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Approximate Solution of the Kuramoto-Shivashinsky Equation on an Unbounded Domain*

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Abstract The main goal of this paper is to approximate the Kuramoto-Shivashinsky (K-S for short) equation on an unbounded domain near a change of bifurcation, where a band of dominant pattern is changing stability. This leads to a slow modulation of the dominant pattern. Here we consider PDEs with quadratic nonlinearities and derive rigorously the modulation equation, which is called the Ginzburg-Landau (G-L for short) equation, for the amplitudes of the dominating modes.

Keywords Multi-scale analysis, Modulation equation, Kuramoto-Shivashinsky equation, Ginzburg-Landau equation
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1 Introduction

In this paper, we are interested in the long time behaviour of nonlinear parabolic partial differential equations (PDEs for short) defined on unbounded domain near its change of stability. First aim here is to derive rigorously the modulation equation of

$$\partial_t \psi(t,x) = \mathcal{L}\psi(t,x) + \varepsilon^2 \nu \psi(t,x) + \gamma \psi(t,x)\partial_x \psi(t,x) \tag{1.1}$$

in the following type

$$\partial_T A = 4\partial_X^2 A + \nu A - \frac{\gamma^2}{9}|A|^2 A, \tag{1.2}$$

where $\mathcal{L}$ in (1.1) is the linear differential operator $-(1+\partial_x^2)^2$ which has eigenvalues $-\lambda_k = -(1-k^2)^2$ for $k \in \mathbb{R}$ and corresponding to eigenfunctions $\mathrm{e}^{\mathrm{i}kx}$.

Second aim of this paper is to approximate the solution ψ of (1.1) by

$$\psi(t,x) \simeq \varepsilon A(\varepsilon^2 t, \varepsilon x)\mathrm{e}^{\mathrm{i}x} + \varepsilon \overline{A}(\varepsilon^2 t, \varepsilon x)\mathrm{e}^{-\mathrm{i}x}, \tag{1.3}$$

where $A(T,X)$ is the solution of the G-L equation (1.2) with $T = \varepsilon^2 t$ and $X = \varepsilon x$.

The Kuramoto-Sivashinsky (K-S for short) equation (1.1) was proposed in the year 1977 by Kuramoto [11] as a model for phase turbulence in reaction diffusion systems. The equation

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was also developed by Sivashinsky [22] in higher space dimensions in modelling small thermal diffusive instabilities in Laminar flame fronts.

The K-S equation arises in many physical problems including flame front instabilities and reaction-diffusion combustion dynamics (see [22–23]), propagation of concentration waves in chemical physics applications (see [11–13]), and falling film flows (see [7, 10, 24]).

Different approaches have been proposed in literature to investigate and seek the solutions of the K-S equation. Some of these methods are variational iteration methods (see [19]), the lattice Boltzmann technique (see [14]), the tanh function method (see [5]), the method of radial basis functions (see [9]), the local discontinuous Galerkin method (see [25]), Perturbation methods (see [20, 26]), the Chebyshev spectral collocation scheme (see [3]). Recently, there are other various methods that have also been developed to construct exact solutions of the K-S equation (see [4, 6, 27–28]).

Ma and Fuchssteiner [15] used the perturbation method and considered approximate solutions to obtain a complete integrable system. On the other hand, Schneider [20] used the perturbation method and approximated the solution of the K-S equation (1.1) on an unbounded domain via the solution of the G-L equation (1.2). But Schneider's method relies on the high regularity of the modulation equation, as he needed $A \in C_b^{1,4}([0,T] \times \mathbb{R})$.

Mielke and Schneider [16] compared the long-time behaviour of all solutions of equation (1.1) with the long-time behaviour of the solution A of (1.2), and they discussed the question of how the attractor can be described by the attractor of the G-L equation (1.2). For this case, similar results are well known (see for instance [8, 17, 21]).

In this paper, we use the perturbation method to approximate the solutions of the K-S equation (1.1). This method depends on the low regularity of the modulation equation. Unfortunately, some regularity for A is needed. So A must be in $C_b^0([0,T], \mathcal{H}^{\frac{1}{2}+})$.

We will prove the following approximation result for the K-S equation (1.1) via the G-L equation (1.2).

Theorem 1.1 *Let $\psi(t,x)$ be a solution of the K-S equation* (1.1), *$v_A(T,X)$ be the formal approximation defined as*

$$v_A(T,X) = A(T,X)\mathrm{e}^{\mathrm{i}X\frac{1}{\varepsilon}} + \overline{A}(T,X)\mathrm{e}^{-\mathrm{i}X\frac{1}{\varepsilon}}, \tag{1.4}$$

where $A(T,X)$ is the solution of the G-L equation (1.2) *such that $A \in C_b^0([0,T_0], \mathcal{H}^\alpha)$ that $\alpha > \frac{1}{2}$. Suppose that the initial condition $\|\psi(0) - \varepsilon A(0)\mathrm{e}^{\mathrm{i}x} - \varepsilon\overline{A}(0)\mathrm{e}^{-\mathrm{i}x}\|_\infty \le d\widetilde{\phi}_\varepsilon$ for some fixed $d > 0$ and*

$$\widetilde{\phi}_\varepsilon = \begin{cases} \varepsilon^{\alpha-\frac{1}{2}}, & \text{if } \dfrac{1}{2} < \alpha \le 2, \\ \varepsilon^{\frac{3}{2}}, & \text{if } \alpha > 2. \end{cases} \tag{1.5}$$

Then, for each $T_0 > 0$ *there exists a constant* $C > 0$*, depending on* $\sup_{T\in[0,T_0]} \|A(T)\|_\alpha$*, such that*

$$\sup_{t\in[0,\varepsilon^{-2}T_0]} \|\psi(t,x) - \varepsilon v_A(\varepsilon^2 t, \varepsilon x)\|_\infty < C\widetilde{\phi}_\varepsilon. \tag{1.6}$$

The rest of this paper is organized as follows. In next section, we give a formal derivation of the modulation equation. In Section 3, we define $\mathcal{H}^\alpha$ and give the relation between the norm in $\mathcal{H}^\alpha$ and the norm in $C_b^0(\mathbb{R})$. In Section 4, we define the Green's function $G_t(x)$ and give estimates on it. Finally, Section 5 is devoted to the proof of the approximation theorem.

2 Formal Derivation of the Modulation Equation

This section is devoted to derive the modulation equation corresponding to equation (1.1). Let us first rescale the equation (1.1). If

$$\psi(t,x) = \varepsilon u(\varepsilon^2 t, \varepsilon x),$$

then (1.1) becomes

$$\partial_T u = \mathcal{L}_\varepsilon u + \nu u + \gamma u \partial_X u \tag{2.1}$$

with differential operator $\mathcal{L}_\varepsilon = -\varepsilon^{-2}(1 + \varepsilon^2\partial_X^2)^2$ on the slow space $X = \varepsilon x$ and the slow time $T = \varepsilon^2 t$.

Now, let

$$w_A(T,X) = [A\mathrm{e}^{\mathrm{i}x} + \varepsilon[B + \varepsilon M]\mathrm{e}^{2\mathrm{i}x} + \varepsilon^2 H\mathrm{e}^{3\mathrm{i}x}] + \varepsilon^2 J + c.c., \tag{2.2}$$

where the amplitudes $A,\ B,\ M,\ H,\ J$ are functions of T and X and $c.c.$ denotes the complex conjugation of the terms preceding it within the brackets. By plugging (2.2) into (2.1) and using the relation

$$\begin{aligned}\mathcal{L}_\varepsilon(f(X)\mathrm{e}^{\mathrm{i}\frac{n}{\varepsilon}X}) = -[&\varepsilon^{-2}(1-n^2)^2 f + 4\mathrm{i}\varepsilon^{-1}n(1-n^2)f' + (2-6n^2)f'' \\ &+ 4\mathrm{i}\varepsilon n f''' + \varepsilon^2 f'''']\mathrm{e}^{\mathrm{i}\frac{n}{\varepsilon}X},\end{aligned} \tag{2.3}$$

yields

$$\begin{aligned}\partial_T A\mathrm{e}^{\mathrm{i}x} + c.c. = 4A''\mathrm{e}^{\mathrm{i}x} &- \Big[\frac{9}{\varepsilon}B - 24\mathrm{i}B' + 9M\Big]\mathrm{e}^{2\mathrm{i}x} - 64H\mathrm{e}^{3\mathrm{i}x} - J \\ &+ \nu A\mathrm{e}^{\mathrm{i}x} + [\gamma A\partial_X A + \frac{\mathrm{i}\gamma}{\varepsilon}A^2]\mathrm{e}^{2\mathrm{i}x} + 3\mathrm{i}\gamma AB\mathrm{e}^{3\mathrm{i}x} \\ &+ \mathrm{i}\gamma\overline{A}B\mathrm{e}^{\mathrm{i}x} + c.c. + \gamma\partial_X|A|^2 + \mathcal{O}(\varepsilon).\end{aligned}$$

In order to remove all unwanted terms, let

$$B = \frac{\mathrm{i}\gamma}{9}A^2, \quad M = -\frac{19\gamma}{81}A\partial_X A, \quad H = \frac{-\gamma^2}{192}A^3, \quad J = \gamma\partial_X|A|^2. \tag{2.4}$$

Hence,

$$\partial_T A\mathrm{e}^{\mathrm{i}x} + c.c. = \Big[4A'' + \nu A - \frac{\gamma^2}{9}|A|^2 A\Big]\mathrm{e}^{\mathrm{i}x} + c.c. + \mathcal{O}(\varepsilon). \tag{2.5}$$

By equating the coefficients terms of $\mathrm{e}^{\mathrm{i}x}$ in (2.5), we obtain

$$\partial_T A = 4A'' + \nu A - \frac{\gamma^2}{9}|A|^2 A + \mathcal{O}(\varepsilon).$$

By neglecting all small terms in ε, we obtain (1.2).

3 The $\mathcal{H}^\alpha$-Spaces

We start this section by giving the definition of the fractional Sobolev space $\mathcal{H}^\alpha$. We rely in this definition on weighted L^2-norms of Fourier transforms.

Definition 3.1 *For $\alpha \in \mathbb{R}$. The space $\mathcal{H}^\alpha$ is defined as*

$$\mathcal{H}^\alpha = \Big\{u : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^{\infty} (1+y^2)^\alpha |\mathcal{F}(u)(y)|^2 \mathrm{d}y < \infty\Big\}$$

with norm

$$\|u\|_\alpha^2 = \int_{-\infty}^{\infty} (1+y^2)^\alpha |\mathcal{F}(u)(y)|^2 \mathrm{d}y,$$

where $\mathcal{F}(u)$ is the Fourier transform of u, defined by

$$\mathcal{F}(u)(k) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(y)\mathrm{e}^{-\mathrm{i}ky} \mathrm{d}y.$$

Note that functions still decay to 0 at ∞ in the space $\mathcal{H}^\alpha$. So if $A \in \mathcal{H}^\alpha$, then the solutions of (1.1) and the solutions of (1.2) still decay to 0 as $|x|$ tends to ∞.

Lemma 3.1 *Let $P(k) \le 0$ be the eigenvalues of the non-positive operator $\mathcal{A}$, where $\mathcal{F}(\mathcal{A}u) = P(\cdot)\mathcal{F}(u)$. Then for $t \ge 0$ and $u \in \mathcal{H}^\alpha$,*

$$\|\mathrm{e}^{t\mathcal{A}} u\|_\alpha \le \|u\|_\alpha. \tag{3.1}$$

It is well known that $\mathrm{e}^{t\mathcal{A}}$, defined by $\mathcal{F}(\mathrm{e}^{t\mathcal{A}}u) = \mathrm{e}^{tP}\mathcal{F}(u)$, generates a contraction semigroup.

Proof From Definition 3.1, we note that (as $\mathrm{e}^{2tP(k)} \le 1$)

$$\|\mathrm{e}^{t\mathcal{A}} u\|_\alpha^2 = \int_{-\infty}^{\infty} (1+y^2)^\alpha \big|\mathrm{e}^{tP(y)}\mathcal{F}(u)(y)\big|^2 \mathrm{d}y \le \|u\|_\alpha^2.$$

The following lemma gives the relation between the norm in $\mathcal{H}^\alpha$ and the norm in $C_b^0(\mathbb{R})$. For the proof see [2] or Theorem 5.4 in [1].

Lemma 3.2 *For $\alpha > \frac{1}{2}$, there is a constant $C > 0$, such that*

$$\|u\|_\infty \le C\|u\|_\alpha \quad \textit{for all } u \in \mathcal{H}^\alpha. \tag{3.2}$$

To estimate the nonlinearity, we need the following lemma which states that the space $\mathcal{H}^\alpha$ is up to the constant a Banach algebra for $\alpha > \frac{1}{2}$. For the proof, see Theorem 4 in [18].

Lemma 3.3 *For $\alpha > \frac{1}{2}$ and $m \in \mathbb{N}$, there exists a positive constant C, such that*

$$\|u^m\|_\alpha \le C\|u\|_\alpha^m \quad \text{for } u \in \mathcal{H}^\alpha. \tag{3.3}$$

In the proof we will need to following generalization of Gronwall's lemma.

Lemma 3.4 *Let $a, b \ge 0$, $0 \le \rho, \eta < 1$ and $0 < T < \infty$. Then, there exists a constant $M = M(a, b, \eta, T)$, such that for all integrable functions $u : [0, T] \to \mathbb{R}$ satisfying*

$$0 \le u(t) \le at^{-\rho} + b\int_0^t (t-s)^{-\eta} u(s)\mathrm{d}s \quad \text{for } 0 \le t \le T,$$

it follows that

$$u(t) \le aMt^{-\rho} \quad \text{for } \ 0 \le t \le T.$$

4 Semigroup and Green's Function Estimation

In this section, we give the definition of the Green's functions $G_t(x)$ for the operator $\mathcal{L}$. Then, we give estimates on it, where we follow the ideas of Collet and Eckmann [8], but for a different operator. Now, we define the Green's functions $G_t(x)$ as follows.

Definition 4.1 *Define $G_t(x)$ as*

$$G_t(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-t(1-2k^2+k^4)}\mathrm{d}k \tag{4.1}$$

for $t > 0$.

It is easy to verify that for the semigroup we have

$$\mathrm{e}^{t\mathcal{A}}u = G_t \star u = \int_{\mathbb{R}} G_t(z - \cdot)u(z)\mathrm{d}z.$$

The next lemma states that $G_t(x)$ is bounded regards to $\|\cdot\|_{L^1}$.

Lemma 4.1 *Let $t > 0$. Then there exists a positive constant C such that*

$$\|\partial_x^r G_t(x)\|_{L^1} \le C\min\{1, t^{-\frac{1}{4}r}\} \quad \text{for } r = 0, 1. \tag{4.2}$$

Before we complete the proof of this lemma, we state and prove the next two lemmas.

Lemma 4.2 *Define $g_\tau(y)$ as*

$$g_\tau(y) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}my}\mathrm{e}^{-Q_1(m,\tau)}\mathrm{d}m,$$

where $Q_1(m,\tau) = \tau^{-2} - 2m^2 + \tau^2 m^4$, then for $0 < \tau \le 1$, there exists a constant $C > 0$, such that

$$\|(1+y^2)\partial_x^r g_\tau(y)\|_\infty \le C\tau^{-r} \quad \text{for } r = 0, 1.$$

Proof By using integration by parts, we obtain

$$\begin{aligned}(1+y^2)g_\tau(y) &= (\mathrm{i})^r\int_{-\infty}^{\infty}P_1(m,\tau)\mathrm{e}^{\mathrm{i}my}\mathrm{e}^{-Q_1(m,\tau)}\mathrm{d}m\\ &= \int_0^{\infty}P_1(m,\tau)\mathrm{e}^{\mathrm{i}my}\mathrm{e}^{-Q_1(m,\tau)}\mathrm{d}m+\int_{-\infty}^{0}P_1(m,\tau)\mathrm{e}^{\mathrm{i}my}\mathrm{e}^{-Q_1(m,\tau)}\mathrm{d}m\\ &:= \mathrm{I}_1+\mathrm{I}_2,\end{aligned}$$

where

$$P_1(m,\tau)=m^r[1+Q_1''-Q_1'^2]+rm^{r-2}[2mQ_1'-(r-1)].$$

We note, for $m\geq 0$ and $0<\tau\leq 1$, that

$$Q_1(m,\tau)=(m\tau-1)^2\underbrace{(m+\tau^{-1})^2}_{\geq\tau^{-2}}\geq(m-\tau^{-1})^2$$

and

$$\begin{aligned}P_1(m,\tau) &= m^r[-3+12\tau^2m^2-16m^2\tau^2(m-\tau^{-1})^2(1+\tau m)^2]\\ &\quad -8rm^r(m-\tau^{-1})(1+\tau m)-(r-1)m^{r-2}.\end{aligned}$$

Thus

$$|P_1(m+\tau^{-1},\tau)|\leq C\tau^{-r}(1+m^{r+6})\quad\text{for } r=0,1.$$

Now we bound I_1 and I_2 separately. For the first integral I_1, we obtain

$$\begin{aligned}\mathrm{I}_1 &= \int_0^{\infty}P_1(m,\tau)\mathrm{e}^{\mathrm{i}my}\mathrm{e}^{-Q_1(m,\tau)}\mathrm{d}m\\ &= \int_{-\tau^{-1}}^{\infty}P_1(m+\tau^{-1},\tau)\mathrm{e}^{\mathrm{i}(m+\tau^{-1})y}\mathrm{e}^{-Q_1(m+\tau^{-1},\tau)}\mathrm{d}r\\ &\leq \int_{-\tau^{-1}}^{\infty}P_1(m+\tau^{-1},\tau)\mathrm{e}^{\mathrm{i}(m+\tau^{-1})y}\mathrm{e}^{-m^2}\mathrm{d}r,\end{aligned}$$

where we used the substitution $m=m-\tau^{-1}$, thus

$$\begin{aligned}|\mathrm{I}_1| &\leq \int_{-\tau^{-1}}^{\infty}|P_1(m+\tau^{-1},\tau)|\mathrm{e}^{-m^2}\mathrm{d}m\leq\tau^{-r}\int_{-\tau^{-1}}^{\infty}(c+cm^{r+6})\mathrm{e}^{-m^2}\mathrm{d}m\\ &\leq\tau^{-r}\int_{-\infty}^{\infty}(c+cm^{r+6})\mathrm{e}^{-m^2}\mathrm{d}m=C\tau^{-r}\quad\text{for } r=0,1.\end{aligned}$$

For the second integral I_2. By replacing m by $-m$, we obtain

$$\mathrm{I}_2=\int_0^{\infty}(-1)^rP_1(m,\tau)\mathrm{e}^{-\mathrm{i}my}\mathrm{e}^{-Q_1(m,\tau)}\mathrm{d}m\quad\text{for } r=0,1.$$

Analogously for the first integral, we derive

$$|\mathrm{I}_2|\leq C\tau^{-r}\quad\text{for } r=0,1.$$

Hence for $0 < \tau \leq 1$, we obtain

$$\|(1+y^2)\partial_x^r g_\tau(y)\|_\infty = \sup_y |(4+y^2)\partial_x^r g_\tau(y)| \leq C\tau^{-r} \quad \text{for } r = 0, 1.$$

Lemma 4.3 *Define $h_\eta(y)$ as*

$$h_\eta(y) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{-Q_2(k,\eta)}\mathrm{d}k,$$

where $Q_2(k,\eta) = \eta^4 - 2\eta^2k^2 + k^4$, then for $0 < \eta < 1$, there exists a constant $C > 0$, such that

$$\|(1+y^2)\partial_y^r h_\eta(y)\|_\infty \leq C \quad \textit{for } r = 0, 1.$$

Proof By using integration by parts, we obtain

$$\begin{aligned}
(1+y^2)\partial_y^r h_\eta(y) &= (\mathrm{i})^r \int_{-\infty}^{\infty} P_2(k,\eta)\mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{-Q_2(k,\eta)}\mathrm{d}k \\
&= (\mathrm{i})^r \int_{1}^{\infty} P_2(k,\eta)\mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{-Q_2(k,\eta)}\mathrm{d}k + (\mathrm{i})^r \int_{-\infty}^{-1} P_2(k,\eta)\mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{-Q_2(k,\eta)}\mathrm{d}k \\
&\quad + (\mathrm{i})^r \int_{-1}^{1} P_2(k,\eta)\mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{-Q_2(k,\eta)}\mathrm{d}k \\
&:= \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3,
\end{aligned}$$

where

$$\begin{aligned}
P_2(k,\eta) &= k^r[1 + Q_2'' - Q_2'^2] + rk^{r-2}[2kQ_2' - (r-1)] \\
&= k^r[1 + 12k^2 - 4\eta^2 - 16k^6 + 32k^4\eta^2 - 16k^2\eta^4] \\
&\quad - 8rk^r(k^2 - \eta^2) - (r-1)k^{r-2}
\end{aligned}$$

for $r = 0, 1$. We note for $0 < \eta < 1$ and $k \geq 1$ that

$$Q_2(k,\eta) = (k-\eta)^2\underbrace{(k+\eta)^2}_{\geq 1} \geq (k-\eta)^2$$

and

$$|P_2(k,\tau)| \leq c(1 + k^{r+6}) \quad \text{for } r = 0, 1.$$

Now, We will bound the terms II_1, II_2 and II_3 separately. To bound II_1 and II_2, we follow the same steps as in the case of Lemma 4.2. To bound the third term:

$$\begin{aligned}
|\mathrm{II}_3| &\leq \int_{-1}^{1} |P_2(k,\eta)||\mathrm{e}^{-Q_2(k,\eta)}|\mathrm{d}k \leq \int_{-1}^{1} |P_2(k,\eta)|\mathrm{d}k \\
&\leq c\int_{-1}^{1} (1 + k^{r+6})\mathrm{d}k = C \quad \text{for } r = 0, 1.
\end{aligned}$$

Hence for $0 < \eta < 1$, we obtain, for $r = 0, 1$,

$$\|(1+y^2)\partial_y^r h_\eta(y)\|_\infty = \sup_y |(1+y^2)\partial_y^r h_\eta(y)| \leq C.$$

Proof of Lemma 4.1 We consider two cases in order to prove (4.2).

Case 1 $t \geq 1$: We note for $\tau = t^{-\frac{1}{2}}$ that

$$G_t(x) = \tau g_\tau(\tau x)$$

and

$$\partial_x^r G_t(x) = \tau^{r+1} \partial_x^r g_\tau(\tau x).$$

Hence

$$\begin{aligned}
\|\partial_x^r G_t(x)\|_{L^1} &= \int_{-\infty}^{\infty} |\partial_x^r G_t(x)| \mathrm{d}x = \int_{-\infty}^{\infty} |\tau \partial_x^r g_\tau(\tau x)| \mathrm{d}x \\
&= \tau^r \int_{-\infty}^{\infty} |\partial_y^r g_\tau(y)| \mathrm{d}y = \tau^r \int_{-\infty}^{\infty} \frac{1}{1+y^2} |(1+y^2) \partial_x^r g_\tau(y)| \mathrm{d}y \\
&\leq \tau^r \|(1+y^2) g_\tau(y)\|_\infty \int_{-\infty}^{\infty} \frac{1}{1+y^2} \mathrm{d}y \\
&\leq C\tau^r \|(1+y^2) g_\tau(y)\|_\infty \quad \text{for } r = 0, 1,
\end{aligned}$$

where $y = \tau x$. By using Lemma 4.2, we obtain for $t \geq 1$,

$$\|\partial_x^r G_t(x)\|_{L^1} \leq C \quad \text{for } r = 0, 1. \tag{4.3}$$

Case 2 $t \in (0, 1)$: We note for $\eta = t^{\frac{1}{4}}$ that

$$G_t(x) = \eta^{-1} h_\eta(\eta^{-1} x)$$

and

$$\partial_x^r G_t(x) = \eta^{-1} \partial_x^r h_\eta(\eta^{-1} x) \quad \text{for } r = 0, 1.$$

Hence,

$$\begin{aligned}
\|\partial_x^r G_t(x)\|_{L^1} &= \int_{-\infty}^{\infty} |\eta^{-1} \partial_x^r h_\eta(\eta^{-1} x)| \mathrm{d}x \\
&= \eta^{-r} \int_{-\infty}^{\infty} \frac{1}{1+y^2} |(1+y^2) h_\eta(y)| \mathrm{d}y \\
&\leq \eta^{-r} \|(1+y^2) h_\eta(y)\|_\infty \int_{-\infty}^{\infty} \frac{1}{1+y^2} \mathrm{d}y \\
&\leq C\eta^{-r} \|(1+y^2) h_\eta(y)\|_\infty,
\end{aligned}$$

where $y = \eta x$. By using Lemma 4.3, we have for $t \in (0, 1)$,

$$\|\partial_x^r G_t(x)\|_{L^1} \leq C t^{-\frac{1}{4} r} \quad \text{for } r = 0, 1. \tag{4.4}$$

Combining (4.3) and (4.4), yields (4.2) for $t > 0$.

Lemma 4.4 *For $t \geq 0$, there is a constant $C > 0$, such that*

$$\|e^{t\mathcal{L}}\partial_x^r u\|_\infty \leq C(1 + t^{-\frac{1}{4}r})\|u\|_\infty \quad \text{for } r = 0, 1 \text{ and } u \in C_b^0(\mathbb{R}). \tag{4.5}$$

Proof Let $\mathcal{F}_t^{-1}$ denote the inverse Fourier transform. Then,

$$\begin{aligned} e^{t\mathcal{L}}\partial_x^r u(x) &= \mathcal{F}_t^{-1}\mathcal{F}_t(e^{t\mathcal{L}}\partial_x^r u(x)) = \mathcal{F}_t^{-1}(e^{-t\lambda_k}\partial_x^r \mathcal{F}_t(u(x))) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mathrm{i}k)^r e^{\mathrm{i}k(x-y)}e^{-t\lambda_k}u(y)\mathrm{d}y\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}u(y)\partial_x^r G_t(x-y)\mathrm{d}y. \end{aligned} \tag{4.6}$$

Thus

$$\|e^{t\mathcal{L}}\partial_x^r u(x)\|_\infty \leq C\|u(x)\|_\infty\|\partial_x^r G_t(x-y)\|_{L^1}.$$

Using Lemma 4.1, yields (4.5).

Corollary 4.1 *For $T \geq 0$, there is a positive constant C, such that*

$$\|e^{T\mathcal{L}_\varepsilon}\partial_X^r u\|_\infty \leq C\min\{\varepsilon^{-r}, \varepsilon^{-\frac{1}{2}r}T^{-\frac{1}{4}r}\}\|u\|_\infty \quad \text{for } r = 0, 1 \text{ and } u \in C_b^0(\mathbb{R}).$$

Proof

$$\begin{aligned} e^{T\mathcal{L}_\varepsilon}\partial_X^r u(X) &= e^{-\varepsilon^{-2}T(1+(\varepsilon\partial_X)^2)^2}\partial_X^r u(X) \\ &= \varepsilon^{-r}e^{-\varepsilon^{-2}T(1+\partial_x^2)^2}\partial_x^r u(\varepsilon x) \\ &= \varepsilon^{-r}e^{\varepsilon^{-2}T\mathcal{L}}\partial_x^r u(\varepsilon x) = \varepsilon^{-r}e^{t\mathcal{L}}\partial_x^r u(X). \end{aligned}$$

By using Lemma 4.4, we get for $r = 0, 1$,

$$\begin{aligned} \|e^{T\mathcal{L}_\varepsilon}\partial_x^r u\|_\infty &= \varepsilon^{-r}\|e^{t\mathcal{L}}\partial_x^r u_\varepsilon\|_\infty \leq C\varepsilon^{-r}\min\{1, t^{-\frac{1}{4}r}\}\|u\|_\infty \\ &= C\min\{\varepsilon^{-r}, \varepsilon^{-\frac{1}{2}r}T^{-\frac{1}{4}r}\}\|u\|_\infty. \end{aligned}$$

The next lemma allows us to replace the semigroup $e^{T\mathcal{L}_\varepsilon}$ by the semigroup $e^{4T\partial_X^2}$, when they are applied to $Ae^{\mathrm{i}\varepsilon^{-1}X}$.

Lemma 4.5 *For $T > 0$ and $A \in \mathcal{H}^\alpha$ with $\alpha > \frac{1}{2}$, there is a positive constant C, such that*

$$\sup_{X\in\mathbb{R}}|e^{T\mathcal{L}_\varepsilon}A(X)e^{\mathrm{i}\varepsilon^{-1}X} - (e^{4T\partial_X^2}A)(X)e^{\mathrm{i}\varepsilon^{-1}X}| \leq C\|A\|_\alpha\phi_\varepsilon,$$

where ϕ_ε is defined as

$$\phi_\varepsilon = \begin{cases} \varepsilon^2, & \text{if } \alpha > \dfrac{5}{2}, \\ \varepsilon^{\alpha-\frac{1}{2}}, & \text{if } \dfrac{1}{2} < \alpha \leq \dfrac{5}{2}. \end{cases} \tag{4.7}$$

Proof We can write $e^{t\mathcal{L}}A(\varepsilon x)e^{ix}$ as a convolution with the Green's function of $\mathcal{L}$, as in (4.6),

$$\begin{aligned}
e^{t\mathcal{L}}A(\varepsilon x)e^{ix} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(x-y)}e^{-t\lambda_k}A(\varepsilon y)e^{iy}\mathrm{d}y\mathrm{d}k \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k-1)(x-y)}e^{-t\lambda_k}A(\varepsilon y)\mathrm{d}y\mathrm{d}ke^{ix} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(\varepsilon x-y)}e^{-t\lambda\varepsilon_{k+1}}A(y)\mathrm{d}y\mathrm{d}ke^{ix},
\end{aligned}$$

where we used the substitution $k=\varepsilon^{-1}(k-1)$ and $y=\varepsilon y$. Hence,

$$\begin{aligned}
e^{t\mathcal{L}}A(\varepsilon x)e^{ix} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(\varepsilon x-y)}e^{-T(\varepsilon k^2+2k)^2}A(y)\mathrm{d}y\mathrm{d}ke^{ix} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{F}(A)(k)e^{i\varepsilon kx}e^{-T(\varepsilon k^2+2k)^2}\mathrm{d}ke^{ix} \\
&\le \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}|\mathcal{F}(A)(k)|e^{-T(\varepsilon^2k^4+4k^2)}\mathrm{d}k \\
&\quad + \frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{2\varepsilon}}^{0}|\mathcal{F}(A)(k)|e^{-T(\varepsilon^2k^4+2k^2)}\mathrm{d}k \\
&\quad + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-1}{2\varepsilon}}|\mathcal{F}(A)(k)|e^{-T(\varepsilon k^2+2k)^2}\mathrm{d}k, \qquad (4.8)
\end{aligned}$$

where we used $|e^{-4\varepsilon Tk^3}|\le 1$ for all $k\ge 0$ for the first integral and $-2k^2(2\varepsilon k+1)\le 0$ for the second integral. Analogously, we can write $(e^{4t\partial_x^2}A)(\varepsilon x)e^{ix}$ as

$$(e^{4t\partial_x^2}A)(\varepsilon x)e^{ix} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(\varepsilon x-y)}e^{-4Tk^2}A(y)\mathrm{d}y\mathrm{d}ke^{ix}.$$

Thus,

$$\begin{aligned}
(e^{4T\partial_X^2}A)(X)e^{i\varepsilon^{-1}X} &\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-4Tk^2}|\mathcal{F}(A)(k)|\mathrm{d}k \\
&\le \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-4Tk^2}|\mathcal{F}(A)(k)|\mathrm{d}k + \frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{2\varepsilon}}^{0}e^{-2Tk^2}|\mathcal{F}(A)(k)|\mathrm{d}k \\
&\quad + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-1}{2\varepsilon}}e^{-4Tk^2}|\mathcal{F}(A)(k)|\mathrm{d}k. \qquad (4.9)
\end{aligned}$$

Define I_1 as the difference between the first two integral of (4.8) and (4.9)

$$\mathrm{I}_1 = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}|\mathcal{F}(A)(k)|[e^{-T(\varepsilon^2k^4+4k^2)}-e^{-4Tk^2}]\mathrm{d}k.$$

Using Cauchy-Schwarz inequality

$$|\mathrm{I}_1|^2 \le C\|A\|_\alpha^2\int_0^{\infty}\frac{1}{(1+k^2)^\alpha}e^{-8Tk^2}[e^{-\varepsilon^2Tk^4}-1]^2\mathrm{d}k.$$

Using the following inequality

$$|e^x-1|\le |x|\max\{1,e^x\}, \qquad (4.10)$$

which follows directly from the intermediate value theorem, in order to obtain

$$|\mathrm{I}_1|^2 \leq C\varepsilon^4 \|A\|_\alpha^2 \int_0^\infty \frac{k^4}{(1+k^2)^\alpha} \mathrm{e}^{-8Tk^2}[Tk^2]^2 \mathrm{d}k.$$

Now, we can use the fact

$$\sup_{z>0} z^m \mathrm{e}^{-z} \leq \infty \quad \text{for all } m \geq 0, \tag{4.11}$$

to get

$$\begin{aligned}
|\mathrm{I}_1|^2 &\leq C\varepsilon^4 \|A\|_\alpha^2 \int_0^\infty \frac{k^4}{(1+k^2)^\alpha} \mathrm{d}k \\
&= 2\varepsilon^4 C\|A\|_\alpha^2 \Big[\int_0^1 \frac{k^4}{(1+k^2)^\alpha}\mathrm{d}k + \int_1^\infty \frac{k^4}{(1+k^2)^\alpha}\mathrm{d}k\Big] \\
&\leq 2\varepsilon^4 C\|A\|_\alpha^2 \Big[1 + \int_1^\infty k^{4-2\alpha}\mathrm{d}k\Big] \\
&\leq C\varepsilon^4 \|A\|_\alpha^2 \quad \text{for } \alpha > \frac{5}{2}.
\end{aligned} \tag{4.12}$$

Analogously, we define I_2 as the difference between the second two integral of (4.8) and (4.9) in order to obtain

$$|\mathrm{I}_2|^2 \leq C\varepsilon^4 \|A\|_\alpha^2 \quad \text{for } \alpha > \frac{5}{2}. \tag{4.13}$$

Now, define I_3 as the difference between the second two integral of (4.8) and (4.9)

$$\mathrm{I}_3 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-1}{2\varepsilon}} |\mathcal{F}(A)(k)|[\mathrm{e}^{-T(\varepsilon k^2+2k)^2} - \mathrm{e}^{-4Tk^2}]\mathrm{d}k.$$

Using Cauchy-Schwarz inequality

$$|\mathrm{I}_3|^2 \leq C\|A\|_\alpha^2 \int_{-\infty}^{\frac{-1}{2\varepsilon}} \frac{\mathrm{d}k}{(1+k^2)^\alpha},$$

where we used $|\mathrm{e}^{-T(\varepsilon k^2+2k)^2} - \mathrm{e}^{-4Tk^2}|^2 \leq 2|\mathrm{e}^{-T(\varepsilon k^2+2k)^2}|^2 + 2|\mathrm{e}^{-4Tk^2}|^2 \leq 4$. Thus

$$|\mathrm{I}_3|^2 \leq C\varepsilon^{2\alpha-1} \|A\|_\alpha^2 \quad \text{for } \alpha > \frac{1}{2}. \tag{4.14}$$

Now

$$\mathrm{e}^{t\mathcal{L}} A(\varepsilon x)\mathrm{e}^{\mathrm{i}x} - (\mathrm{e}^{4T\partial_X^2} A)(X)\mathrm{e}^{\mathrm{i}\varepsilon^{-1}X} \leq \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \tag{4.15}$$

We finish the proof by taking $|\cdot|$ on both sides of (4.15) and using (4.12)–(4.14).

Lemma 4.6 *Let $n \in \mathbb{R} \setminus \{\pm 1\}$ and $r \in \{0,1\}$. If $T > 0$ and $M \in \mathcal{H}^\alpha$, $\forall \alpha > \frac{1}{2}$, then there exists a positive constant C, such that*

$$\sup_{X\in\mathbb{R}} |\mathrm{e}^{T\mathcal{L}_\varepsilon} \partial_X^r M(X) \mathrm{e}^{\mathrm{i}n\varepsilon^{-1}X}| \leq C\Theta_r(\varepsilon)\|M\|_\alpha, \tag{4.16}$$

where

$$\Theta_r(\varepsilon) = (\varepsilon^{-\frac{3}{2}r}) \exp\Big(-\frac{9}{16}\varepsilon^{-2}T\Big) + \varepsilon^{\alpha-\frac{1}{2}-r}.$$

Proof Writing $\mathrm{e}^{t\mathcal{L}}M(\varepsilon x)\mathrm{e}^{inx}$ as

$$\begin{aligned}\mathrm{e}^{t\mathcal{L}}M(\varepsilon x)\mathrm{e}^{inx} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k(x-y)}\mathrm{e}^{-t\lambda_k}M(\varepsilon y)\mathrm{e}^{iny}\mathrm{d}y\mathrm{d}k\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(k-n)(x-y)}\mathrm{e}^{-t\lambda_k}M(\varepsilon y)\mathrm{d}y\mathrm{d}k\mathrm{e}^{inx}\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k(\varepsilon x-y)}\mathrm{e}^{-t\lambda_{\varepsilon k+n}}M(y)\mathrm{d}y\mathrm{d}k\mathrm{e}^{inx},\end{aligned}$$

where we used the substitution $y=\varepsilon y$ and $k=\varepsilon^{-1}(k-n)$. Hence, using the definition of λ_k,

$$\mathrm{e}^{t\mathcal{L}}M(\varepsilon x)\mathrm{e}^{inx} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k(\varepsilon x-y)}\mathrm{e}^{-t[1-(\varepsilon k+n)^2]^2}M(y)\mathrm{d}y\mathrm{d}k\mathrm{e}^{inx}.$$

Thus, using $X=\varepsilon x$,

$$\mathrm{e}^{T\mathcal{L}_\varepsilon}M(X)\mathrm{e}^{\mathrm{i}n\varepsilon^{-1}X} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k(X-y)}\mathrm{e}^{-t[1-(\varepsilon k+n)^2]^2}M(y)\mathrm{d}y\mathrm{d}k\mathrm{e}^{\mathrm{i}n\varepsilon^{-1}X}.$$

Taking $|\cdot|$ on both sides and using Cauchy-Schwarz inequality, to obtain

$$|\mathrm{e}^{T\mathcal{L}_\varepsilon}(M(X)\mathrm{e}^{in\varepsilon^{-1}X})|^2 \le C\|M\|_\alpha^2\int_{-\infty}^{\infty}\frac{1}{(1+k^2)^\alpha}\mathrm{e}^{-2t[1-(\varepsilon k+n)^2]^2}\mathrm{d}k. \tag{4.17}$$

Now, we want to bound this integral

$$\mathrm{II} = \int_{-\infty}^{\infty}\Phi(k)\mathrm{d}k \le \int_0^{\frac{1}{2\varepsilon}}\Phi(k)\mathrm{d}k + \int_{\frac{-1}{2\varepsilon}}^{0}\Phi(k)\mathrm{d}k + 2\int_{\frac{1}{2\varepsilon}}^{\infty}\frac{1}{(1+k^2)^\alpha}\mathrm{d}k$$

with

$$\Phi(k) = \frac{1}{(1+k^2)^\alpha}\mathrm{e}^{-2tq(k)} \text{ and } q(k) = [1-(\varepsilon k+n)^2]^2.$$

Now, let us bound $q(k)$ on $[0,\pm\frac{1}{2\varepsilon}]$. There are multiple cases depending on n and k.

Case 1 $n=0$ and $k\in\big[-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}\big]$. Since $|k|\le\frac{1}{2\varepsilon}$, we obtain

$$q(k) = [1-\varepsilon^2k^2]^2 \ge \frac{9}{16}.$$

Case 2 $n\ge 2$ and $k\in\big[0,\frac{1}{2\varepsilon}\big]$ $\big($or $n\le -2$ and $k\in\big[\frac{-1}{2\varepsilon},0\big]\big)$. Since $\varepsilon k\ge 0$, we obtain

$$q(k) = [n+\varepsilon k+1]^2[n+\varepsilon k-1]^2 \ge [n+1]^2[n-1]^2 \ge 9.$$

Case 3 $n\ge 2$ and $k\in[\frac{-1}{2\varepsilon},0]$ $\big($or $n\le -2$ and $k\in\big[0,\frac{1}{2\varepsilon}\big]\big)$. Since $k\le\frac{1}{2\varepsilon}$, then

$$q(k) = [\varepsilon k+n-1]^2[1+n+\varepsilon k]^2 \ge \frac{25}{16}.$$

Combining all cases, yields

$$q(k) \ge \frac{9}{16} \quad \text{for } k\in\Big[0,\frac{1}{2\varepsilon}\Big].$$

Thus, for $\alpha > \frac{1}{2}$ we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \Phi(k)\mathrm{d}k &\le 2\int_0^{\frac{1}{2\varepsilon}} \frac{1}{(1+k^2)^\alpha} \mathrm{e}^{-\frac{9}{8}t}\mathrm{d}k + 2\int_{\frac{1}{2\varepsilon}}^{\infty} \frac{1}{(1+k^2)^\alpha}\mathrm{d}k \\ &\le C[\mathrm{e}^{-\frac{9}{8}t} + C\varepsilon^{2\alpha-1}]. \end{aligned} \tag{4.18}$$

Plugging (4.18) into (4.17), yields

$$\|\mathrm{e}^{T\mathcal{L}_\varepsilon} M(X)\mathrm{e}^{\mathrm{i}n\varepsilon^{-1}X}\|_\infty \le C\|M\|_\alpha \Big(\exp\Big(-\frac{9}{16}\varepsilon^{-2}T\Big) + \varepsilon^{\alpha-\frac{1}{2}}\Big). \tag{4.19}$$

Now, consider (4.16) with $r=1$. From Corollary 4.1, we obtain

$$\begin{aligned}\|\mathrm{e}^{T\mathcal{L}_\varepsilon}\partial_X M(X)\mathrm{e}^{\mathrm{i}n\varepsilon^{-1}X}\|_\infty &= \|\mathrm{e}^{\frac{1}{2}T\mathcal{L}_\varepsilon}\partial_X(\mathrm{e}^{\frac{1}{2}T\mathcal{L}_\varepsilon} M(X)\mathrm{e}^{\mathrm{i}n\varepsilon^{-1}X})\|_\infty \\ &\le C(\varepsilon^{-1} + \varepsilon^{\frac{-1}{2}}T^{\frac{-1}{4}})\|\mathrm{e}^{\frac{1}{2}T\mathcal{L}_\varepsilon} M(X)\mathrm{e}^{\mathrm{i}n\varepsilon^{-1}X}\|_\infty. \end{aligned}$$

Using (4.19), yields

$$\|\mathrm{e}^{T\mathcal{L}_\varepsilon}\partial_X M(X)\mathrm{e}^{\mathrm{i}n\varepsilon^{-1}X}\|_\infty \le C\Theta_1(\varepsilon)\|M\|_\alpha. \tag{4.20}$$

We finish the proof by collecting (4.19) and (4.20).

5 Proof of the Approximation Theorem

We present in this section the proof of the approximation theorem.

Definition 5.1 *Define the residual $\rho(T)$ as*

$$\rho(T) = w_A(T) - \mathrm{e}^{T\mathcal{L}_\varepsilon} w_A(0) - \nu\int_0^T \mathrm{e}^{(T-s)\mathcal{L}_\varepsilon} w_A \mathrm{d}s - \frac{1}{2}\gamma\int_0^T \mathrm{e}^{(T-s)\mathcal{L}_\varepsilon}\partial_X w_A^2 \mathrm{d}s, \tag{5.1}$$

where w_A is defined in (2.2).

Lemma 5.1 *If $\sup\limits_{[0,T]}\|A\|_\alpha \le C$, for $\alpha > \frac{1}{2}$, then*

$$\|\rho(T)\|_\infty \le C\varepsilon^{-1}\widetilde{\phi}_\varepsilon, \tag{5.2}$$

where $\widetilde{\phi}_\varepsilon$ is defined in (1.5).

Proof We obtain from (1.4)

$$\begin{aligned}\rho(T) &= A(T)\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{T\mathcal{L}_\varepsilon}A(0)\mathrm{e}^{\mathrm{i}x} - \nu\int_0^T \mathrm{e}^{(T-\tau)\mathcal{L}_\varepsilon} A\mathrm{e}^{\mathrm{i}x}\mathrm{d}\tau + c.c. \\ &\quad - \frac{1}{2}\gamma\int_0^T \mathrm{e}^{(T-\tau)\mathcal{L}_\varepsilon}\partial_X(A\mathrm{e}^{\mathrm{i}x} + B\mathrm{e}^{\mathrm{i}2x} + c.c.)^2\mathrm{d}\tau + \mathcal{O}(\varepsilon) \\ &= A(T)\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{T\mathcal{L}_\varepsilon}A(0)\mathrm{e}^{\mathrm{i}x} - \nu\int_0^T \mathrm{e}^{(T-\tau)\mathcal{L}_\varepsilon} A\mathrm{e}^{\mathrm{i}x}\mathrm{d}\tau + c.c. \\ &\quad - \frac{1}{2}\gamma\int_0^T \mathrm{e}^{(T-\tau)\mathcal{L}_\varepsilon}\partial_X(A^2\mathrm{e}^{2\mathrm{i}x})\mathrm{d}\tau - \gamma\int_0^T \mathrm{e}^{(T-\tau)\mathcal{L}_\varepsilon}\partial_X(|A|^2)\mathrm{d}\tau \end{aligned}$$

$$- \gamma \mathrm{i} \int_0^T \mathrm{e}^{(T-\tau)\mathcal{L}_\varepsilon} \overline{A} B \mathrm{e}^{\mathrm{i}x} \mathrm{d}\tau + \mathcal{O}(\varepsilon).$$

Using Lemma 4.5 for $\alpha > \frac{1}{2}$, we obtain

$$\begin{aligned}
\rho(T) = A(T) - \mathrm{e}^{4T\partial_X^2} A(0) &- \int_0^T \mathrm{e}^{4(T-\tau)\partial_X^2} [\nu A \mathrm{d}s - \mathrm{i}\gamma \overline{A} B] \mathrm{d}\tau \cdot \mathrm{e}^{\mathrm{i}x} \\
&- \frac{1}{2}\gamma \int_0^T \mathrm{e}^{(T-\tau)\mathcal{L}_\varepsilon} \partial_X (A^2 \mathrm{e}^{2\mathrm{i}x}) \mathrm{d}\tau + c.c. \\
&- \gamma \int_0^T \mathrm{e}^{(T-s)\mathcal{L}_\varepsilon} \partial_X |A|^2 \mathrm{d}s + \mathcal{O}(\phi_\varepsilon),
\end{aligned}$$

where ϕ_ε is defined in (4.7) from (1.2), we have

$$\begin{aligned}
\rho(T) = &- \frac{1}{2}\gamma \int_0^T \mathrm{e}^{(T-\tau)\mathcal{L}_\varepsilon} \partial_X (A^2 \mathrm{e}^{2\mathrm{i}x}) \mathrm{d}\tau + c.c. \\
&- \gamma \int_0^T \mathrm{e}^{(T-\tau)\mathcal{L}_\varepsilon} \partial_X |A|^2 \mathrm{d}\tau + \mathcal{O}(\phi_\varepsilon).
\end{aligned}$$

Taking $\|\cdot\|_\infty$ on both sides and using Lemma 4.6 to get

$$\begin{aligned}
\|\rho(T)\|_\infty &\le C \int_0^T \|\mathrm{e}^{(T-\tau)\mathcal{L}_\varepsilon} \partial_X (A^2 \mathrm{e}^{2\mathrm{i}x})\|_\infty \mathrm{d}\tau + C \int_0^T \|\mathrm{e}^{(T-\tau)\mathcal{L}_\varepsilon} \partial_X |A|^2\|_\infty \mathrm{d}\tau + C\phi_\varepsilon \\
&\le C \sup_{\tau\in[0,T]} \|A\|_\alpha [\varepsilon^{\frac{1}{2}} + \varepsilon^{\alpha - \frac{3}{2}}] + C\phi_\varepsilon \\
&\le C\varepsilon^{-1}\widetilde{\phi}_\varepsilon.
\end{aligned}$$

In the end, we use the results previously obtained to prove the approximation Theorem 1.1 for the solution of the Kuramoto-Shivashinsky equation (1.1).

Proof of the Approximation Theorem 1.1 Using multiplier theorem to separate the critical and non critical modes in Fourier space by so called the mode filters. Fix $0 < \sigma \le \frac{1}{8}$. Let χ be a C_0^∞ cut off function with

$$\chi(k) = \begin{cases} 1, & \text{if } k \in \mathrm{I}_c = [-\sigma - 1, \sigma - 1] \cup [1 - \sigma, 1 + \sigma], \\ 0, & \text{if } k \in \mathbb{R} \backslash ([-2\sigma - 1, 2\sigma - 1] \cup [1 - 2\sigma, 1 + 2\sigma]). \end{cases}$$

Then, we define the mode filter for the critical modes by

$$E_c u = \mathcal{F}^{-1} \chi \mathcal{F} u,$$

and for the stable modes by

$$E_s = 1 - E_c.$$

Since E_c and E_s are not projections, we define auxiliary mode filters P_c and P_s satisfying $P_c E_c = E_c$ and $P_s E_s = E_s$ by

$$P_c u = \mathcal{F}^{-1} \chi_c \mathcal{F} u,$$

where χ_c is a C_0^∞ cut off function defined as

$$\chi_c(k) = \begin{cases} 1, & \text{if } k \in [-2\sigma - 1, 2\sigma - 1] \cup [1 - 2\sigma, 1 + 2\sigma], \\ 0, & \text{if } k \in \mathbb{R} \backslash ([-3\sigma - 1, 3\sigma - 1] \cup [1 - 3\sigma, 1 + 3\sigma]), \end{cases}$$

and by

$$P_s u = \mathcal{F}^{-1}(1 - \chi_s)\mathcal{F}u,$$

where χ_s is a C_0^∞ cut off function defined as

$$\chi_s(k) = \begin{cases} 1, & \text{if } k \in \Big[-\dfrac{\sigma}{2} - 1, \dfrac{\sigma}{2} - 1\Big] \cup \Big[1 - \dfrac{\sigma}{2}, 1 + \dfrac{\sigma}{2}\Big], \\ 0, & \text{if } k \in \mathbb{R} \backslash ([-\sigma - 1, \sigma - 1] \cup [1 - \sigma, 1 + \sigma]). \end{cases}$$

Hence, due to disjoint supports in Fourier space, we can use that

$$E_c((E_c u) \cdot (E_c v)) = 0. \tag{5.3}$$

Now define

$$\psi(t) = \varepsilon w_A + \varepsilon^2 R = \varepsilon w_{A_c} + \varepsilon^2 w_{A_s} + \varepsilon^2 R_c + \varepsilon^3 R_s, \tag{5.4}$$

where $w_{A_c} = E_c w_A$, $w_{A_s} = E_s w_A$, $R_c = E_c R$ and $R_s = E_s R$, for short.

Substituting (5.4) into (1.1), we obtain

$$\begin{aligned} \partial_t R_c + \varepsilon \partial_t R_s = {} & \mathcal{L} R_c + \varepsilon^2 \nu R_c + \varepsilon^2 R_c \partial_x R_c + \varepsilon \mathcal{L} R_s + \varepsilon^3 \nu R_s + \varepsilon^4 R_s \partial R_s \\ & + \varepsilon \partial_x (w_{A_c} w_{A_s}) + \varepsilon \partial_x (w_{A_c} R_c) + \varepsilon^2 \partial_x (w_{A_c} R_s) + \varepsilon^2 \partial_x (w_{A_s} R_c) \\ & + \varepsilon^3 \partial_x (w_{A_s} R_s) + \varepsilon^3 \partial_x (R_c R_s) - \varepsilon^{-1} \partial_t \rho_c(\varepsilon^2 t) - \partial_t \rho_s(\varepsilon^2 t), \end{aligned}$$

where $\rho(T)$ is defined in (5.1) and $\rho_c = E_c \rho$, $\rho_s = E_s \rho$.

Applying the operators P_c and P_s, we obtain the following system

$$\begin{aligned} \partial_t R_c = {} & \mathcal{L} R_c + \varepsilon^2 \nu R_c + \varepsilon \partial_x \mathcal{B}_c(R_c) + \varepsilon^2 \partial_x \mathcal{N}_c(R_c) \\ & + \varepsilon^3 \partial_{xc}(R_s) + \varepsilon \partial_x W_c(w_{A_s}) - \varepsilon^{-1} \partial_t \rho_c(\varepsilon^2 t) \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \partial_t R_s = {} & \mathcal{L} R_s + \varepsilon^2 \nu R_s + \partial_x \mathcal{B}_s(R_c) + \varepsilon \partial_x \mathcal{N}_s(R_c) \\ & + \varepsilon^2 \partial_{xs}(R_s) + \partial_x W_s(w_{A_s}) - \varepsilon^{-1} \partial_t \rho_s(\varepsilon^2 t), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \mathcal{B}(R_c) &= \frac{1}{2} \varepsilon R_c^2 + w_{A_c} R_c, & \mathcal{N}(R_c) &= w_{A_c} R_s + w_{A_s} R_c, \\ (R_s) &= w_{A_s} R_s + R_c R_s + \frac{1}{2} \varepsilon R_s^2, & W(w_{A_s}) &= \frac{1}{2} (\varepsilon - 1) w_{A_s}^2 + w_{A_c} w_{A_s}. \end{aligned}$$

Applying E_c into both sides of (5.5) and using (5.3), yields

$$\partial_t R_c = \mathcal{L}R_c + \varepsilon^2 \nu R_c + \varepsilon^2 \partial_x \mathcal{N}_c(R_c) + \varepsilon^3 \partial_{xc}(R_s) \\ + \varepsilon \partial_x W_c(w_{A_s}) - \varepsilon^{-1}\partial_t \rho_c(\varepsilon^2 t). \tag{5.7}$$

Integrating from 0 to t, we obtain

$$R_c(t) = \mathrm{e}^{t\mathcal{L}} R_c(0) + \int_0^t \mathrm{e}^{(t-\tau)\mathcal{L}}[\varepsilon^2 \nu R_c + \varepsilon^2 \partial_x \mathcal{N}_c(R_c) + \varepsilon^3 \partial_{xc}(R_s)](\tau)\mathrm{d}\tau \\ + \varepsilon \int_0^t \mathrm{e}^{(t-\tau)\mathcal{L}}[\partial_x W_c(w_{A_s})](\tau)\mathrm{d}\tau - \varepsilon^{-1}\rho_c(T).$$

Taking $\|\cdot\|_\infty$ on both sides and using Lemma 3.1, yields

$$\|R_c(t)\|_\infty \le \|R_c(0)\|_\infty + \varepsilon^{-1}\|\rho_c(\varepsilon^2 t)\|_\infty + C\varepsilon^2 \int_0^t \|R_c\|_\infty \mathrm{d}\tau \\ + C\varepsilon^2 \int_0^t \{1 + (t-\tau)^{-\frac{1}{4}}\}\|\mathcal{N}_c(R_c)\|_\infty \mathrm{d}\tau \\ + C\varepsilon^3 \int_0^t \{1 + (t-\tau)^{-\frac{1}{4}}\}\|_c(R_s)\|_\infty \mathrm{d}\tau \\ + C\varepsilon \int_0^t \{1 + (t-\tau)^{-\frac{1}{4}}\}\|W_c(w_{A_s})\|_\infty \mathrm{d}\tau.$$

It is easy to bound $\mathcal{N}_c(R_c)$, $c(R_s)$ and $W_c(w_{A_s})$ as follows

$$\|\mathcal{N}(R_c)\|_\infty \le C\|R_s\|_\infty + C\|R_c\|_\infty, \\ \|W_c(w_{A_s})\|_\infty \le C, \\ \|_c(R_s)\|_\infty \le C\|R_s\|_\infty + C\|R_c\|_\infty \|R_s\|_\infty + C\varepsilon \|R_s\|_\infty^2.$$

Thus, by using Lemma 3.4 (Gronwall's lemma) with $\|R_c(0)\|_\infty \le C\varepsilon^{-2}\widetilde{\phi}_\varepsilon$, we obtain

$$\|R_c(t)\|_\infty \le C\varepsilon^{-2}\widetilde{\phi}_\varepsilon. \tag{5.8}$$

Analogously, for (5.6) we obtain

$$\|R_s(t)\|_\infty \le C\varepsilon^{-2}\widetilde{\phi}_\varepsilon + \int_0^t \{1 + (t-\tau)^{-\frac{1}{4}}\}\|R_c\|_\infty \mathrm{d}\tau \\ \le C\varepsilon^{-2}\widetilde{\phi}_\varepsilon + C \sup_{\tau\in[0,t]} \|R_c(\tau)\|_\infty \\ \le C\varepsilon^{-2}\widetilde{\phi}_\varepsilon,$$

where we used (5.8). Hence,

$$\sup_{[0,\varepsilon^{-2}T]} \|R\|_\infty \le \sup_{[0,\varepsilon^{-2}T]} \|R_s\|_\infty + \varepsilon \sup_{[0,\varepsilon^{-2}T]} \|R_s\|_\infty \le C\varepsilon^{-2}\widetilde{\phi}_\varepsilon. \tag{5.9}$$

Since

$$\psi(t) = \varepsilon w_A + \varepsilon^2 R,$$

we have

$$\begin{aligned}\sup_{t\in[0,\varepsilon^{-2}T_0]}\|\psi(t)-\varepsilon v_A(\varepsilon^2 t)\|_\infty &\le \varepsilon \sup_{T\in[0,T_0]}\|v_A(T)-w_A(T)\|_\infty+\varepsilon^2\sup_{t\in[0,\varepsilon^{-2}T_0]}\|R(t)\|_\infty\\ &\le \varepsilon^2\sup_{T\in[0,T_0]}\|B(T)\mathrm{e}^{2\mathrm{i}x}\|_\infty+\varepsilon^3\sup_{T\in[0,T_0]}\|H(T)\mathrm{e}^{3\mathrm{i}x}\|_\infty\\ &\quad+\varepsilon^3\sup_{T\in[0,T_0]}\|J(T)\|_\infty+\varepsilon^2\sup_{t\in[0,\varepsilon^{-2}T_0]}\|R(t)\|_\infty\\ &\le C\widetilde{\phi}_\varepsilon.\end{aligned}$$

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