega^{-1}\omega_2^{11}\omega_1^{13}s_1^{\frac{\lambda_1^2(\mu)}{\lambda_1^1(\mu)}+\beta_1}\delta-(\omega\omega_1^{32})^{-1}\omega_2^{11}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}y_1^1\\ \quad+\omega^{-1}\omega_2^{11}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu+M_2^3\mu+\text{h.o.t.}=0.\end{cases}\tag{2.15}$$

(III) $\omega_2^{13} \neq 0,\ \omega_2^{33} = 0$

$$
\begin{cases}
s_1^{\beta_1}\delta - (\omega_1^{11})^{-1} s_2^{\frac{1}{\beta_2}}\delta + M_1^1\mu + \text{h.o.t.} = 0, \\
s_2^{\frac{\rho_2^2}{\rho_2^1}}\delta + \omega^{-1}\omega_2^{31}\omega_1^{13} s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1}\delta + (\omega\omega_1^{32})^{-1}\omega_2^{31} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} y_1^1 \\
\qquad -\omega^{-1}\omega_2^{31} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3\mu + M_2^1\mu + \text{h.o.t.} = 0, \\
s_2 y_1^1 + \omega^{-1}\omega_2^{13} s_1\delta + M_2^3\mu + \text{h.o.t.} = 0.
\end{cases}
\tag{2.16}
$$

Remark 2.2 Note that the solutions of (2.14)–(2.16) lose the uniqueners, and the solutions demonstrate different kinds of dynamical patterns corresponding to the different parameter regions, then equations (2.14)–(2.16) are called the bifurcation equation.

Remark 2.3 For the first two cases, by some simple computation, we can obtain similar bifurcation results to that given in [17]; so we omit it. While, in case (III), we will show that there are different bifurcation phenomena from that discussed in [17]. Therefore, we will only focus on the third case.

3 Bifurcation Results

In this section we will study the bifurcation problem of the loop Γ under all hypotheses (H_1)–(H_5). The existence, coexistence and noncoexistence of periodic orbit, homoclinic loop and heteroclinic loop are discussed by studying the corresponding bifurcation equation. By establishing of local maps F_1^0 and F_2^0, we know that if $s_1 = s_2 = 0$, then the heteroclinic loop of system (1.1) is persistent; if $s_1 = 0,\ s_2 > 0$, then the system (1.1) has a loop homoclinic to p_1; if $s_1 > 0,\ s_2 = 0$, then the system (1.1) has a loop homoclinic to p_2; if $s_1 > 0,\ s_2 > 0$, the system (1.1) has a periodic orbit. Then, we need only to consider the nonnegative solution s_1 and s_2 of the bifurcation equation.

Now we consider the persistence of the heteroclinic loop under small perturbation.

Theorem 3.1 *Suppose that hypotheses* (H_1)*–*(H_5) *are satisfied, and* $\mathrm{Rank}(M_1^1, M_2^1, M_2^3) = 3$, $\omega_2^{13} \neq 0,\ \omega_2^{33} = 0$, *then there exists an* $(l-3)$*-dimensional surface*

$$
L_{12}(y_1^1) = \{\mu :\ M_1^1\mu + \text{h.o.t.} = M_2^1\mu + \text{h.o.t.} = M_2^3\mu + \text{h.o.t.} = 0\}
$$

with a normal plane spanned by $\sum\limits_{12} = \mathrm{span}\{M_1^1, M_2^1, M_2^3\}$, *such that the system* (1.1) *has a unique heteroclinic loop* $\Gamma^\mu(y_1^1) = \Gamma_1^\mu \cup \Gamma_2^\mu$ *connecting* p_1 *and* p_2 *as* $\mu \in L_{12}$, $0 < |\mu| \ll 1$ *and* $|y_1^1| \ll 1$. *Furthermore, the persistent heteroclinic orbit* Γ_1^μ *has no orbit-flip as* $t \to +\infty$ *if* $y_1^1 \neq 0$.

Proof If $s_1 = s_2 = 0$ is the solution of the bifurcation equations (2.16), then we have

$$
\begin{cases}
M_1^1\mu + \text{h.o.t.} = 0, \\
M_2^1\mu + \text{h.o.t.} = 0, \\
M_2^3\mu + \text{h.o.t.} = 0.
\end{cases}
$$

If $\mathrm{Rank}(M_1^1, M_2^1, M_3^2) = 3$, then

$$
L_{12}(v_1^1) = \{\mu :\ M_1^1\mu + \text{h.o.t.} = M_2^1\mu + \text{h.o.t.} = M_3^2\mu + \text{h.o.t.} = 0\}
$$

is a codimension 3 surface with normal plane spanned by $\{M_1^1, M_2^1, M_3^2\}$ at $\mu = 0$ such that the system (1.1) has a unique heterodimensional loop near Γ as $\mu \in L_{12}(v_1^1)$, $0 < |\mu| \ll 1$, and $|y_1^1| \ll 1$. In addition, because the y axis corresponds to the leading stable eigendirection, we easily get to know that if $y_1^1 \neq 0$, then Γ_1^μ enters p_2 along y axis, that is, it can not exhibit orbit flip near Γ_1^μ as $t \to +\infty$.

A corresponding results about the existence of the homoclinic orbit connecting p_i is contained in the next theorems.

Theorem 3.2 *Suppose that hypotheses* (H_1)–(H_5) *are valid,* $\mathrm{Rank}(M_1^1, M_2^1, M_2^3) = 3$, *and* $\omega_2^{13} \neq 0$, $\omega_2^{33} = 0$, *then the following results are true:*

(1) *If* $\rho_2^2 > \lambda_2^1$, *then in the region* $R_1^2 = \{\mu \mid \omega_1^{11} M_1^1 \mu > 0, M_2^1 \mu < 0\}$, *there exists an* $(l-1)$*-dimensional surface*

$$L_1^2 = \{\mu \mid W_1(\mu) \triangleq (\omega_1^{11} \delta^{-1} M_1^1 \mu)^{\frac{\beta_2 \rho_2^2}{\rho_2^1}} \delta + M_2^1 \mu + \text{h.o.t.} = 0,\ |M_2^3 \mu| \ll |M_1^1 \mu|^{\beta_2}\}$$

with a normal vector M_2^1 *at* $\mu = 0$, *which is tangent to the surface* $L_{12}(v_1^1)$ *at* $\mu = 0$, *such that the system* (1.1) *has a unique loop* Γ_1^2 *homoclinic to* p_1 *near* Γ *as* $\mu \in L_1^2$ *and* $0 < |\mu| \ll 1$.

(2) *In the region* $R_2^1 = \{\mu \mid M_1^1 \mu < 0,\ \omega \omega_2^{13} M_2^3 \mu < 0\}$, *there exists an* $(l-2)$*-dimensional bifurcation surface* $L_2^1(y_1^1) \cap H_2^1(y_1^1)$, *such that the system* (1.1) *has a unique loop* Γ_2^1 *homoclinic to* p_2 *near* Γ *as* $\mu \in L_2^1(y_1^1) \cap H_2^1(y_1^1) \subset R_2^1$, $0 < |\mu| \ll 1$ *and* $|y_1^1| \ll 1$, *where*

$$\begin{aligned} L_2^1(y_1^1) &= \{\mu \mid W_2(\mu) \triangleq [\omega(\omega_2^{13})^{-1} \delta^{-1} M_2^3 \mu]^{\beta_1} \delta + M_1^1 \mu + \text{h.o.t.} = 0\}, \\ H_2^1(y_1^1) &= \{\mu \mid (\omega \omega_1^{32})^{-1} \omega_2^{31} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} y_1^1 - \omega^{-1} \omega_2^{31} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3 \mu - \omega^{-1} \omega_2^{31} \omega_1^{13} s_1^{\frac{\lambda_1^2}{\lambda_1^1} + \beta_1} \delta \\ &\qquad + M_2^1 \mu + \text{h.o.t.} = 0,\ s_1 = -\omega(\omega_2^{13})^{-1} \delta^{-1} M_2^3 \mu,\ |y_1^1| \ll 1\}. \end{aligned}$$

Proof (1) Assume that (2.16) has a solution $s_1 = 0,\ s_2 > 0$, then it can be simplified into the following form:

$$\begin{cases} -(\omega_1^{11})^{-1} s_2^{\frac{1}{\beta_2}} + M_1^1 \mu + \text{h.o.t.} = 0, \\ s_2^{\frac{\rho_2^2}{\rho_2^1}} \delta + M_2^1 \mu + \text{h.o.t.} = 0, \\ s_2 y_1^1 + M_2^3 \mu + \text{h.o.t.} = 0. \end{cases} \tag{3.1}$$

Obviously, in the region defined by R_1^2 and $|M_2^3 \mu| \ll |M_1^1 \mu|^{\beta_2}$, the third equation has a unique small solution

$$y_1^1(s_2, \mu) = \frac{M_2^3 \mu}{(\omega_1^{11} M_1^1 \mu)^{\beta_2}} + \text{h.o.t.}, \quad |y_1^1| \ll 1.$$

Therefore, (3.1) determines an $(l-1)$ dimensional surface L_1^2 which is perpendicular to M_2^1 at $\mu = 0$. On the basis of $\mathrm{Rank}(M_1^1, M_2^1, M_2^3) = 3$, L_1^2 is well defined. Now (2.16) has a solution $s_1 = 0,\ s_2 > 0$ as $\mu \in L_1^2,\ 0 < |\mu| \ll 1,\ |y_1^1| \ll 1$. That is, the system (1.1) has a unique homoclinic orbit Γ_1^2 connecting p_1 near Γ.

(2) Assume that $s_1 > 0,\ s_2 = 0$ is a solution of (2.16), then (2.16) is reduced to

$$\begin{cases} s_1^{\beta_1} \delta + M_1^1 \mu + \text{h.o.t.} = 0, \\ (\omega \omega_1^{32})^{-1} \omega_2^{31} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} y_1^1 - \omega^{-1} \omega_2^{31} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3 \mu + \omega^{-1} \omega_2^{31} \omega_1^{13} s_1^{\frac{\lambda_1^2}{\lambda_1^1} + \beta_1} \delta + M_2^1 \mu + \text{h.o.t.} = 0, \\ \omega^{-1} \omega_2^{13} s_1 \delta + M_2^3 \mu + \text{h.o.t.} = 0. \end{cases} \tag{3.2}$$

In the region given by R_2^1, the third equation has a unique solution s_1, then substituting it into the first two equations, we obtain that

$$[-\omega(\omega_2^{13})^{-1}\delta^{-1}M_2^3\mu]^{\beta_1}\delta+M_1^1\mu+\text{h.o.t.}=0,$$
$$(\omega\omega_1^{32})^{-1}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}y_1^1-\omega^{-1}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu+\omega^{-1}\omega_2^{31}\omega_1^{13}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1}\delta+M_2^1\mu+\text{h.o.t.}=0,$$

where $s_1=-\omega(\omega_2^{13})^{-1}\delta^{-1}M_2^3\mu$. Therefore, the system (3.2) determines an $(l-2)$ dimensional surface $L_2^1(y_1^1)\cap H_2^1(y_1^1)$ with the normal surface $\Sigma=\text{span}\{M_1^1,M_2^1\}$ at $\mu=0$. We see that $L_2^1(y_1^1)\cap H_2^1(y_1^1)$ is tangent to $L_{12}(y_1^1)$ at $\mu=0$. Now the system (2.16) has a solution $s_1>0,\ s_2=0$ as $\mu\in L_2^1(y_1^1)\cap H_2^1(y_1^1)\subset R_2^1,\ 0<|\mu|\ll 1$ and $|y_1^1|\ll 1$. The system (1.1) then possesses a homoclinic loop $\Gamma_2^1(y_1^1)$ connecting p_2 near Γ.

Next, relying on the analysis for the bifurcation equations (2.16), we discuss the coexistence of the heterodimensional cycle, homoclinic orbit and periodic orbit under small perturbation.

Theorem 3.3 *Suppose that hypotheses* (H_1)–(H_5) *are valid,* $\text{Rank}(M_1^1,M_2^1,M_2^3)=3,\ \omega_2^{13}\neq 0,\ \omega_2^{33}=0$, *then for* $0<|\mu|\ll 1$, *we have that*

(1) *the system* (1.1) *does not have any homoclinic orbit coexisting with the persistent heterodimensional cycle* Γ^μ *as* $\mu\in L_{12}(y_1^1)$;

(2) *if* $\rho_1^1(\rho_2^2+\rho_2^1)>\lambda_2^1(\lambda_1^2+\lambda_1^1),\ \omega\omega_1^{11}\omega_1^{32}\omega_2^{13}M_1^3\mu<0$, *then the system* (1.1) *has a unique periodic orbit coexisting with* Γ^μ *as* $\mu\in L_{12}(y_1^1)$.

Proof If $\omega_2^{13}\neq 0,\ \omega_2^{33}=0$ and $\mu\in L_{12}(y_1^1),\ |\mu|\ll 1$, then (2.16) gives

$$\begin{cases} s_1^{\beta_1}\delta-(\omega_1^{11})^{-1}s_2^{\frac{1}{\beta_2}}\delta+\text{h.o.t.}=0,\\ s_2^{\frac{\rho_2^2}{\rho_2^1}}\delta+\omega^{-1}\omega_2^{31}\omega_1^{13}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1}\delta+(\omega\omega_1^{32})^{-1}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}y_1^1\\ \qquad -\omega^{-1}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu+\text{h.o.t.}=0,\\ s_2y_1^1+\omega^{-1}\omega_2^{13}s_1\delta+\text{h.o.t.}=0. \end{cases}\tag{3.3}$$

(1) By the first equation of (3.3), we have $s_2=\omega_1^{11}s_1^{\beta_1\beta_2}+\text{h.o.t.}$ It is obvious that $s_2\geq 0$ if $s_1\geq 0$ and $\omega_1^{11}>0$, and $s_1=0$ if and only if $s_2=0$, so we conclude that (1.1) does not have any homoclinic loops for $\mu\in L_{12}(y_1^1)$.

(2) On the other hand, by the third equation of (3.3), we have

$$y_1^1=-\frac{\omega^{-1}\omega_2^{13}s_1\delta}{s_2}+\text{h.o.t.}=-\frac{\omega^{-1}\omega_2^{13}\delta}{\omega_1^{11}}s_1^{1-\beta_1\beta_2}+\text{h.o.t.}$$

By (H_5), we have $\beta_1\beta_2<1$, then $0<s_1\ll 1$ implies that $|y_1^1|\ll 1$. Substituting the expressions of $s_2,\ y_1^1$ into the second equation, we obtain

$$(\omega_1^{11})^{\frac{\rho_2^2}{\rho_2^1}+1}s_1^{\beta_1\beta_2(\frac{\rho_2^2}{\rho_2^1}+1)}\delta-\omega^{-2}(\omega_1^{32})^{-1}\omega_2^{31}\omega_2^{13}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+1}\delta-\omega^{-1}\omega_1^{11}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1\beta_2}M_1^3\mu$$
$$+\omega^{-1}\omega_1^{11}\omega_2^{31}\omega_1^{13}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1\beta_2+\beta_1}\delta+\text{h.o.t.}=0.$$

Assume $\beta_1\beta_2(\frac{\rho_2^2}{\rho_2^1}+1) > \frac{\lambda_1^2}{\lambda_1^1}+1$, namely, $\rho_1^1(\rho_2^2+\rho_2^1) > \lambda_2^1(\lambda_1^2+\lambda_1^1)$, the above equation is now changed into the following form:

$$\omega^{-1}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1\beta_2}[(\omega\omega_1^{32})^{-1}\omega_2^{13}\delta s_1^{1-\beta_1\beta_2}+\omega_1^{11}M_1^3\mu]+\text{h.o.t.}=0,$$

which has exactly two nonnegative solutions

$$s_1=0,\quad s_1=\Big[-\frac{\omega_1^{11}M_1^3\mu}{(\omega\omega_1^{32})^{-1}\omega_2^{13}\delta}\Big]^{\frac{1}{1-\beta_1\beta_2}}+\text{h.o.t.}$$

If $\omega\omega_1^{11}\omega_1^{32}\omega_2^{13}M_1^3\mu<0$, then combining with $s_2=\omega_1^{11}s_1^{\beta_1\beta_2}+\text{h.o.t.}$, we know that the system (1.1) has exactly one periodic orbit besides the persistent heterodimensional cycle as $\mu\in L_{12}(y_1^1)$.

Theorem 3.4 *Suppose that hypotheses* (H_1)–(H_5) *are valid,* $\mathrm{Rank}(M_1^1,M_2^1,M_2^3)=3$, $\rho_2^1+\rho_2^2>2\lambda_2^1$, $\frac{\lambda_1^2}{\lambda_1^1}+1<\beta_1$, $\omega_2^{13}\neq 0$, $\omega_2^{33}=0$, $\omega_1^{11}>0$ *and* $|M_1^1\mu|^{1-\beta_1\beta_2}\ll|M_1^3\mu|^{\beta_1}$, *then for* $\mu\in L_1^2$ *and* $0<|\mu|\ll 1$, *the following results hold:*

(1) *If* $\omega\omega_1^{11}\omega_1^{32}\omega_2^{31}M_1^3\mu>0$, $\omega_1^{11}\omega_1^{32}M_1^3\mu(\omega_1^{32}\alpha^{\beta_2}M_1^3\mu+M_2^3\mu)>0$, *where* $\alpha=\omega_1^{11}\delta^{-1}M_1^1\mu$, *then the system* (1.1) *has no periodic orbits coexisting with the homoclinic loop* Γ_1^2 *for* $\mu\in L_1^2$.

(2) *If* $\omega\omega_1^{11}\omega_1^{32}\omega_2^{31}M_1^3\mu>0$ *or* $(\omega\omega_1^{11}\omega_1^{32}\omega_2^{31}M_1^3\mu<0)$, $\omega_1^{11}\omega_1^{32}M_1^3\mu(M_2^3\mu+\omega_1^{32}\alpha^{\beta_2}M_1^3\mu)<0$, *where* $\alpha=\omega_1^{11}\delta^{-1}M_1^1\mu$, $|M_2^3\mu|\ll|M_1^3\mu||M_1^1\mu|^{\beta_2}$, *then the system* (1.1) *has exactly one periodic orbit coexisting with the homoclinic loop* Γ_1^2 *near* Γ *for* $\mu\in L_1^2$.

(3) *If* $\omega\omega_1^{11}\omega_1^{32}\omega_2^{31}M_1^3\mu<0$, $\omega_1^{11}\omega_1^{32}M_1^3\mu(\omega_1^{32}\alpha^{\beta_2}M_1^3\mu+M_2^3\mu)>0$, *take*

$$\begin{aligned}\Delta=&-\omega^{-1}\omega_2^{13}\delta(\beta_1^{-1}-1)\Big(-\frac{\omega^{-1}\omega_2^{13}\delta}{\beta_1\beta_2\omega_1^{11}\omega_1^{32}\alpha^{\beta_2-1}M_1^3\mu}\Big)^{\frac{1}{\beta_1-1}}\\&+\omega_1^{32}\alpha^{\beta_2}M_1^3\mu+M_2^3\mu+\text{h.o.t.},\end{aligned}$$

then we have the following results:

(a) *When* $\omega\omega_2^{13}\Delta<0$, *the system* (1.1) *has no periodic orbits for* $\mu\in L_1^2$.

(b) *When* $\Delta=0$, *the system* (1.1) *has a double periodic orbit for* $\mu\in L_1^2$.

(c) *When* $\omega\omega_2^{13}\Delta>0$, *the system* (1.1) *has exactly two periodic orbits for* $\mu\in L_1^2$.

Proof Under the hypotheses, the third equation of (2.16) shows that

$$s_2y_1^1=-\omega^{-1}\omega_2^{13}s_1\delta-M_2^3\mu+\text{h.o.t.}\tag{3.4}$$

Substituting it into the second equation of (2.16), we have

$$\begin{aligned}H(s_1,s_2,\mu)\triangleq\ &s_2^{\frac{\rho_2^2}{\rho_2^1}+1}\delta-\omega^{-2}(\omega_1^{32})^{-1}\omega_2^{13}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+1}\delta-(\omega\omega_1^{32})^{-1}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_2^3\mu\\&-\omega^{-1}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}s_2M_1^3\mu+s_2M_2^1\mu+\text{h.o.t.}=0.\end{aligned}$$

On the other hand, if $0\le s_1\ll 1$, $\mu\in L_1^2$ and $0<|\mu|\ll 1$, by the first equation of (2.16), we have

$$s_2=(\omega_1^{11}s_1^{\beta_1}+\omega_1^{11}\delta^{-1}M_1^1\mu)^{\beta_2}+\text{h.o.t.}\tag{3.5}$$

Let $s_1^{\beta_1} = t,\ \omega_1^{11}\delta^{-1}M_1^1\mu = \alpha$, then we obtain the following form:

$$\begin{cases} s_2 \approx \alpha^{\beta_2} + \beta_2\omega_1^{11}\alpha^{\beta_2-1}t + \text{h.o.t.} \\ s_2^{\frac{\rho_2^2}{\rho_2^1}} \approx \alpha^{\frac{\beta_2\rho_2^2}{\rho_2^1}} + \dfrac{\beta_2\rho_2^2}{\rho_2^1}\omega_1^{11}\alpha^{\frac{\beta_2\rho_2^2}{\rho_2^1}-1}t + \text{h.o.t.} \end{cases} \tag{3.6}$$

Substitute the expressions of $s_2,\ s_2^{\frac{\rho_2^2}{\rho_2^1}}$ into $H(s_1, s_2, \mu)$. Due to $\alpha^{\frac{\beta_2\rho_2^2}{\rho_2^1}}\delta + M_2^1\mu + \text{h.o.t.} = 0$ as L_1^2, then we obtain that

$$\begin{aligned} H(s_1,\mu) = {}& \delta\frac{\beta_2\rho_2^2}{\rho_2^1}\omega_1^{11}\alpha^{\frac{\beta_2\rho_2^2}{\rho_2^1}+\beta_2-1}s_1^{\beta_1} + \frac{\beta_2^2\rho_2^2}{\rho_2^1}(\omega_1^{11})^2\alpha^{\frac{\beta_2\rho_2^2}{\rho_2^1}+\beta_2-2}s_1^{2\beta_1}\delta \\ & - \delta\omega^{-2}(\omega_1^{32})^{-1}\omega_2^{13}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+1} - (\omega\omega_1^{32})^{-1}\omega_2^{31}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_2^3\mu \\ & - \omega^{-1}\omega_2^{31}\alpha^{\beta_2}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu - \beta_2\omega^{-1}\omega_2^{31}\omega_1^{11}\alpha^{\beta_2-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1}M_1^3\mu + \text{h.o.t.} \end{aligned}$$

By $\rho_2^1 + \rho_2^2 > 2\lambda_2^1,\ \frac{\lambda_1^2}{\lambda_1^1} + 1 < \beta_1$, the above function can be simplified into

$$\begin{aligned} \widetilde{H}(s_1,\mu) &= \beta_2\omega_1^{11}\omega_1^{32}\alpha^{\beta_2-1}M_1^3\mu s_1^{\beta_1} + \omega^{-1}\omega_2^{13}\delta s_1 + \omega_1^{32}\alpha^{\beta_2}M_1^3\mu + M_2^3\mu + \text{h.o.t.} \\ &\triangleq N(s_1,\mu) - L(s_1,\mu) = 0, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} N(s_1,\mu) &= \beta_2\omega_1^{11}\omega_1^{32}\alpha^{\beta_2-1}M_1^3\mu s_1^{\beta_1} + \text{h.o.t.}, \\ L(s_1,\mu) &= -\omega^{-1}\omega_2^{13}\delta s_1 - \omega_1^{32}\alpha^{\beta_2}M_1^3\mu - M_2^3\mu + \text{h.o.t.} \end{aligned}$$

Then we have

$$\begin{aligned} \widetilde{H}(0,\mu) &= \omega_1^{32}\alpha^{\beta_2}M_1^3\mu + M_2^3\mu + \text{h.o.t.}, \\ \widetilde{H}'_{s_1}(s_1,\mu) &= \beta_1\beta_2\omega_1^{11}\omega_1^{32}\alpha^{\beta_2-1}M_1^3\mu s_1^{\beta_1-1} + \omega^{-1}\omega_2^{13}\delta + \text{h.o.t.} \end{aligned}$$

If $\omega\omega_1^{11}\omega_2^{13}\omega_1^{32}M_1^3\mu < 0$, by $|M_1^1\mu|^{1-\beta_2} \ll |M_1^1\mu|^{1-\beta_1\beta_2} \ll |M_1^3\mu|^{\beta_1} \ll |M_1^3\mu|$, we know that $\widetilde{H}'_{s_1}(s_1,\mu)$ has a unique small positive solution

$$s_1 \triangleq \overline{s} = \Big(-\frac{\omega_2^{13}\delta\alpha^{1-\beta_2}}{\beta_1\beta_2\omega\omega_1^{11}\omega_1^{32}M_1^3\mu}\Big)^{\frac{1}{\beta_1-1}} + \text{h.o.t.}$$

If $\omega\omega_1^{11}\omega_2^{13}\omega_1^{32}M_1^3\mu > 0$, then $\widetilde{H}'_{s_1}(s_1,\mu) \neq 0$.

(1) When $\omega\omega_1^{11}\omega_2^{13}\omega_1^{32}M_1^3\mu > 0,\ \omega_1^{11}\omega_1^{32}M_1^3\mu(\omega_1^{32}\alpha^{\beta_2}M_1^3\mu + M_2^3\mu) > 0$, then the straight line L, and the curve N can not intersect in the half plane for $s_1 > 0$, that is $H(s_1,\mu) = 0$ has no positive solution. Therefore, the system (1.1) has no periodic orbit as $\mu \in L_1^2$.

(2) When $\omega\omega_1^{11}\omega_2^{13}\omega_1^{32}M_1^3\mu > 0$ (or $\omega\omega_1^{11}\omega_2^{13}\omega_1^{32}M_1^3\mu < 0$), $\omega_1^{11}\omega_1^{32}M_1^3\mu(M_2^3\mu + \omega_1^{32}\alpha^{\beta_2}M_1^3\mu) < 0$, then the straight line L and the curve N intersect one positive point, that is, $H(s_1,\mu) = 0$ has one positive root. Next we will show this positive root is small enough.

Without loss of generality, we assume $\omega_1^{11}\omega_1^{32}M_1^3\mu > 0,\ \omega\omega_2^{13} > 0,\ \omega_1^{32}\alpha^{\beta_2}M_1^3\mu + M_2^3\mu < 0$, then we have

$$\widetilde{H}(0,\mu) < 0, \quad \widetilde{H}'_{s_1}(s_1,\mu) > 0, \quad \widetilde{H}(\widetilde{s}_1,\mu) = \omega^{-1}\omega_2^{13}\delta\widetilde{s}_1 > 0,$$

where
$$0<\widetilde{s}_1=\Big(-\frac{\omega_1^{32}\alpha^{\beta_2}M_1^3\mu+M_2^3\mu}{\beta_2\omega_1^{11}\omega_1^{32}\alpha^{\beta_2-1}M_1^3\mu}\Big)^{\frac{1}{\beta_1}}=\Big(-\frac{\omega_1^{32}\alpha M_1^3\mu+\alpha^{1-\beta_2}M_2^3\mu}{\beta_2\omega_1^{11}\omega_1^{32}M_1^3\mu}\Big)^{\frac{1}{\beta_1}}.$$
By $|M_2^3\mu|\ll|M_1^3\mu||M_1^1\mu|^{\beta_2}$, we have
$$|M_1^1\mu|^{1-\beta_2}|M_2^3\mu|\ll|M_1^3\mu||M_1^1\mu|^{\beta_2}|M_1^1\mu|^{1-\beta_2}=|M_1^1\mu||M_1^3\mu|\ll|M_1^3\mu|,$$
which guarantees that $\widetilde{s}_1\ll 1$. Then, we know that $H(s_1,\mu)$ has a unique small positive solution s_1 satisfying $0<s_1<\widetilde{s}_1\ll 1$. Also, by (H$_5$) and $\rho_2^1+\rho_2^2>2\lambda_2^1$, we know that the expansion of $s_2^{\frac{\rho_2^2}{\rho_2^1}}$ in (3.6) is meaningful, while by $|M_2^3\mu|\ll|M_1^3\mu||M_1^1\mu|^{\beta_2}$ and the expression of $\widetilde{s}_1$, we have $s_1=o(|M_1^1\mu|^{\frac{1}{\beta_1}})$, which guarantees that the expansion of s_2 in (3.6) is meaningful.

Therefore, the system (1.1) has one unique periodic orbit coexisting with the homoclinic loop Γ_1^2 near Γ for $\mu\in L_1^2$ and $|y_1^1|\ll 1$.

(3) When $\omega\omega_1^{11}\omega_2^{13}\omega_1^{32}M_1^3\mu<0$, $\omega_1^{11}\omega_1^{32}M_1^3\mu(\omega_1^{32}\alpha^{\beta_2}M_1^3\mu+M_2^3\mu)>0$, without loss of generality, we assume $\omega_1^{11}\omega_1^{32}M_1^3\mu>0$, $\omega\omega_2^{13}<0$, $\omega_1^{32}\alpha^{\beta_2}M_1^3\mu+M_2^3\mu>0$, then we have $\widetilde{H}(0,\mu)>0$, $\widetilde{H}''_{s_1s_1}(s_1,\mu)>0$ and $\widetilde{H}(\overline{s},\mu)=\Delta$, where
$$\begin{aligned}\Delta\triangleq&-\omega^{-1}\omega_2^{13}\delta(\beta_1^{-1}-1)\Big(-\frac{\omega^{-1}\omega_2^{13}\delta}{\beta_1\beta_2\omega_1^{11}\omega_1^{32}\alpha^{\beta_2-1}M_1^3\mu}\Big)^{\frac{1}{\beta_1-1}}\\&+\omega_1^{32}\alpha^{\beta_2}M_1^3\mu+M_2^3\mu+\text{h.o.t.}\end{aligned}$$

If $\widetilde{H}(\overline{s},\mu)=\Delta>0$, the straight line L and the curve N can not insect in the half plane; if $\widetilde{H}(\overline{s},\mu)=\Delta=0$, the straight line L is tangent to the curve N at point $s_1=\overline{s}$, that is, $s_1=\overline{s}$ is a double positive zero point of $\widetilde{H}(s,\mu)=0$; if $\widetilde{H}(\overline{s},\mu)=\Delta<0$, the straight line L intersects the curve N at exact two points $0<s'<\overline{s}<s''$, which means $\widetilde{H}(s,\mu)=0$ has two positive solutions.

With the analysis above, we know that each positive zero point s_1 of $\widetilde{H}(s,\mu)=0$ corresponds to a unique pair of positive solutions (s_1,s_2) of the bifurcation equation (2.16). Then we obtain the conclusions.

4 Example

In this section, an example of vector field is given to show the existence of the system which has a nontransversal heterodimensional cycle with one orbit flip, and demonstrate how to use the method given in this paper to discuss the bifurcation problem.

Consider the following three-dimensional system
$$\dot{z}=f(z)+g(z,\mu),\tag{4.1}$$
and its unperturbed system
$$\dot{z}=f(z),\tag{4.2}$$
where $z=(z_1,z_2,z_3)^*\in\mathbb{R}^3$, $\mu=(\mu_1,\mu_2,\mu_3)^*\in\mathbb{R}^3$, $g(z,0)=0$, $0<|\mu|\ll 1$, and
$$f(z)=\begin{pmatrix}-(z_1-1)(z_1+1)+3(z_1^2+z_2^2-1)\\-z_1z_2\\\frac{1}{3}(7-8z_1)z_3\end{pmatrix},$$

$$g(z,\mu)=\begin{pmatrix}\mu_1(z_1^2-1)\\ \mu_2(z_1^2+z_2^2-1)\\ \mu_3(z_1-1)(z_1+1)^2\end{pmatrix}.$$

When $\mu=0$, the system (4.2) has equilibria

$$p_1=(-1,0,0),\quad p_2=(1,0,0),$$

and a heteroclinic cycle $\Gamma=\Gamma_1\cup\Gamma_2$ connecting p_1 and p_2, where

$$\Gamma_1\subset W_1^u\cap W_2^s:\{z=r_1(t)\mid z_1^2+z_2^2=1,\ z_3=0,\ z_2\geqslant 0,\ t\in\mathbb{R}\}$$

and

$$\Gamma_2\subset W_1^s\cap W_2^u:\{z=r_2(t)\mid z_2=z_3=0,\ z_1\in(-1,1),\ t\in\mathbb{R}\},$$

which is expressed by $\Gamma_i=\{z=r_i(t),\ t\in R\},\ i=1,2$. Here

$$r_1(t)=(z_{11},z_{12},z_{13})(t)=\Big(\frac{1-\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}},\ 2(2+\mathrm{e}^{2t}+\mathrm{e}^{-2t})^{-\frac{1}{2}},\ 0\Big),$$
$$r_2(t)=(z_{21},z_{22},z_{23})(t)=\Big(\frac{1-\mathrm{e}^{4t}}{1+\mathrm{e}^{4t}},\ 0,\ 0\Big),$$

which satisfies $r_1(-\infty)=r_2(+\infty)=P_1$, $r_1(+\infty)=r_2(-\infty)=P_2$ (see Figure 3).

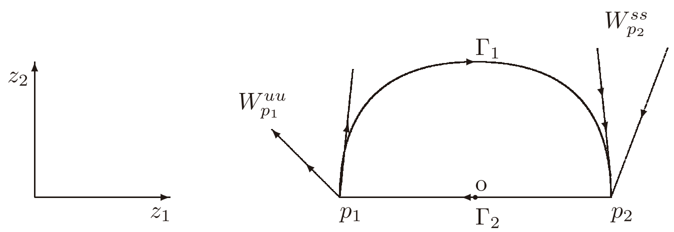

Figure 3 Γ_1 with orbit flip in positive direction.

Since

$$Df(z)=\begin{pmatrix}4z_1 & 6z_2 & 0\\ -z_2 & -z_1 & 0\\ -\frac{8}{3}z_3 & 0 & \frac{1}{3}(7-8z_1)\end{pmatrix},$$

then we have

$$Df(p_1)=\mathrm{diag}(-4,1,5),\quad Df(p_2)=\mathrm{diag}\Big(4,-1,-\frac{1}{3}\Big),$$

which means $\Gamma=\Gamma_1\cup\Gamma_2$ is a heterodimensional cycle and Γ_1 has orbit flip in positive direction; in other words, heteroclinic orbit Γ_1 enters the equilibrium p_2 along the strong stable direction z_2 as $t\to+\infty$. Notice that $T_{r_1(t)}W^s_{p_2}\to\mathrm{span}\{(0,1,0)^*,\ (0,0,1)^*\}$, as $t\to-\infty$, where $(0,1,0,)^*$, $(0,0,1)^*$ are the unit eigenvectors of p_1 corresponding to the positive eigenvalue 1, 5, respectively. Then the 2-dimensional unstable manifolds of p_1 coincide with the 2-dimensional stable manifolds of p_2, that is, Γ_1 is a nontransversal orbit.

Let $0<\delta\ll 1$ and T_i $(i=1,2)$ be large enough such that

$$r_1(-T_1)=(-\sqrt{1-\delta^2},\delta,0)^*,\quad r_1(T_1)=(\sqrt{1-\delta^2},\delta,0)^*,$$
$$r_2(-T_2)=(1-\delta,0,0)^*,\qquad r_2(T_2)=(-1+\delta,0,0)^*,$$

then we have

$$T_1 = \ln \frac{\delta}{1-\sqrt{1-\delta^2}} = \ln \frac{2}{\delta(1+O(\delta^2))}, \quad T_2 = \frac{1}{4}(\ln(2-\delta) - \ln \delta).$$

Now we consider the linear variational system of unperturbed system (4.2) along Γ_i $(i = 1, 2)$:

$$\dot{z} = Df(r_i(t))z, \tag{4.3}$$

and its adjoint system

$$\dot{z} = -(Df(r_i(t)))^* z, \tag{4.4}$$

where

$$Df(r_1(t)) = \begin{pmatrix} 4z_{11}(t) & 6z_{12}(t) & 0 \\ -z_{12}(t) & -z_{11}(t) & 0 \\ 0 & 0 & \frac{1}{3}(7-8z_{11}(t)) \end{pmatrix},$$

$$Df(r_2(t)) = \begin{pmatrix} 4z_{21}(t) & 0 & 0 \\ 0 & -z_{21}(t) & 0 \\ 0 & 0 & \frac{1}{3}(7-8z_{21}(t)) \end{pmatrix}.$$

Next we discuss the persistent of the heterodimensional cycle of (4.2), by a similar computation given in Section 2, we know that the persistent of the heterodimensional cycle is related with elements in $Z_i(T_i)$, $Z_i(-T_i)$ $(i = 1, 2)$ as well as M_1^1, M_2^1, M_2^3. Firstly, we consider the fundamental solution matrix $Z_1(t)$ and $\Phi_1(t)$.

One fundamental solution matrix for (4.3) is

$$Z_1(t) = \begin{pmatrix} u_{11}(t) & u_{21}(t) & 0 \\ u_{12}(t) & u_{22}(t) & 0 \\ 0 & 0 & u_{33} \end{pmatrix},$$

take $\Phi_i(t) = (Z_i^{-1}(t))^* = (\varphi_i^1, \varphi_i^2, \varphi_i^3)$. By Liouville formula, we have

$$D = \det \begin{vmatrix} u_{11}(t) & u_{21}(t) \\ u_{12}(t) & u_{22}(t) \end{vmatrix} = \det \begin{vmatrix} u_{11}(-T_1) & u_{21}(-T_1) \\ u_{12}(-T_1) & u_{22}(-T_1) \end{vmatrix} \cdot \mathrm{e}^{\int_{-T_1}^{t} \frac{3(1-\mathrm{e}^{-2s})}{1+\mathrm{e}^{-2s}} \mathrm{d}s}.$$

By $\Phi_i^*(t)Z_i(t) = \mathrm{Id}$, we have

$$\varphi_1^{1*}(t) = (u_{22}(t), -u_{21}(t), 0)/D,$$

where $D = u_{11}(t)u_{22}(t) - u_{12}(t)u_{21}(t) = \big[\frac{\delta(\mathrm{e}^t+\mathrm{e}^{-t})}{2}\big]^3$. Notice that

$$g_\mu(r_1(t), 0) = \begin{pmatrix} z_{11}^2(t) - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (z_{11}(t)-1)(z_{11}(t)+1)^2 \end{pmatrix},$$

then with the expression of $u_{22}(t) = \dot{z}_{12}(t)$, we have

$$M_1^1 = \int_{-T_1}^{T_1} \varphi_1^{1*} g_\mu(r_1(t), 0)\, \mathrm{d}t = \Big(\frac{64}{\delta^3} \int_0^{+\infty} \frac{x^7(1-x^2)}{(x^2+1)^7}\, \mathrm{d}x, 0, 0\Big).$$

Next we consider $Z_2(t)$ and $\Phi_2(t)$. By $Df(r_2(t))$, we obtain one fundamental solution matrix for (4.3) as follows:

$$Z_2(t) = \operatorname{diag}(C_1 \mathrm{e}^{4t}(1+\mathrm{e}^{4t})^{-2},\ C_2 \mathrm{e}^{-t}(1+\mathrm{e}^{4t})^{\frac{1}{2}},\ C_3 \mathrm{e}^{-\frac{t}{3}}(1+\mathrm{e}^{4t})^{\frac{4}{3}}).$$

Thus, we obtain

$$Z_2(t) = \begin{pmatrix} 0 & C_1 \mathrm{e}^{4t}(1+\mathrm{e}^{4t})^{-2} & 0 \\ 0 & 0 & C_3 \mathrm{e}^{-\frac{t}{3}}(1+\mathrm{e}^{4t})^{\frac{4}{3}} \\ C_2 \mathrm{e}^{-t}(1+\mathrm{e}^{4t})^{\frac{1}{2}} & 0 & 0 \end{pmatrix}$$

for $t \leqslant -T_2$, and

$$Z_2(t) = \begin{pmatrix} C_2 \mathrm{e}^{-t}(1+\mathrm{e}^{4t})^{\frac{1}{2}} & 0 & 0 \\ 0 & C_1 \mathrm{e}^{4t}(1+\mathrm{e}^{4t})^{-2} & 0 \\ 0 & 0 & C_3 \mathrm{e}^{-\frac{t}{3}}(1+\mathrm{e}^{4t})^{\frac{4}{3}} \end{pmatrix}$$

for $t \geqslant T_2$. By the initial values $Z_2(-T_2)$ given in Section 2, we have

$$\begin{aligned} C_1 &= \Big(\frac{\delta}{2-\delta}\Big)^{-4}\Big[1+\Big(\frac{\delta}{2-\delta}\Big)^4\Big]^2, \\ C_2 &= \Big(\frac{\delta}{2-\delta}\Big)\Big[1+\Big(\frac{\delta}{2-\delta}\Big)^4\Big]^{-\frac{1}{2}}, \\ C_3 &= \Big(\frac{\delta}{2-\delta}\Big)^{\frac{1}{3}}\Big[1+\Big(\frac{\delta}{2-\delta}\Big)^4\Big]^{-\frac{4}{3}}. \end{aligned}$$

Correspondingly, by performing the coordinates transformation in the small neighborhood of P_i, we have

$$\Phi_2(t) = \begin{pmatrix} 0 & C_1^{-1} \mathrm{e}^{-4t}(1+\mathrm{e}^{4t})^2 & 0 \\ C_2^{-1} \mathrm{e}^{t}(1+\mathrm{e}^{4t})^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & C_3^{-1} \mathrm{e}^{\frac{t}{3}}(1+\mathrm{e}^{4t})^{-\frac{4}{3}} \end{pmatrix}$$

for $t \in \mathbb{R}$. Note that

$$g_\mu(r_2(t), 0) = \begin{pmatrix} z_{21}^2(t)-1 & 0 & 0 \\ 0 & z_{21}^2(t)-1 & 0 \\ 0 & 0 & (z_{21}(t)-1)(z_{21}(t)+1)^2 \end{pmatrix}.$$

Hence, we can calculate

$$\begin{aligned} M_2^1 &= \int_{-T_2}^{T_2} \varphi_2^{1*} g_\mu(r_2(t),0)\,\mathrm{d}t = \Big(0, -\frac{1}{C_2}\int_0^{+\infty} \frac{x^{\frac{1}{4}}}{(1+x)^{\frac{5}{2}}}\,\mathrm{d}x, 0\Big), \\ M_2^3 &= \int_{-T_2}^{T_2} \varphi_2^{3*} g_\mu(r_2(t),0)\,\mathrm{d}t = \Big(0, 0, -\frac{1}{C_3}\int_0^{+\infty} \frac{x^{\frac{1}{12}}}{(1+x)^{\frac{13}{3}}}\,\mathrm{d}x\Big). \end{aligned}$$

With M_1^1, M_2^1, M_2^3 being specifically given above, then by Theorem 3.1, the system (4.1) has a unique heterodimensional loop $\Gamma^\mu = \Gamma_1^\mu \cup \Gamma_2^\mu$ as $\mu \in L_{12}$ and $0 < |\mu| \ll 1$. To illustrate other results concerning homoclinic bifurcation, periodic bifurcation, we need more information, which will cause much more complicated computation. However, the idea and procedure are more or less the same as this one.

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