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Bingham Flows in Periodic Domains of Infinite Length

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(*Dedicated to Philippe G. Ciarlet on the occasion of his* 80*th birthday*)

Abstract The Bingham fluid model has been successfully used in modeling a large class of non-Newtonian fluids. In this paper, the authors extend to the case of Bingham fluids the results previously obtained by Chipot and Mardare, who studied the asymptotics of the Stokes flow in a cylindrical domain that becomes unbounded in one direction, and prove the convergence of the solution to the Bingham problem in a finite periodic domain, to the solution of the Bingham problem in the infinite periodic domain, as the length of the finite domain goes to infinity. As a consequence of this convergence, the existence of a solution to a Bingham problem in the infinite periodic domain is obtained, and the uniqueness of the velocity field for this problem is also shown. Finally, they show that the error in approximating the velocity field in the infinite domain with the velocity in a periodic domain of length 2ℓ has a polynomial decay in ℓ, unlike in the Stokes case (see [Chipot, M. and Mardare, S., Asymptotic behaviour of the Stokes problem in cylinders becoming unbounded in one direction, *Journal de Mathématiques Pures et Appliquées*, **90**(2), 2008, 133–159]) where it has an exponential decay. This is in itself an important result for the numerical simulations of non-Newtonian flows in long tubes.

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1 Introduction

The Bingham fluid model has been proposed by Bingham [2] in 1916 to model plastic flows. Bingham fluids behave at high stresses like a Newtonian fluid, however at low stresses they do not deform. In other words, the shear rate depends linearly on the shear stress only past a certain value of the shear stress, called yield stress; below the yield stress there is no shear. More precisely, the stress tensor σ is given by $\sigma = -pI + \tau$, with

$$\begin{aligned} &\tau = 2\mu D(u) + \sqrt{2}g\frac{D(u)}{|D(u)|}, && \text{if } |D(u)| \neq 0, \\ &|\tau| \leq \sqrt{2}g, && \text{if } |D(u)| = 0, \end{aligned}$$

where u is the velocity, g is the yield stress and the shear rate tensor D is defined by $D_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)$. Thus the flow region of a Bingham fluid can have zones under stress but with no

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deformation, where the fluid behaves like a rigid body. Thus the actual flow region is unknown and in some sense the problem is a "free boundary" problem that leads to a variational inequality formulation (see [5, 10]).

The Bingham model has been successfully used for describing non-Newtonian flows like the flow of drilling mud, lava and paint, flow of avalanches and landslides (see [8, 11]), blood flow in arteries (see [16]), metal deformation for various metal processing techniques like wire drawing (see [6–7, 12]), magneto-rheological or electro-rheological fluids (see [13–14, 17]). More complicated models for such non-Newtonian flows account for shear-thinning or shear-thickening phenomena. For a mechanical and mathematical presentation of non-Newtonian fluids we refer to the recent comprehensive book [5].

In this paper we extend, to the case of Bingham fluids, the results previously obtained by Chipot and Mardare [3] who studied the asymptotics of the Stokes flow in a cylindrical domain that becomes unbounded in one direction.

In Section 2 we introduce the main notations and the variational inequality for the flow of a Bingham fluid in a pipe of finite length 2ℓ, that admits (see [5, 10]) a solution (u_ℓ, p_ℓ), the velocity u_ℓ being unique. What we are interested in is the behavior of u_ℓ as ℓ goes to infinity.

The main results are stated in Section 3. Theorem 3.1 states the convergence of the solution u_ℓ in a periodic domain of length 2ℓ, to the solution u_∞ in the infinite periodic domain, in the strong $\mathbb{H}^1$-norm; here we also state that the error has a polynomial decay. We should note that the same type of error in the Stokes case (see [3]) was shown to have an exponential decay. In addition, we also formulate Theorem 3.2 that states that any weak-$L^2/\mathbb{R}$ limit point of the family $(p_\ell)_{\ell>0}$ is a pressure p_∞ corresponding to the Bingham problem in the infinite pipe. We note that the polynomial decay is an important result in itself: For computational purposes in large arrays of cylindrical pipes (as in models of blood flow in arteries and small vessels), one can assume a Poiseuille type of flow away from the bifurcations; thus one needs to be able to estimate the error made by approximating with the Poiseuille flow away from the bifurcations. For the sake of simplicity, we consider homogeneous Dirichlet boundary conditions on the whole boundary of the periodic domain. However, the results remain valid if we consider non-homogeneous Dirichlet boundary conditions (see Remark 2.1 in Section 2 for more details). This allows to consider cases of flows with non-zero flux, which is important from the point of view of fluid mechanics. In fact, what is important here is that we have Dirichlet boundary conditions on the lateral boundary of the domain, hence we can even consider other types of boundary conditions on the two ends of the pipe.

Section 4 is dedicated to some useful lemmas. In particular, in Lemma 4.1, we prove an arithmetic inequality necessarily satisfied by the first term of a finite increasing sequence of positive real numbers, that satisfies a recursive inequality; this particular recursive inequality is related to the Bingham constitutive equation and the result is essential in the proof of Theorem 3.1. As a consequence of this lemma, we derive Corollary 4.1 that is used to obtain the uniqueness of the velocity for the limit problem. We should note here that this corollary can have a direct proof, independent of Lemma 4.1; however, for convenience, we choose here not to show the direct proof.

In Section 5 we first prove in Theorem 5.1 a Cauchy-type condition for the sequence of solutions u_ℓ, that uses in an essential way the result from Lemma 4.1. Next we provide the proofs for the main theorems stated in Section 3. The existence and uniqueness of the solution u_∞ to the Bingham problem in the infinite periodic domain is a new result, which is not obvious, and is contained in the proof of Theorems 3.1; in particular the function u_∞ is constructed piecewise and the problem it satisfies is then identified.

2 The Setting

Let $N \geq 2$ and denote by $(e_1, \cdots, e_N)$ the canonical basis of $\mathbb{R}^N$. For $x \in \mathbb{R}^N$, we denote $x = (x_1, x')$ with $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$. If $x, y \in \mathbb{R}^k$, then $x \cdot y$ denotes the usual scalar product in $\mathbb{R}^k$. For a measurable subset A of $\mathbb{R}^k$, we denote by $|A|$ its k-dimensional Lebesgue measure.

Throughout this paper, we use the following notations:

(1) Q is a bounded domain contained in $(0,1)\times\mathbb{R}^{N-1}$ with a Lipschitz continuous boundary,

(2) $\Omega_\infty = \int \big(\bigcup\limits_{k\in\mathbb{Z}} \overline{Q}_k \big)$, where $Q_k = Q + ke_1$,

(3) $\Omega_\ell = \Omega_\infty \cap \{|x_1| < \ell\}$, for any nonnegative real number ℓ.

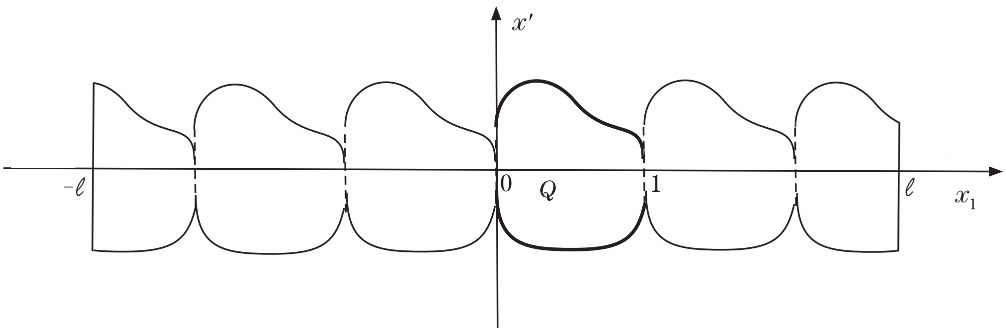

Figure 1 The domain Ω_ℓ.

Throughout this paper, we denote by $D(v)$ the symmetric part of the velocity gradient ∇v, given by

$$D(v) = \big(D_{ij}(v)\big)_{i,j\in\{1,\cdots,N\}} = \Big(\frac{1}{2}\Big(\frac{\partial v^j}{\partial x_i} + \frac{\partial v^i}{\partial x_j}\Big)\Big)_{i,j\in\{1,\cdots,N\}} \tag{2.1}$$

for every vector field $v = (v^1, \cdots, v^N)$.

Let us introduce the following problem

$$\begin{cases} -\operatorname{div}\big(\tau(u_\ell)\big) + \nabla p_\ell = f & \text{in } \Omega_\ell, \\ \operatorname{div} u_\ell = 0 & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell, \end{cases} \tag{2.2}$$

where

$$\tau(u) = 2\mu D(u) + \sqrt{2}g\frac{D(u)}{|D(u)|}, \quad \text{if } |D(u)| \neq 0, \tag{2.3}$$

$$|\tau(u)| \leq \sqrt{2}g, \quad \text{if } |D(u)| = 0, \tag{2.4}$$

and where

$$|D(u)| = \Big(\sum_{i,j=1}^{N} (D_{ij}(u))^2 \Big)^{\frac{1}{2}},$$

and μ and g are positive constants, representing the dynamic viscosity and respectively the yield stress.

Remark 2.1 For simplifying the presentation of the paper, we have only considered homogeneous Dirichlet boundary conditions on $\partial\Omega_\ell$. However, it is easily seen from the proofs in Section 5 that the main results remain valid if one has non-homogeneous Dirichlet boundary conditions. Indeed, what is important in the proof of Theorem 5.1 (see Section 5) is that

$$u_\ell - u_s = 0 \quad \text{on the lateral boundary } \ \partial\Omega_\ell \setminus \{|x_1| = \ell\}$$

and

$$\int_{\{x_1=t\}} (u_\ell^1 - u_s^1)(x_1, x')\,\mathrm{d}x' = 0 \quad \text{for a.e. } t \in (-\ell, \ell).$$

This remain valid if the boundary condition in (2.2) is replaced by $u_\ell = h$ on $\partial\Omega_\ell$, where h belongs to $(H^1_{\mathrm{loc}}(\Omega_\infty))^N$ and satisfies $\operatorname{div} h = 0$ in Ω_∞, and

$$\|h\|_{H^1(\Omega_\ell)} \le \widetilde{C}(1 + \ell^{\widetilde{b}}) \quad \text{for all } \ell > 0,$$

for some constants $\widetilde{C}, \widetilde{b} > 0$ (the last inequality is needed in order to obtain the estimate (5.1) in the proof of Theorem 5.1, see Section 5).

In particular, if $h = 0$ on $\partial\Omega_\infty$, we end up with the following situation which is interesting from the point of view of fluid mechanics: u_ℓ vanishes on the lateral boundary $\partial\Omega_\ell \setminus \{|x_1| = \ell\}$, but not necessarily on the two ends $\partial\Omega_\ell \cap \{x_1 = -\ell\}$ and $\partial\Omega_\ell \cap \{x_1 = \ell\}$, and there exists a constant F (the flux), not necessarily zero, such that

$$\int_{\{x_1=t\}} u_\ell^1(x_1, x')\,\mathrm{d}x' = F \quad \text{for a.e. } t \in (-\ell, \ell)$$

(see Remark 2.2).

Problem (2.2) models the flow of a Bingham fluid in the bounded domain Ω_ℓ. We are interested in its asymptotic behavior as ℓ goes to infinity.

Before giving the variational formulation of (2.2) let us introduce some functional spaces, which will be used throughout the paper.

For any open set $\mathcal{O}$ of $\mathbb{R}^N$, we set

(1) $\mathbb{L}^2(\mathcal{O}) = (L^2(\mathcal{O}))^N$,

(2) $\mathbb{H}^1(\mathcal{O}) = (H^1(\mathcal{O}))^N$,

(3) $\mathbb{H}^1_0(\mathcal{O}) = (H^1_0(\mathcal{O}))^N$,

(4) $\widehat{\mathbb{H}}^1_0(\mathcal{O}) = \{v \in \mathbb{H}^1_0(\mathcal{O}) \mid \operatorname{div} v = 0 \text{ in } \mathcal{O}\}$,

(5) $\widehat{\mathbb{H}}^1_{\mathrm{loc}}(\mathcal{O}) = \{v \in \mathbb{H}^1_{\mathrm{loc}}(\mathcal{O}) = (H^1_{\mathrm{loc}}(\mathcal{O}))^N;\ \operatorname{div} v = 0\}$,

where the functions in $H^1_0(\mathcal{O})$ are extended by zero outside $\mathcal{O}$.

We suppose that the body forces satisfy

$$f = (f^1, \cdots, f^N) \in \mathbb{L}^2(\Omega_\ell), \quad \forall \ell > 0. \tag{2.5}$$

Following [10], for every $\ell > 0$ we associate to problem (2.2) the variational inequality below:

$$\begin{cases} u_\ell \in \widehat{\mathbb{H}}_0^1(\Omega_\ell), \\ \mu \displaystyle\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla(u_\ell - v)\,\mathrm{d}x + g\sqrt{2}\int_{\Omega_\ell} (|D(u_\ell)| - |D(v)|)\,\mathrm{d}x \\ \le \displaystyle\int_{\Omega_\ell} f\cdot(u_\ell - v)\,\mathrm{d}x, \quad \forall v \in \widehat{\mathbb{H}}_0^1(\Omega_\ell), \end{cases} \tag{2.6}$$

where "$\cdot$" denotes the usual scalar product of matrices, given by

$$A\cdot B = \sum_{i,j} a_{ij}b_{ij} \quad \text{for every pair } (A,B) \text{ of } N\times N \text{ matrices.}$$

It is known that this problem admits a unique solution u_ℓ.

In the sequel we write the variational formulation (2.6) under the following equivalent form, obtained by replacing v by $u_\ell - v$:

$$\begin{cases} u_\ell \in \widehat{\mathbb{H}}_0^1(\Omega_\ell), \\ \mu \displaystyle\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v\,\mathrm{d}x + g\sqrt{2}\int_{\Omega_\ell} (|D(u_\ell)| - |D(u_\ell - v)|)\,\mathrm{d}x \\ \le \displaystyle\int_{\Omega_\ell} f\cdot v\,\mathrm{d}x, \quad \forall v \in \widehat{\mathbb{H}}_0^1(\Omega_\ell). \end{cases} \tag{2.7}$$

Remark 2.2 Let $v = (v^1, \cdots, v^N) \in \widehat{\mathbb{H}}_0^1(\Omega_\ell)$ for some $\ell > 0$. Then, applying the Green formula on the set $\Omega_\ell \cap \big((t,\ell)\times\mathbb{R}^{N-1}\big)$ for any $t\in(-\ell,\ell)$, we have

$$\int_{\{x_1=t\}} v^1(x_1,x')\,\mathrm{d}x' = 0 \quad \text{for a.e.} t \in (-\ell,\ell).$$

3 Statement of the Main Results

The main result of this paper can be stated as follows.

Theorem 3.1 *For any $\ell > 0$, let u_ℓ be the solution of problem* (2.7)*. Assume that for some constants $c_0 \ge 0$ and $b \in \mathbb{R}$ the function f satisfies*

$$\|f\|_{L^2(\Omega_\ell)} \le c_0(1+\ell^b) \quad \textit{for any } \ell > 0. \tag{3.1}$$

Then

$$u_\ell \to u_\infty \quad \textit{strongly in } \mathbb{H}^1\big(\Omega_a\big) \quad \textit{for any } a > 0, \tag{3.2}$$

the limit u_∞ being the unique solution of the following problem:

$$\begin{cases} u_\infty \in \widehat{\mathbb{H}}_{\mathrm{loc}}^1(\Omega_\infty), \\ \mu \displaystyle\int_{\Omega_\infty} \nabla u_\infty \cdot \nabla v\,\mathrm{d}x + g\sqrt{2}\int_{\Omega_\infty} (|D(u_\infty)| - |D(u_\infty - v)|)\,\mathrm{d}x \\ \le \displaystyle\int_{\Omega_\infty} f\cdot v\,\mathrm{d}x \quad \textit{for any } v \in \widehat{\mathbb{H}}_0^1(\Omega_a) \textit{ and for any } a > 0, \\ u_\infty = 0 \quad \textit{on } \partial\Omega_\infty, \\ \displaystyle\int_{\{x_1=s\}} u_\infty^1(x_1,x')\,\mathrm{d}x' = 0 \quad \textit{for a.e. } s \in \mathbb{R}, \\ \|\nabla u_\infty\|_{L^2(\Omega_\ell)} \le c_1(1+\ell^{b_1}) \quad \textit{for any } \ell > 0 \textit{ and for some } c_1 \ge 0,\ b_1 \in \mathbb{R}. \end{cases} \tag{3.3}$$

Moreover, for every $\alpha \in \left(0, \frac{1}{2}\right)$, there exist two constants $c \geq 0$ depending only on α and $L_0 > 0$ independent of ℓ (but depending on μ, g, Q, α, b and c_0), such that

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq c\ell^{-\alpha} \quad \text{for any } \ell \geq L_0. \tag{3.4}$$

Theorem 3.1 will be proved in Section 5 and needs some preliminary tools, given in the next section.

Remark 3.1 (1) The existence and uniqueness of the solution u_∞ to problem (3.3) in the infinite periodic domain Ω_∞ is a new result which is not obvious. It is contained in the proof of Theorem 3.1 as a byproduct of the method used therein to prove the convergence of u_ℓ.

(2) By comparing with the case of the Stokes system treated in [3], we see that in Theorem 3.1 we obtain a much lower rate of convergence for the solution to the Bingham problem in the periodic domain Ω_ℓ: We recall that in [3] it was proved that the solution u_ℓ to the Stokes problem in Ω_ℓ converges exponentially to its limit u_∞, which is the solution to a Stokes problem in the infinite domain Ω_∞, i.e.,

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq Ce^{-\alpha\ell} \quad \text{for any } \ell > 0,$$

for some positive constant α.

We can trace the origin of this difference in the proof of Theorem 5.1, more precisely in the non-homogeneity of inequality (5.10). As expected, this non-homogeneity is induced by the supplementary nonlinear term appearing in the variational inequality (2.7) associated to the Bingham equation: Note that in (2.7) the case $g = 0$ corresponds to the variational equation associated to the Stokes problem.

(3) As an immediate consequence of the uniqueness of the solution to problem (3.3), we can prove that if f is 1-periodic in the x_1-direction, then the velocity u_∞ has the same property. Hence, as for the Stokes problem (see [3]), the periodicity of the data implies the periodicity of the velocity corresponding to the problem in the infinite pipe.

We turn now our attention to the pressure associated to problem (2.7).

It is known that there exists $p_\ell \in L^2(\Omega_\ell)$ such that the pair (u_ℓ, p_ℓ) is a solution of the following variational problem:

$$\begin{cases} u_\ell \in \widehat{\mathbb{H}}^1_0(\Omega_\ell), \\ \mu \displaystyle\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v \,\mathrm{d}x + g\sqrt{2} \int_{\Omega_\ell} (|D(u_\ell)| - |D(u_\ell - v)|) \,\mathrm{d}x - \int_{\Omega_\ell} p_\ell \operatorname{div} v \,\mathrm{d}x \\ \leq \displaystyle\int_{\Omega_\ell} f \cdot v \,\mathrm{d}x, \quad \forall v \in \mathbb{H}^1_0(\Omega_\ell). \end{cases} \tag{3.5}$$

Remark 3.2 Unlike the pressure in the Stokes problem, the pressure p_ℓ corresponding to the Bingham problem (3.5) is not unique up to an additive constant. For instance, if $f = 0$, then any pair (u_ℓ, p_ℓ) with $u_\ell = 0$ and $p_\ell \in L^\infty(\Omega_\ell)$ such that $\|p_\ell\|_{L^\infty(\Omega_\ell)/\mathbb{R}} \leq g\sqrt{\frac{2}{N}}$ is a solution to problem (3.5).

Nevertheless, we can show the following result.

Theorem 3.2 *For any $\ell > 0$, let (u_ℓ, p_ℓ) be a solution of problem* (3.5). *Under the assumptions of Theorem* 3.1, *for any $a > 0$, there exists a constant C independent of ℓ such* *that*

$$\|p_\ell\|_{L^2(\Omega_a)/\mathbb{R}} \le C \quad \text{for any } \ell \ge a. \tag{3.6}$$

Moreover, if $\{\ell_n\}$ *is a sequence which tends to* $+\infty$ *and such that*

$$p_{\ell_n} \rightharpoonup p_\infty \quad \text{weakly in } L^2(\Omega_a)/\mathbb{R} \text{ for any } a > 0, \tag{3.7}$$

for some $p_\infty \in L^2_{\mathrm{loc}}(\Omega_\infty)$, *then the pair* (u_∞, p_∞) *satisfies the following variational limit problem:*

$$\begin{cases} \mu \displaystyle\int_{\Omega_\infty} \nabla u_\infty \cdot \nabla v \,\mathrm{d}x + g\sqrt{2}\int_{\Omega_\infty} (|D(u_\infty)| - |D(u_\infty - v)|)\,\mathrm{d}x - \int_{\Omega_\infty} p_\infty \operatorname{div} v \,\mathrm{d}x \\ \le \displaystyle\int_{\Omega_\infty} f \cdot v \,\mathrm{d}x \quad \text{for any } v \in \mathbb{H}^1_0(\Omega_a) \text{ and for any } a > 0. \end{cases} \tag{3.8}$$

Remark 3.3 The convergence in (3.7) means that for any $a > 0$, by considering p_{ℓ_n} and p_∞ as elements of $L^2(\Omega_a)/\mathbb{R}$, we can find representatives $\widetilde{p}_{\ell_n}$ and $\widetilde{p}_\infty$ such that $\widetilde{p}_{\ell_n} \rightharpoonup \widetilde{p}_\infty$ weakly in $L^2(\Omega_a)$.

4 A Main Tool

In this section we prove an arithmetic inequality, which plays an important role in the proof of Theorem 3.1 and seems to be interesting by itself. We also prove a useful consequence (see Corollary 4.1).

Lemma 4.1 *Let* C, $\widetilde{C}_0 \ge 0$, $\gamma \in \mathbb{R}$ *and* $h \in \mathbb{N}$, $h \ge 2$ *be given. Then there exists* $m_0 = m_0(C, \widetilde{C}_0, \gamma, h) \in \mathbb{N}^*$ *such that for any finite sequence* $\{a_k\}$ *containing:*

$m^h + 1$ *elements, with* $m \ge m_0$, *and satisfying*

$$\begin{cases} \text{(i) } 0 \le a_0 \le a_1 \le \cdots \le a_{m^h} \le \widetilde{C}_0(1 + m^\gamma), \\ \text{(ii) } a_k \le C\big((a_{k+1} - a_k) + \sqrt{a_{k+1} - a_k}\big), \quad \forall k \in \{0, 1, \cdots, m^h - 1\}, \end{cases} \tag{4.1}$$

we have

$$a_0 \le \frac{1}{m^{h-1}}. \tag{4.2}$$

Proof We argue by contradiction assuming that

$$a_0 > \frac{1}{m^{h-1}}. \tag{4.3}$$

If k is such that $a_{k+1} - a_k \ge 1$, then $\sqrt{a_{k+1} - a_k} \le a_{k+1} - a_k$. Hence, from (4.1),

$$a_k \le 2C(a_{k+1} - a_k). \tag{4.4}$$

If $0 \le a_{k+1} - a_k < 1$, then $a_{k+1} - a_k \le \sqrt{a_{k+1} - a_k}$ so that, using again (4.1),

$$a_k \le 2C\sqrt{a_{k+1} - a_k}\,,$$

which, taking into account the monotonicity of $\{a_k\}$, implies that

$$a_0 a_k \le a_k^2 \le 4C^2(a_{k+1} - a_k).$$

This gives

$$a_k \le \frac{4C^2}{a_0}(a_{k+1} - a_k) \le 4C^2 m^{h-1}(a_{k+1} - a_k). \tag{4.5}$$

Let us now choose $m_1 \in \mathbb{N}^*$ such that $2Cm_1^{h-1} \ge 1$. Then if $m \ge m_1$ from (4.4)–(4.5) we derive

$$a_k \le 4C^2 m^{h-1}(a_{k+1} - a_k) \quad \text{for any } k = 0, \cdots, m^h - 1,$$

which is equivalent to

$$a_k \le \frac{4C^2 m^{h-1}}{1 + 4C^2 m^{h-1}} a_{k+1} \quad \text{for any } k = 0, \cdots, m^h - 1.$$

Starting from $k = 0$, by iteration we deduce that

$$a_0 \le \Big(\frac{4C^2 m^{h-1}}{1 + 4C^2 m^{h-1}}\Big)^{m^h} a_{m^h} \le \widetilde{C}_0(1 + m^\gamma)\Big(\frac{4C^2 m^{h-1}}{1 + 4C^2 m^{h-1}}\Big)^{m^h} = \frac{\widetilde{C}_0(1 + m^\gamma)}{\big(1 + \frac{1}{4C^2 m^{h-1}}\big)^{m^h}}. \tag{4.6}$$

Let us consider now the sequence $\{b_m\}_{m\in\mathbb{N}}$ defined by

$$0 \le b_m = \frac{\widetilde{C}_0 m^{h-1}(1 + m^\gamma)}{\big(1 + \frac{1}{4C^2 m^{h-1}}\big)^{m^h}} \quad \text{for any } m \in \mathbb{N}.$$

Observe that since $\lim\limits_{m\to\infty} \big(1 + \frac{1}{4C^2 m^{h-1}}\big)^{4C^2 m^{h-1}} = \mathrm{e} > 2$, one has $\big(1 + \frac{1}{4C^2 m^{h-1}}\big)^{4C^2 m^{h-1}} \ge 2$ for m large enough, and therefore

$$b_m = \frac{\widetilde{C}_0 m^{h-1}(1 + m^\gamma)}{\big(1 + \frac{1}{4C^2 m^{h-1}}\big)^{4C^2 m^{h-1} \frac{m}{4C^2}}} \le \frac{\widetilde{C}_0 m^{h-1}(1 + m^\gamma)}{2^{\frac{m}{4C^2}}}$$

for m sufficiently large. Consequently, $\lim\limits_{m\to\infty} b_m = 0$, hence there exists $m_2 \in \mathbb{N}$ such that $b_m \le 1$ for every $m \ge m_2$.

This, together with (4.6) implies that for $m \ge m_0 := \max\{m_1, m_2\}$ we have

$$a_0 \le \frac{b_m}{m^{h-1}} \le \frac{1}{m^{h-1}},$$

which contradicts (4.3) and ends the proof.

The following corollary will be used when studying the uniqueness of the limit problem.

Corollary 4.1 *Let $\{\alpha_k\}_{k\in\mathbb{N}}$ be a nonnegative, non-decreasing sequence satisfying*

$$\begin{cases} \text{(i) } \alpha_k \le C_0(1 + k^\lambda), \quad \forall k \in \mathbb{N}, \\ \text{(ii) } \alpha_k \le C\big((\alpha_{k+1} - \alpha_k) + \sqrt{\alpha_{k+1} - \alpha_k}\big), \quad \forall k \in \mathbb{N} \end{cases} \tag{4.7}$$

for some constants $C, C_0, \lambda \ge 0$. Then

$$\alpha_k = 0, \quad \forall k \in \mathbb{N}. \tag{4.8}$$

Proof Let $k_0 \in \mathbb{N}$ be fixed and set $a_k = \alpha_{k_0+k}$.

For every $m \in \mathbb{N}$ such that $m \geq \sqrt{k_0}$, thanks to (4.7)(i) we have

$$a_{m^2} = \alpha_{k_0+m^2} \leq C_0(1 + (k_0 + m^2)^\lambda) \leq C_0(1 + (2m^2)^\lambda) \leq 2^\lambda C_0(1 + m^{2\lambda}),$$

which shows that (4.1)(i) is satisfied for $\{a_0, \cdots, a_{m^2}\}$, with $h = 2$, $\widetilde{C}_0 = 2^\lambda C_0$ and $\gamma = 2\lambda$. Clearly (4.7)(ii) implies (4.1)(ii), so that from Lemma 4.1 we have

$$\alpha_{k_0} = a_0 \leq \frac{1}{m} \quad \text{for any } m \geq \max\{\sqrt{k_0}, m_0\},$$

where $m_0 = m_0(C, 2^\lambda C_0, 2\lambda, 2) = m_0(C, C_0, \lambda)$ is the one given by Lemma 4.1.

Hence, $a_{k_0} = 0$, which implies (4.8), since k_0 is arbitrary in $\mathbb{N}$.

5 Proof of the Main Results

This section is devoted to the proof of our main results, Theorems 3.1–3.2. For Theorem 3.1, the most difficult part consists in proving the existence of a limit for u_ℓ in $\mathbb{H}^1(\Omega_{\ell_0})$ for any fixed ℓ_0. We follow the ideas of [4] and [3] which need some important modifications due to the non-linearity of the problem.

We first prove a Cauchy condition for u_ℓ in the following theorem, which also provides some accurate estimates.

Theorem 5.1 *For any $\ell > 0$, let u_ℓ be the solution of problem* (2.7).

Under the assumptions of Theorem 3.1, *there exists a constant $C_1 > 0$, depending only on c_0, μ and Q, such that*

$$\|\nabla u_\ell\|_{L^2(\Omega_\ell)} \leq C_1(1 + \ell^b) \quad \textit{for every } \ell > 0. \tag{5.1}$$

Moreover, for any $\alpha \in \left(0, \frac{1}{2}\right)$, there exists $L_0 > 0$ independent of ℓ and s (but depending on μ, g, Q, α, b and c_0) such that the following Cauchy condition holds:

$$\|\nabla(u_\ell - u_s)\|_{L^2(\Omega_{\ell/2})} \leq c\,\ell^{-\alpha} \quad \textit{for every } \ s > \ell \geq L_0, \tag{5.2}$$

where $c \geq 0$ is a constant depending only on α.

Proof Taking u_ℓ as test function in the variational inequality (2.7), we have

$$\mu\|\nabla u_\ell\|^2_{L^2(\Omega_\ell)} \leq \|f\|_{L^2(\Omega_\ell)}\|u_\ell\|_{L^2(\Omega_\ell)} \leq c(Q)\|f\|_{L^2(\Omega_\ell)}\|\nabla u_\ell\|_{L^2(\Omega_\ell)}, \tag{5.3}$$

where $c(Q)$ is the Poincaré constant in Ω_ℓ, which can be chosen depending only on Q (and independent of ℓ). Estimate (5.1) follows then from assumption (3.1) on f.

From now on, the real numbers s and ℓ are such that $s > \ell > 0$.

In order to prove the Cauchy condition (5.2), let us first note that for any $s > \ell > 0$, the difference $u_\ell - u_s$ satisfies the following variational inequality:

$$\begin{aligned} &\mu \int_{\Omega_\ell} \nabla(u_\ell - u_s) \cdot \nabla v \, \mathrm{d}x + g\sqrt{2} \int_{\Omega_\ell} (|D(u_\ell)| \\ &- |D(u_\ell - v)| + |D(u_s)| - |D(u_s + v)|) \, \mathrm{d}x \leq 0, \quad \forall v \in \widehat{\mathbb{H}}^1_0(\Omega_\ell). \end{aligned} \tag{5.4}$$

Indeed, it suffices (for a fixed $v \in \widehat{\mathbb{H}}^1_0(\Omega_\ell)$) to take v as test function in the variational inequality (2.7) satisfied by u_ℓ, then take $-v$ in the variational inequality satisfied by u_s and to sum up the two inequalities. Note that $u_s = u_s + v$ in $\Omega_s \setminus \overline{\Omega}_\ell$, hence $|D(u_s)| = |D(u_s + v)|$ in $\Omega_s \setminus \overline{\Omega}_\ell$.

Following [3] and [9], we build a good test function for the variational inequality (5.4).

First we remark that $\mathrm{div}(u_\ell - u_s) = 0$, but $u_\ell - u_s = -u_s \neq 0$ on $\partial\Omega_\ell \cap \{|x_1| = \ell\}$. In order to obtain a function in $\mathbb{H}^1_0(\Omega_\ell)$, we multiply $u_\ell - u_s$ by the cut-off function in the variable x_1, $\rho : \mathbb{R} \to \mathbb{R}$ whose graph is depicted below, for $\ell_1 \in \mathbb{N}^*$, $\ell_1 \leq \ell - 1$.

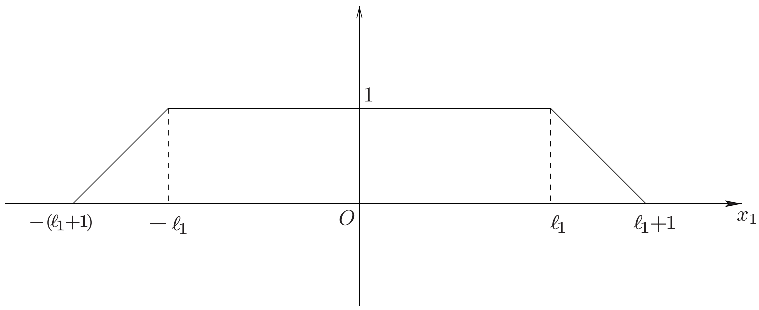

Figure 2 The function ρ.

Then we have

$$\mathrm{div}(\rho(u_\ell - u_s)) = \rho\ \mathrm{div}(u_\ell - u_s) + \rho'(u^1_\ell - u^1_s) = \rho'(u^1_\ell - u^1_s)$$

and therefore, the divergence of $\rho(u_\ell - u_s)$ may not vanish on $\Omega_{\ell_1+1} \setminus \overline{\Omega}_{\ell_1}$.

Thus, for any $\ell_1 \in \mathbb{N}^*$, we set

$$\mathcal{D}_{\ell_1} = \Omega_{\ell_1+1} \setminus \overline{\Omega}_{\ell_1} \tag{5.5}$$

and

$$\mathcal{D}^+_{\ell_1} = \mathcal{D}_{\ell_1} \cap \{x_1 > 0\} = Q_{\ell_1}, \quad \mathcal{D}^-_{\ell_1} = \mathcal{D}_{\ell_1} \cap \{x_1 < 0\} = Q_{-(\ell_1+1)}$$

and notice that $\mathcal{D}_{\ell_1}$ is the union of the disjoint connected sets $\mathcal{D}^+_{\ell_1}$ and $\mathcal{D}^-_{\ell_1}$, which are both translated sets of the same set Q.

Moreover, thanks to Remark 2.2 and the definition of $\mathcal{D}^+_{\ell_1}$, we have the following equality:

$$\begin{aligned}\int_{\mathcal{D}^+_{\ell_1}} \rho'(u^1_\ell - u^1_s)\,\mathrm{d}x &= -\int_{\mathcal{D}^+_{\ell_1}} (u^1_\ell - u^1_s)\,\mathrm{d}x \\ &= \int_{\ell_1}^{\ell_1+1} \int_{\mathcal{D}^+_{\ell_1} \cap \{x_1 = t\}} (u^1_s - u^1_\ell)\,\mathrm{d}x'\mathrm{d}t = 0.\end{aligned}$$

In the same manner, we obtain

$$\int_{\mathcal{D}^-_{\ell_1}} \rho'(u^1_\ell - u^1_s)\,\mathrm{d}x = 0.$$

Using a classical result (see for instance [1, 15]), this allows us to construct a function $\beta \in \mathbb{H}_0^1(\mathcal{D}_{\ell_1})$ such that

$$\begin{cases} \operatorname{div}\beta = \rho'(u_\ell^1 - u_s^1) & \text{in } \mathcal{D}_{\ell_1}, \\ \|\nabla\beta\|_{L^2(\mathcal{D}_{\ell_1})} \le C\|u_\ell^1 - u_s^1\|_{L^2(\mathcal{D}_{\ell_1})}, & C = C(Q) \text{ independent of } \ell,\, \ell_1. \end{cases} \tag{5.6}$$

More specifically, we construct the function β separately on each connected component of $\mathcal{D}_{\ell_1}$, i.e., on $\mathcal{D}_{\ell_1}^+$ and $\mathcal{D}_{\ell_1}^-$. Or these domains are both translated sets of Q. Since the constant C appearing in (5.6) is stable with respect to translations, it depends only on the domain Q and it is therefore independent on ℓ and ℓ_1.

Extending β by 0 outside $\mathcal{D}_{\ell_1}$ we obtain that

$$v_{s,\ell} \doteq \rho(u_\ell - u_s) - \beta \in \widehat{\mathbb{H}}_0^1(\Omega_\ell), \tag{5.7}$$

which is now a good test function for inequality (5.4).

Therefore

$$\begin{aligned} &\mu \int_{\Omega_{\ell_1+1}} \nabla(u_\ell - u_s)\cdot\nabla v_{s,\ell}\,\mathrm{d}x \\ &+ g\sqrt{2}\int_{\mathcal{D}_{\ell_1}} \big(|D(u_\ell)| - |D(u_\ell - v_{s,\ell})| + |D(u_s)| - |D(u_s + v_{s,\ell})|\big)\,\mathrm{d}x \le 0, \end{aligned}$$

since

$$u_\ell - v_{s,\ell} = u_\ell - \rho(u_\ell - u_s) + \beta = \begin{cases} u_s & \text{in } \Omega_{\ell_1}, \\ u_\ell & \text{in } \Omega_\ell \setminus \overline{\Omega}_{\ell_1+1} \end{cases}$$

and

$$u_s + v_{s,\ell} = u_s + \rho(u_\ell - u_s) - \beta = \begin{cases} u_\ell & \text{in } \Omega_{\ell_1}, \\ u_s & \text{in } \Omega_\ell \setminus \overline{\Omega}_{\ell_1+1}. \end{cases}$$

Observe that β vanishes outside $\mathcal{D}_{\ell_1}$ and that for any $u, v \in \mathbb{H}^1(\mathcal{D}_{\ell_1})$,

$$|D(u)| - |D(v)| \le |D(u) - D(v)| = |D(u - v)| \quad \text{in } \mathcal{D}_{\ell_1}.$$

Consequently,

$$\begin{aligned} &\mu \int_{\Omega_{\ell_1+1}} \nabla(u_\ell - u_s)\cdot\nabla\big(\rho(u_\ell - u_s)\big)\,\mathrm{d}x \\ &\le \mu \int_{\mathcal{D}_{\ell_1}} \nabla(u_\ell - u_s)\cdot\nabla\beta\,\mathrm{d}x \\ &\quad + g\sqrt{2}\int_{\mathcal{D}_{\ell_1}} \big(|D(u_\ell - v_{s,\ell})| - |D(u_\ell)| + |D(u_s + v_{s,\ell})| - |D(u_s)|\big)\,\mathrm{d}x \\ &\le \mu \int_{\mathcal{D}_{\ell_1}} |\nabla(u_\ell - u_s)|\,|\nabla\beta|\,\mathrm{d}x + 2g\sqrt{2}\int_{\mathcal{D}_{\ell_1}} |D(v_{s,\ell})|\,\mathrm{d}x. \end{aligned}$$

We develop now $\nabla(\rho(u_\ell - u_s))$. Then, noticing that ρ' vanishes outside $\mathcal{D}_{\ell_1}$ and using the

Cauchy-Schwarz inequality in $\mathcal{D}_{\ell_1}$, we derive

$$
\begin{aligned}
&\mu \int_{\Omega_{\ell_1+1}} \rho |\nabla(u_\ell - u_s)|^2 \, \mathrm{d}x \\
&\le -\mu \int_{\mathcal{D}_{\ell_1}} \frac{\partial(u_\ell - u_s)}{\partial x_1} \cdot \rho'(u_\ell - u_s) \, \mathrm{d}x \\
&\quad + \mu \int_{\mathcal{D}_{\ell_1}} |\nabla(u_\ell - u_s)| \, |\nabla \beta| \, \mathrm{d}x + 2g\sqrt{2} \int_{\mathcal{D}_{\ell_1}} |D(v_{s,\ell})| \, \mathrm{d}x \\
&\le \mu \int_{\mathcal{D}_{\ell_1}} \Big| \frac{\partial(u_\ell - u_s)}{\partial x_1} \Big| \, |u_\ell - u_s| \, \mathrm{d}x + \mu \|\nabla(u_\ell - u_s)\|_{L^2(\mathcal{D}_{\ell_1})} \|\nabla \beta\|_{L^2(\mathcal{D}_{\ell_1})} \\
&\quad + 2g\sqrt{2} \int_{\mathcal{D}_{\ell_1}} |D(v_{s,\ell})| \, \mathrm{d}x \\
&\le C(\|\nabla(u_\ell - u_s)\|^2_{L^2(\mathcal{D}_{\ell_1})} + \|\nabla(u_\ell - u_s)\|_{L^2(\mathcal{D}_{\ell_1})} \|u^1_\ell - u^1_s\|_{L^2(\mathcal{D}_{\ell_1})}) \\
&\quad + 2g\sqrt{2} \int_{\mathcal{D}_{\ell_1}} |\nabla v_{s,\ell}| \, \mathrm{d}x,
\end{aligned} \tag{5.8}
$$

where in the last inequality we used the Poincaré inequality, the estimate (5.6) and the inequality $|D(v)| \le |\nabla v|$ in $\mathcal{D}_{\ell_1}$ for any $v \in \mathbb{H}^1(\mathcal{D}_{\ell_1})$, which follows from definition (2.1) of $D(v)$.

From (5.7), using the Cauchy-Schwarz inequality in $\mathcal{D}_{\ell_1}$, we obtain

$$
\begin{aligned}
\int_{\mathcal{D}_{\ell_1}} |\nabla v_{s,\ell}| \, \mathrm{d}x &\le \sqrt{|\mathcal{D}_{\ell_1}|} \, \|\nabla(\rho(u_\ell - u_s) - \beta)\|_{L^2(\mathcal{D}_{\ell_1})} \\
&\le \sqrt{2|Q|} \, (\|\nabla(\rho(u_\ell - u_s))\|_{L^2(\mathcal{D}_{\ell_1})} + \|\nabla \beta\|_{L^2(\mathcal{D}_{\ell_1})}).
\end{aligned} \tag{5.9}
$$

We compute once again $\nabla\big(\rho(u_\ell - u_s)\big)$ and apply (5.6) and the Poincaré inequality in (5.9).

This, used in (5.8) together with the remark that $\rho = 1$ in Ω_{ℓ_1} and $\rho \ge 0$ in Ω_{ℓ_1+1}, implies that there exists a constant C depending only on Q, g and μ, such that

$$
\begin{aligned}
\|\nabla(u_\ell - u_s)\|^2_{L^2(\Omega_{\ell_1})} &\le C \|\nabla(u_\ell - u_s)\|^2_{L^2(\mathcal{D}_{\ell_1})} \\
&\quad + \frac{4g\sqrt{|Q|}}{\mu} (\|\nabla(\rho(u_\ell - u_s))\|_{L^2(\mathcal{D}_{\ell_1})} + \|\nabla \beta\|_{L^2(\mathcal{D}_{\ell_1})}) \\
&\le C(\|\nabla(u_\ell - u_s)\|^2_{L^2(\mathcal{D}_{\ell_1})} + \|\nabla(u_\ell - u_s)\|_{L^2(\mathcal{D}_{\ell_1})}).
\end{aligned} \tag{5.10}
$$

Let $\ell_0 \in \mathbb{N}^*$ be fixed such that $\ell_0 \le \ell - 1$. Then for any $k \in \mathbb{N}$ satisfying $\ell_0 + k \le \ell - 1$, we can take $\ell_1 = \ell_0 + k$ in the previous inequality.

We also remark that

$$
\|\nabla(u_\ell - u_s)\|^2_{L^2(\mathcal{D}_{\ell_1})} = \|\nabla(u_\ell - u_s)\|^2_{L^2(\Omega_{\ell_1+1})} - \|\nabla(u_\ell - u_s)\|^2_{L^2(\Omega_{\ell_1})}.
$$

By denoting $a_k = \|\nabla(u_\ell - u_s)\|^2_{L^2(\Omega_{\ell_0+k})}$, in view of (5.10) we obtain that for all these k we have the following inequality:

$$
a_k \le C((a_{k+1} - a_k) + \sqrt{a_{k+1} - a_k}\,). \tag{5.11}
$$

Let $h \in \mathbb{N}$, $h \ge 2$ be fixed.

We first show that if

$$2(m^h+1) \le \ell < s \le 2((m+1)^h+1) \quad \text{for some } m \in \mathbb{N}^*,$$

and $\ell_0 \in \mathbb{N}^*$ is such that $\ell_0 \le \left[\frac{\ell}{2}\right]+1$ ($[\cdot]$ denotes the integer part), then the finite sequence

$$\{a_k\}_{k=0}^{m^h}, \quad a_k = \|\nabla(u_\ell - u_s)\|^2_{L^2(\Omega_{\ell_0+k})}$$

satisfies the hypotheses of Lemma 4.1.

Inequality (4.1)(ii) follows from (5.11). Moreover, we notice that the finite sequence $\{a_k\}$ is non-negative and non-decreasing. Let us prove that the last inequality in (4.1)(i) is also verified.

Indeed, this inequality is a consequence of inequality (5.1) and of the fact that $\ell_0 + m^h \le \ell$:

$$\begin{aligned}
\sqrt{a_{m^h}} &= \|\nabla(u_\ell - u_s)\|_{L^2(\Omega_{\ell_0+m^h})} \le \|\nabla(u_\ell - u_s)\|_{L^2(\Omega_\ell)} \le \|\nabla u_\ell\|_{L^2(\Omega_\ell)} + \|\nabla u_s\|_{L^2(\Omega_\ell)} \\
&\le \|\nabla u_\ell\|_{L^2(\Omega_\ell)} + \|\nabla u_s\|_{L^2(\Omega_s)} \le C_1(1+\ell^b+1+s^b) \\
&\le 2C_1(1+2^b((m+1)^h+1)^b) \le C_{b,h}(1+m^{hb}).
\end{aligned}$$

Consequently, the desired inequality is satisfied with $\gamma = 2hb$ and a constant $\widetilde{C}_0$ depending only on C_1, b and h.

Thus, if $m \ge m_0(C, \widetilde{C}_0, \gamma, h)$, then by Lemma 4.1, we have that

$$a_0 = \|\nabla(u_\ell - u_s)\|^2_{L^2(\Omega_{\ell_0})} \le \frac{1}{m^{h-1}},$$

hence

$$\|\nabla(u_\ell - u_s)\|_{L^2(\Omega_{\ell_0})} \le \frac{1}{m^{\frac{h-1}{2}}} \tag{5.12}$$

for any ℓ and s satisfying

$$2(m^h+1) \le \ell < s \le 2((m+1)^h+1)$$

and any $\ell_0 \in \mathbb{N}^*$ such that $\ell_0 \le \left[\frac{\ell}{2}\right]+1$.

Let us observe that for any $\ell \ge 2(m_0^h+1)$, there exists $m \in \mathbb{N}^*$, $m \ge m_0$ such that

$$2(m^h+1) \le \ell \le 2((m+1)^h+1). \tag{5.13}$$

In order to prove (5.2), let us choose $s > \ell \ge 2(m_0^h+1)$. Since $s > \ell$, there exists an integer $q \in \mathbb{N}$ such that

$$2[(m+q)^h+1] \le s \le 2[(m+q+1)^h+1].$$

Then we have

$$\begin{aligned}
\|\nabla(u_\ell - u_s)\|_{L^2(\Omega_{\ell_0})} &\le \|\nabla(u_\ell - u_{2((m+1)^h+1)})\|_{L^2(\Omega_{\ell_0})} \\
&\quad + \sum_{i=1}^{q-1} \|\nabla(u_{2((m+i)^h+1)} - u_{2((m+i+1)^h+1)})\|_{L^2(\Omega_{\ell_0})} \\
&\quad + \|\nabla(u_{2((m+q)^h+1)} - u_s)\|_{L^2(\Omega_{\ell_0})} \\
&\le \sum_{i=0}^{q} \frac{1}{(m+i)^{\frac{h-1}{2}}} \le \sum_{i=0}^{+\infty} \frac{1}{(m+i)^{\frac{h-1}{2}}} = \sum_{k=m}^{+\infty} \frac{1}{k^{\frac{h-1}{2}}} \le \int_{m-1}^{+\infty} \frac{\mathrm{d}y}{y^{\frac{h-1}{2}}} \\
&= \frac{2}{(h-3)(m-1)^{\frac{h-3}{2}}} \le \frac{c_h}{m^{\frac{h-3}{2}}},
\end{aligned}$$

where in the second inequality we have used (5.12) and for the last computations we assumed that $h \geq 4$.

From (5.13) one has $(m+1)^h \geq \frac{\ell}{2} - 1$ so that $m \geq \left(\frac{\ell}{2} - 1\right)^{\frac{1}{h}} - 1 \geq C_h \ell^{\frac{1}{h}}$. Thus, taking $L_0 = 2(m_0^h + 1)$, we obtain

$$\|\nabla(u_\ell - u_s)\|_{L^2(\Omega_{\ell_0})} \leq \frac{c}{\ell^{\frac{h-3}{2h}}}, \quad \forall s > \ell \geq L_0.$$

Let now $\alpha \in \left(0, \frac{1}{2}\right)$ be the constant given in the statement of the theorem and choose h such that $\frac{h-3}{2h} = \frac{1}{2} - \frac{3}{2h} \geq \alpha$ and $\ell_0 = \left[\frac{\ell}{2}\right] + 1$. We derive

$$\|\nabla(u_\ell - u_s)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq \|\nabla(u_\ell - u_s)\|_{L^2(\Omega_{\ell_0})} \leq \frac{c}{\ell^{\frac{h-3}{2h}}} \leq \frac{c}{\ell^{\alpha}}, \quad \forall s > \ell \geq L_0,$$

since $\frac{\ell}{2} \leq \ell_0$. This gives the desired Cauchy inequality.

Proof of Theorem 3.1 It is divided into four steps.

Step 1 Proof of a Cauchy condition for u_ℓ.

From Theorem 5.1 we deduce that for every $\ell_0 > 0$, we have

$$\|\nabla(u_n - u_m)\|_{L^2(\Omega_{\ell_0})} \leq c\, n^{-\alpha}, \quad \forall n, m \in \mathbb{N}^*,\ m > n \geq \max\{L_0, 2\ell_0\}, \tag{5.14}$$

which implies using the Poincaré inequality in Ω_{ℓ_0} that $\{u_n\}_{n\in\mathbb{N}^*}$ is a Cauchy sequence in $\mathbb{H}^1(\Omega_{\ell_0})$. Then the sequence $\{u_n\}$ converges to some $u_\infty^{\ell_0}$ in the Banach space $\mathbb{H}^1(\Omega_{\ell_0})$.

This together with Theorem 5.1, implies that for every $\ell_0 > 0$ there exists a function $u_\infty^{\ell_0} \in \mathbb{H}^1(\Omega_{\ell_0})$ such that

$$u_\ell \to u_\infty^{\ell_0} \quad \text{strongly in } \mathbb{H}^1(\Omega_{\ell_0}). \tag{5.15}$$

Step 2 Construction of the limit function u_∞ and proof of estimate (3.4).

In this step we prove that there exists a function u_∞ such that $u_\infty = u_\infty^{\ell_0}$ on each Ω_{ℓ_0}.

According to Step 1, we have in particular that for every $n \in \mathbb{N}^*$ there exists a function $u_\infty^n \in \mathbb{H}^1(\Omega_n)$ such that

$$u_\ell \to u_\infty^n \quad \text{strongly in } \mathbb{H}^1(\Omega_n).$$

Then, for $n < m$ we have that $u_\infty^n = u_\infty^m$ a.e. in $\Omega_n \subset \Omega_m$, so that by setting

$$u_\infty = \begin{cases} u_\infty^1 & \text{in } \Omega_1, \\ u_\infty^n & \text{in } \Omega_n \setminus \Omega_{n-1} \text{ for } n \geq 2, \end{cases}$$

we obtain

$$u_\ell \to u_\infty \quad \text{strongly in } \mathbb{H}^1(\Omega_{\ell_0}) \quad \text{for every } \ell_0 > 0. \tag{5.16}$$

Now, passing to the limit in (5.2) as $s \to +\infty$, for a fixed ℓ, we deduce estimate (3.4).

Step 3 Identification of the problem satisfied by u_∞.

In this step we prove that the limit u_∞ constructed in the previous step is a solution of problem (3.3).

In order to prove that $u_\infty \in \widehat{\mathbb{H}}^1_{\mathrm{loc}}(\Omega_\infty)$ it remains to prove that it is divergence-free. This is a simple consequence of the fact that $\operatorname{div} u_\ell = 0$ in Ω_{ℓ_0} for any $\ell \geq \ell_0$ and of convergence (5.16).

Let us now pass to the limit in the variational inequality (2.7).

Let $\ell_0 > 0$ be fixed and $v \in \widehat{\mathbb{H}}^1_0(\Omega_{\ell_0})$. Then, for any $\ell \geq \ell_0$ the function v belongs to $\widehat{\mathbb{H}}^1_0(\Omega_\ell)$. Therefore using v as test function in (2.7) we have

$$\mu \int_{\Omega_{\ell_0}} \nabla u_\ell \cdot \nabla v \, \mathrm{d}x + g\sqrt{2} \int_{\Omega_{\ell_0}} (|D(u_\ell)| - |D(u_\ell - v)|) \, \mathrm{d}x \leq \int_{\Omega_{\ell_0}} f \cdot v \, \mathrm{d}x,$$

since $u_\ell = u_\ell - v$ outside Ω_{ℓ_0}.

Making $\ell \to +\infty$ in the above inequality and using again convergence (5.16) we derive that

$$\mu \int_{\Omega_{\ell_0}} \nabla u_\infty \cdot \nabla v \, \mathrm{d}x + g\sqrt{2} \int_{\Omega_{\ell_0}} (|D(u_\infty)| - |D(u_\infty - v)|) \, \mathrm{d}x \leq \int_{\Omega_{\ell_0}} f \cdot v \, \mathrm{d}x.$$

Using the fact that v vanishes outside of Ω_{ℓ_0}, this implies the variational inequality in (3.3).

Observe also that as a simple consequence of (5.16) we also have

$$u_\infty = 0 \quad \text{on } \partial\Omega_\infty. \tag{5.17}$$

We now prove that

$$\int_{\{x_1 = s\}} u^1_\infty(x_1, x') \, \mathrm{d}x' = 0 \quad \text{for a.e. } s \in \mathbb{R}. \tag{5.18}$$

Let $\ell_0 > 0$ be fixed. Then, for every $\ell > \ell_0$, by Remark 2.2 we have in particular,

$$\int_{\{x_1 = s\}} u^1_\ell(x_1, x') \, \mathrm{d}x' = 0 \quad \text{for a.e. } s \in (-\ell_0, \ell_0).$$

Passing to the limit as $\ell \to +\infty$, in view of convergence (5.16) and the trace theorem we have (5.18) since ℓ_0 is arbitrary.

Finally, let us show that u_∞ satisfies the estimate for the gradient in (3.3). This follows from (5.1) of Theorem 5.1 and estimate (3.4) proved in Step 2, since for any $\ell \geq \frac{L_0}{2}$, we have

$$\begin{aligned} \|\nabla u_\infty\|_{L^2(\Omega_\ell)} &\leq \|\nabla(u_\infty - u_{2\ell})\|_{L^2(\Omega_\ell)} + \|\nabla u_{2\ell}\|_{L^2(\Omega_{2\ell})} \\ &\leq c(2\ell)^{-\alpha} + C_1(1 + (2\ell)^b) \leq \widetilde{c}_1(1 + \ell^b). \end{aligned}$$

Taking $c_1 = \max\{\widetilde{c}_1, \|\nabla u_\infty\|_{L^2(\Omega_{L_0/2})}\}$, we get

$$\|\nabla u_\infty\|_{L^2(\Omega_\ell)} \leq c_1(1 + \ell^b) \quad \text{for every } \ell > 0,$$

where c_1 depends on μ, g, Q, α, c_0, b, but is independent of ℓ.

Step 4 Uniqueness of the solution to the limit problem.

Let u_∞ and w_∞ be two solutions of problem (3.3).

The same argument used in the proof of Theorem 5.1 to show (5.10), replacing u_ℓ and u_s by u_∞ and w_∞ respectively, gives

$$\|\nabla(u_\infty - w_\infty)\|^2_{L^2(\Omega_{\ell_1})} \leq C \left(\|\nabla(u_\infty - w_\infty)\|^2_{L^2(\mathcal{D}_{\ell_1})} + \|\nabla(u_\infty - w_\infty)\|_{L^2(\mathcal{D}_{\ell_1})}\right) \tag{5.19}$$

for every $\ell_1 \in \mathbb{N}^*$, where $\mathcal{D}_{\ell_1}$ is given by (5.5).

Set now $\alpha_0 = 0$ and $\alpha_k = \|\nabla(u_\infty - w_\infty)\|^2_{L^2(\Omega_k)}$ for $k \in \mathbb{N}^*$. Then, the sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ satisfies the hypotheses of Corollary 4.1, thanks to the estimate of the gradient in (3.3) satisfied by u_∞ and w_∞ and (5.19).

This implies (4.8), that is

$$\|\nabla(u_\infty - w_\infty)\|_{L^2(\Omega_k)} = 0 \quad \text{for any } k \in \mathbb{N}^*,$$

which by the Poincaré inequality gives $u_\infty = w_\infty$. The proof is now complete.

Proof of Theorem 3.2 Let $a > 0$ be fixed and $\ell \geq a$. Let $\widetilde{p}_\ell = p_\ell - \frac{1}{|\Omega_a|}\int_{\Omega_a} p_\ell \, dx$ be the representative of p_ℓ in $L^2(\Omega_a)/\mathbb{R}$ satisfying $\int_{\Omega_a} \widetilde{p}_\ell \, dx = 0$. Then (see for instance [1, 15]) there exists $v_\ell \in \mathbb{H}^1_0(\Omega_a)$ such that

$$\begin{cases} \operatorname{div} v_\ell = \widetilde{p}_\ell \quad \text{in } \ \Omega_a, \\ \|v_\ell\|_{H^1(\Omega_a)} \leq C_a \|\widetilde{p}_\ell\|_{L^2(\Omega_a)}, \end{cases} \tag{5.20}$$

where the constant C_a depends only on Ω_a.

On the other hand, it is obvious that the pair $(u_\ell, \widetilde{p}_\ell)$ also satisfies the variational inequality in (3.5), i.e.,

$$\begin{cases} \quad \mu \displaystyle\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v \, dx + g\sqrt{2} \int_{\Omega_\ell} (|D(u_\ell)| - |D(u_\ell - v)|) \, dx - \int_{\Omega_\ell} \widetilde{p}_\ell \operatorname{div} v \, dx \\ \leq \displaystyle\int_{\Omega_\ell} f \cdot v \, dx, \quad \forall v \in \mathbb{H}^1_0(\Omega_\ell). \end{cases}$$

Taking $(-v_\ell)$ (extended by 0 outside of Ω_a) as test function in the variational inequality above, we get by using the Cauchy-Schwarz inequality, the inequality $|D(v_\ell)| \leq |\nabla v_\ell|$ and (5.20),

$$\begin{aligned} \int_{\Omega_a} \widetilde{p}_\ell^2 \, dx &= \int_{\Omega_a} \widetilde{p}_\ell \operatorname{div} \, v_\ell \, dx \\ &\leq \mu \int_{\Omega_a} \nabla u_\ell \cdot \nabla v_\ell \, dx + g\sqrt{2} \int_{\Omega_a} (|D(u_\ell + v_\ell)| - |D(u_\ell)|) \, dx - \int_{\Omega_a} f \cdot v_\ell \, dx \\ &\leq \mu \|\nabla u_\ell\|_{L^2(\Omega_a)} \|\nabla v_\ell\|_{L^2(\Omega_a)} + g\sqrt{2} \int_{\Omega_a} |D(v_\ell)| \, dx + \|f\|_{L^2(\Omega_a)} \|v_\ell\|_{L^2(\Omega_a)} \\ &\leq C_a \big(\mu \|\nabla u_\ell\|_{L^2(\Omega_a)} + g\sqrt{2}|\Omega_a|^{\frac{1}{2}} + \|f\|_{L^2(\Omega_a)}\big) \|\widetilde{p}_\ell\|_{L^2(\Omega_a)} \\ &\leq C \|\widetilde{p}_\ell\|_{L^2(\Omega_a)}, \end{aligned}$$

where the constant C is independent of ℓ (but depends on Ω_a, $u_{\infty|_{\Omega_a}}$ and $f_{|_{\Omega_a}}$), since thanks to (3.4),

$$\begin{aligned} \|\nabla u_\ell\|_{L^2(\Omega_a)} &\leq \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_a)} + \|\nabla u_\infty\|_{L^2(\Omega_a)} \\ &\leq \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} + \|\nabla u_\infty\|_{L^2(\Omega_a)} \leq c\ell^{-\alpha} + \|\nabla u_\infty\|_{L^2(\Omega_a)} \\ &\leq c(2a)^{-\alpha} + \|\nabla u_\infty\|_{L^2(\Omega_a)} \end{aligned}$$

for $\ell \geq \max\{2a, L_0\}$, and thanks to (5.1),

$$\|\nabla u_\ell\|_{L^2(\Omega_a)} \leq \|\nabla u_\ell\|_{L^2(\Omega_\ell)} \leq C_1(1 + \ell^b) \leq C_1 \max\{1 + a^b, 1 + (\max\{2a, L_0\})^b\},$$

if $a \leq \ell < \max\{2a, L_0\}$.

Therefore, $\|\widetilde{p}_\ell\|_{L^2(\Omega_a)} \leq C$ for all $\ell \geq a$, hence

$$\|p_\ell\|_{L^2(\Omega_a)/\mathbb{R}} \leq C \quad \text{for all } \ell \geq a.$$

Since $L^2(\Omega_a)$ is a Hilbert space, there exists a sequence $\{p_{\widetilde{\ell}_n}\}$ and a function $p^a_\infty \in L^2(\Omega_a)$ such that $p_{\widetilde{\ell}_n} \rightharpoonup p^a_\infty$ weakly in $L^2(\Omega_a)/\mathbb{R}$. Letting a take all the values in $\mathbb{N}^*$ and using a diagonal selection process for a sequence of successive subsequences, we can construct a sequence $\{p_{\ell_n}\}$ and a function $p_\infty \in L^2_{\text{loc}}(\Omega_\infty)$ (for the construction of p_∞ we use a technique that is similar to the one in Step 2 of the proof of Theorem 3.1) such that

$$p_{\ell_n} \rightharpoonup p_\infty \quad \text{weakly in } L^2(\Omega_a)/\mathbb{R} \text{ for any } a > 0.$$

The fact that the pair (u_∞, p_∞) satisfies the variational inequality (3.8) is then simply obtained by passing to the limit (as n goes to $+\infty$) in the variational inequality (3.5) satisfied by the pair (u_{ℓ_n}, p_{ℓ_n}).

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