Chin. Ann. Math. Ser. B
39(2), 2018, 201–212
DOI: 10.1007/s11401-018-1060-x

Chinese Annals of Mathematics, Series B

Gevrey Class Regularity of a Semigroup Associated with a Nonlinear Korteweg-de Vries Equation*

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(*Dedicated to Philippe G. Ciarlet with admiration and friendship on the occasion of his* 80*th birthday*)

Abstract In this paper, the authors consider the Gevrey class regularity of a semigroup associated with a nonlinear Korteweg-de Vries (KdV for short) equation. By estimating the resolvent of the corresponding linear operator, the authors conclude that the semigroup generated by the linear operator is not analytic but of Gevrey class $\delta \in \left(\frac{3}{2}, \infty\right)$ for $t > 0$.

Keywords Korteweg-de Vries equation, Resolvent estimation, Analytic semigroup, Gevrey class
2000 MR Subject Classification 35Q53, 35P05, 47D03

1 Introduction

The motivation of this paper comes from the study of the asymptotic stability for the following nonlinear KdV equation posed on a finite spatial interval

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, & t \in (0, \infty),\ x \in (0, L), \\ y(t, 0) = y(t, L) = y_x(t, L) = 0, & t \in (0, \infty), \\ y(0, x) = y_0(x), & x \in (0, L), \end{cases} \tag{1.1}$$

where the space length $L \in (0, \infty)$.

When studying the asymptotic stability problem of nonlinear systems, a usual way is to firstly do the linearization (around the origin) and study the asymptotic stability of the linearized system. By dropping the nonlinear term yy_x, we obtain the associated linear system of

Manuscript received May 5, 2017. Revised July 2, 2017.
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*This work was supported by the National Natural Science Foundation of China (Nos. 11401021, 11471044, 11771336, 11571257), the LIASFMA, the ANR project Finite4SoS (No. ANR 15-CE23-0007) and the Doctoral Program of Higher Education of China (Nos. 20130006120011, 20130072120008).

(1.1) as follows:

$$\begin{cases} y_t + y_x + y_{xxx} = 0, & t \in (0,\infty),\ x \in (0,L), \\ y(t,0) = y(t,L) = y_x(t,L) = 0, & t \in (0,\infty), \\ y(0,x) = y_0(x), & x \in (0,L). \end{cases} \tag{1.2}$$

Set

$$X := L^2(0,L).$$

Define the linear operator $A : D(A) \to X$ by

$$A\varphi = -\varphi' - \varphi''', \quad \forall \varphi \in D(A) \tag{1.3}$$

with

$$D(A) := \{\varphi \in H^3(0,L);\ \varphi(0) = \varphi(L) = \varphi'(L) = 0\}.$$

It is easy to see that A is closed (see [6]). The well-posedness of (1.2) has been proved by Rosier [6] by showing that the linear operator A is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(0,L)$. Moreover, he introduced in [6] the following set of critical lengths

$$\mathcal{N} := \Big\{2\pi\sqrt{\frac{j^2+l^2+jl}{3}};\ j,l \in \mathbb{N}^*\Big\},$$

when considering the controllability problem where the control acts as Neumann boundary condition at the right end-point.

For the asymptotic stability of system (1.1), it was proved in [5] that when the space length $L \notin \mathcal{N}$, 0 is exponentially stable for the corresponding linearized equation (1.2), which gives the local asymptotic stability around the origin for the nonlinear system (1.1). When $L \in \mathcal{N}$, the exponential stability of (1.2) does not hold because of the existence of a finite-dimensional space of solutions that are completely undamped. This is obtained by analyzing the spectrum of the linear operator A. To be more precise, it was proved in [6] that when $L \in \mathcal{N}$, (1.2) admits a family of non-trivial solutions of the form $y_0(x)\mathrm{e}^{\lambda t}$ for some $\lambda \in \mathrm{i}\mathbb{R}$, where $y_0(x)$ satisfies

$$\begin{cases} \lambda y_0(x) + y_0'(x) + y_0'''(x) = 0, \\ y_0(0) = y_0(L) = y_0'(0) = y_0'(L) = 0. \end{cases}$$

In this case, the origin of (1.2) is not asymptotically stable, and thus the linearization analysis fails. However, it is still very interesting to study the asymptotic properties of the nonlinear system (1.1) in this critical case.

In [2], we proved the existence and smoothness of the center manifold of (1.1) when $L = 2k\pi$, (i.e., taking $j = l = k$ in $\mathcal{N}$), where k is a positive integer such that (see [3, Theorem 8.1, Remark 8.2])

$$(j^2 + l^2 + jl = 3k^2,\ j,l \in \mathbb{N}^*) \Rightarrow (j = l = k). \tag{1.4}$$

In this case, the center manifold is of dimension 1. By analyzing the reduced equation on the center manifold, we showed that the nonlinear system (1.1) is asymptotically stable around the origin with a polynomial decay rate. Using the same method, we proved in [7] the local asymptotic stability for another special critical length $L = 2\pi\sqrt{\frac{7}{3}}$ (i.e., taking $j = 1$, $l = 2$ in $\mathcal{N}$), where the center manifold is of dimension 2.

While studying the existence of the center manifold, it was noticed that the linear operator A does not generate an analytic semigroup, but a semigroup of Gevrey class. The Gevrey class of semigroups have a behavior somewhat "between" that of differentiable semigroups and analytic semigroups. For the convenience of readers, we first give the definition of Gevrey class $\delta > 1$ (see [1, 8]).

Definition 1.1 *Let $T(t)$ be a strongly continuous semigroup on a Banach space X and let $\delta > 1$. We say that $T(t)$ is of Gevrey class δ for $t > t_0$ if $T(t)$ is infinitely differentiable for $t \in (t_0, \infty)$ and, for every compact subset $K \subset (t_0, \infty)$ and each $\theta > 0$, there exists a constant $C = C(\theta, K)$ such that*

$$\|T^{(n)}(t)\| \leqslant C\theta^n (n!)^\delta, \quad \forall t \in K, \ n = 1, 2, \cdots .$$

When $\delta = 1$, $T(t)$ is analytic (see [9]).

With the definition of Gevrey class δ, we give the main result of this article.

Theorem 1.1 *The linear operator A defined by (1.3) does not generate an analytic semigroup but a semigroup of Gevrey class δ for every $\delta > \frac{3}{2}$ for $t > 0$ and for all lengths $L \in (0, \infty)$.*

The organization of the paper is as follows. In Section 2, we present some properties about the spectrum of the linear operator A and give the explicit formula for the resolvent of A. In Section 3, the estimation for the resolvent of A is proved in order to obtain our main result—Theorem 1.1.

2 Preliminary

In [2], we proved that for all $L \in (0, \infty)$, the spectrum of the linear operator A consists only of isolated eigenvalues of finite algebraic multiplicity. Moreover, the following lemma tells that for any fixed $L \in (0, \infty)$, there exists at most a finite number of eigenvalues on the imaginary axis.

Lemma 2.1 *For any fixed $L \in (0, \infty)$, the following assertions hold:*

(1) If $L \notin \mathcal{N}$, there are no eigenvalues on the imaginary axis.

(2) If $L \in \mathcal{N}$, there exists a finite number of pairs $\{j_k, l_k\}_{k=1}^{k=n} \subset \mathbb{N}^* \times \mathbb{N}^*$ with $j_k \geqslant l_k$, such that

$$L = 2\pi\sqrt{\frac{j_k^2 + j_k l_k + l_k^2}{3}}.$$

For each $k \in \{1, \cdots, n\}$, there exists a pair of conjugate eigenvalues of A on the imaginary axis and

$$\sigma_p(A) \cap \mathrm{i}\mathbb{R} = \Big\{\lambda = \pm \mathrm{i}q;\ q = \Big(\frac{2\pi}{3L}\Big)^3 (2j_k + l_k)(j_k - l_k)(2l_k + j_k)\Big\}. \tag{2.1}$$

Here $\sigma_p(A)$ is the set of eigenvalues of A, i.e., the set of $\lambda \in \mathbb{C}$ such that there exists $\varphi \in D(A) \setminus \{0\}$ (but complex-valued), such that $A\varphi = \lambda\varphi$.

Proof We have $\lambda \in \sigma_p(A) \cap \mathrm{i}\mathbb{R}$ if and only if there exists $\varphi \in H^3(0, L) \setminus \{0\}$ such that

$$\begin{cases} \lambda\varphi + \varphi' + \varphi''' = 0, \\ \varphi(0) = \varphi(L) = \varphi'(L) = 0. \end{cases} \tag{2.2}$$

By multiplying (2.2) by $\overline{\varphi}$, and then integrating over $[0, L]$, we obtain

$$\lambda \int_0^L \varphi\overline{\varphi}\mathrm{d}x + \int_0^L \varphi'\overline{\varphi}\mathrm{d}x + \int_0^L \varphi'''\overline{\varphi}\mathrm{d}x = 0. \tag{2.3}$$

Taking the real part of (2.3), we have

$$\int_0^L \frac{\varphi'\overline{\varphi} + \overline{\varphi}'\varphi}{2}\,\mathrm{d}x + \int_0^L \frac{\varphi'''\overline{\varphi} + \overline{\varphi}'''\varphi}{2}\,\mathrm{d}x = 0. \tag{2.4}$$

Integrating by parts in (2.4) and using (2.2), we get

$$\varphi'(0) = 0.$$

Hence, $\lambda \in \sigma_p(A) \cap \mathrm{i}\mathbb{R}$ if and only if there exists $\varphi \in H^3(0, L) \setminus \{0\}$ such that

$$\begin{cases} \lambda\varphi + \varphi' + \varphi''' = 0, \\ \varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L) = 0. \end{cases} \tag{2.5}$$

It was proved by Rosier [6, Lemma 3.5] that when $L \notin \mathcal{N}$, there does not exist $\lambda \in \mathbb{C}$, $\varphi \in H^3(0, L) \setminus \{0\}$ such that (2.5) holds, thus we get that there are no eigenvalues of A on the imaginary axis and the first part of Lemma 2.1 follows. Moreover, when $L \in \mathcal{N}$, we get from the proof of [6, Lemma 3.5] that there exist a finite number of eigenvalues of A on the imaginary axis which are given explicitly in (2.1), thus the rest of this lemma follows.

From Lemma 2.1, we get immediately the following result, where $\rho(A)$ is the resolvent set of the closed operator A, i.e., the set of $\lambda \in \mathbb{C}$ for which $\lambda\mathrm{I} - A$ has a bounded inverse, I denoting the identity map.

Lemma 2.2 *There exists $\omega_0 > 0$ such that $\mathrm{i}\omega \in \rho(A)$ whenever $|\omega| \geqslant \omega_0$.*

Concerning the explicit formula for the resolvent of A, we have the following lemma.

Lemma 2.3 *For each $\lambda \in \rho(A)$, $\lambda \neq \pm(\frac{2\sqrt{3}\mathrm{i}}{9})$, denote by p_i, $i = 1, 2, 3$ the three roots of $\lambda + p + p^3 = 0$, and set*

$$H := \begin{pmatrix} 1 & 1 & 1 \\ \mathrm{e}^{p_1 L} & \mathrm{e}^{p_2 L} & \mathrm{e}^{p_3 L} \\ p_1\mathrm{e}^{p_1 L} & p_2\mathrm{e}^{p_2 L} & p_3\mathrm{e}^{p_3 L} \end{pmatrix}.$$

Then we have $p_i \neq p_j$ when $i \neq j$, and

$$\det(H) = (p_3 - p_2)\mathrm{e}^{-p_1 L} + (p_1 - p_3)\mathrm{e}^{-p_2 L} + (p_2 - p_1)\mathrm{e}^{-p_3 L} \neq 0. \tag{2.6}$$

Moreover, the resolvent of A is given by

$$(\lambda\mathrm{I} - A)^{-1}\psi(x) = I_1(x) + I_2(x) + I_3(x), \quad \forall \psi \in L^2(0, L),$$

where

$$\begin{aligned} I_1(x) = \frac{\mathrm{e}^{p_1 x}}{\det(H)} \times \Big\{ &\frac{(p_3 - p_2)\mathrm{e}^{-p_1 L}}{(p_2 - p_1)(p_3 - p_1)} \int_0^x \mathrm{e}^{-p_1 s}\psi(s)\mathrm{d}s + \Big(\frac{\mathrm{e}^{-p_2 L}}{p_2 - p_1} - \frac{\mathrm{e}^{-p_3 L}}{p_3 - p_1}\Big) \\ &\cdot \int_x^L \mathrm{e}^{-p_1 s}\psi(s)\mathrm{d}s - \frac{\mathrm{e}^{-p_1 L}}{p_2 - p_1} \int_0^L \mathrm{e}^{-p_2 s}\psi(s)\mathrm{d}s + \frac{\mathrm{e}^{-p_1 L}}{p_3 - p_1} \int_0^L \mathrm{e}^{-p_3 s}\psi(s)\mathrm{d}s \Big\}, \end{aligned}$$

$$I_2(x) = \frac{e^{p_2 x}}{\det(H)} \times \Big\{ -\frac{(p_1-p_3)e^{-p_2L}}{(p_3-p_2)(p_2-p_1)} \int_0^x e^{-p_2 s}\psi(s)ds - \frac{e^{-p_2 L}}{p_2-p_1}\int_0^L e^{-p_1 s}\psi(s)ds$$
$$+ \Big(\frac{e^{-p_1 L}}{p_2-p_1} + \frac{e^{-p_3 L}}{p_3-p_2}\Big)\int_x^L e^{-p_2 s}\psi(s)ds - \frac{e^{-p_2 L}}{p_3-p_2}\int_0^L e^{-p_3 s}\psi(s)ds\Big\},$$
$$I_3(x) = \frac{e^{p_3 x}}{\det(H)} \times \Big\{ \frac{(p_2-p_1)e^{-p_3L}}{(p_3-p_2)(p_3-p_1)} \int_0^x e^{-p_3 s}\psi(s)ds + \frac{e^{-p_3 L}}{p_3-p_1}\int_0^L e^{-p_1 s}\psi(s)ds$$
$$- \frac{e^{-p_3 L}}{p_3-p_2}\int_0^L e^{-p_2 s}\psi(s)ds + \Big(-\frac{e^{-p_1 L}}{p_3-p_1} + \frac{e^{-p_2 L}}{p_3-p_2}\Big)\int_x^L e^{-p_3 s}\psi(s)ds\Big\}.$$

Proof Suppose that $\lambda \in \rho(A)$. Then for any $\psi \in L^2(0, L)$, there exists a $\varphi \in D(A)$ such that

$$(\lambda I - A)\varphi = \psi,$$

i.e.,

$$\begin{cases} \lambda\varphi + \varphi' + \varphi''' = \psi, \\ \varphi(0) = \varphi(L) = \varphi'(L) = 0. \end{cases} \tag{2.7}$$

We consider the homogeneous differential equation associated with (2.7)

$$\lambda\varphi + \varphi' + \varphi''' = 0. \tag{2.8}$$

The characteristic equation of (2.8) is $\lambda + p + p^3 = 0$. Using our notation, we have

$$\begin{cases} p_1 + p_2 + p_3 = 0, \\ p_1p_2 + p_2p_3 + p_1p_3 = 1, \\ -p_1p_2p_3 = \lambda. \end{cases} \tag{2.9}$$

Note that if $\lambda + p + p^3 = 0$ has multiple root, we can assume without loss of generality that $p_1 = p_2 \neq p_3$. In this case, we can obtain $\lambda = \pm\big(\frac{2\sqrt{3}i}{9}\big)$. Therefore, if $\lambda \neq \pm\big(\frac{2\sqrt{3}i}{9}\big)$, it follows that $p_i \neq p_j$ when $i \neq j$, and consequently, the general solution of (2.8) is given by

$$\varphi(x) = C_1 e^{p_1 x} + C_2 e^{p_2 x} + C_3 e^{p_3 x}.$$

Here and hereafter, we denote by C_1, C_2 and C_3 arbitrary constants. Using the method of variation of constant, the general solution of (2.7) is supposed to be

$$\varphi(x) = C_1(x)e^{p_1 x} + C_2(x)e^{p_2 x} + C_3(x)e^{p_3 x} \tag{2.10}$$

with $C_1(x)$, $C_2(x)$ and $C_3(x)$ satisfying

$$\begin{cases} C_1'(x)e^{p_1 x} + C_2'(x)e^{p_2 x} + C_3'(x)e^{p_3 x} = 0, \\ C_1'(x)(p_1 e^{p_1 x}) + C_2'(x)(p_2 e^{p_2 x}) + C_3'(x)(p_3 e^{p_3 x}) = 0, \\ C_1'(x)(p_1^2 e^{p_1 x}) + C_2'(x)(p_2^2 e^{p_2 x}) + C_3'(x)(p_3^2 e^{p_3 x}) = \psi(x). \end{cases} \tag{2.11}$$

We deduce from (2.11) that

$$C_1(x) = \frac{\displaystyle\int_0^x e^{-p_1 s}\psi(s)ds}{(p_2-p_1)(p_3-p_1)} + C_1, \tag{2.12}$$

$$C_2(x) = -\frac{\int_0^x e^{-p_2 s}\psi(s)ds}{(p_3 - p_2)(p_2 - p_1)} + C_2, \tag{2.13}$$

$$C_3(x) = \frac{\int_0^x e^{-p_3 s}\psi(s)ds}{(p_3 - p_2)(p_3 - p_1)} + C_3. \tag{2.14}$$

We now determine C_1, C_2 and C_3 by using the boundary conditions in (2.7). Using (2.7), together with (2.10) and (2.12)–(2.14) gives

$$H(C_1, C_2, C_3)^{\mathrm{T}} = (0, K_1, K_2)^{\mathrm{T}} \tag{2.15}$$

and

$$K_1 = -\frac{e^{p_1 L}\int_0^L e^{-p_1 x}\psi(x)dx}{(p_2 - p_1)(p_3 - p_1)} + \frac{e^{p_2 L}\int_0^L e^{-p_2 x}\psi(x)dx}{(p_3 - p_2)(p_2 - p_1)} - \frac{e^{p_3 L}\int_0^L e^{-p_3 x}\psi(x)dx}{(p_3 - p_2)(p_3 - p_1)},$$

$$K_2 = -\frac{p_1 e^{p_1 L}\int_0^L e^{-p_1 x}\psi(x)dx}{(p_2 - p_1)(p_3 - p_1)} + \frac{p_2 e^{p_2 L}\int_0^L e^{-p_2 x}\psi(x)dx}{(p_3 - p_2)(p_2 - p_1)} - \frac{p_3 e^{p_3 L}\int_0^L e^{-p_3 x}\psi(x)dx}{(p_3 - p_2)(p_3 - p_1)}.$$

Since $\lambda \in \rho(A)$, it is clear that $\det(H) \neq 0$. Then, we can determine C_1, C_2 and C_3 through (2.15) uniquely. To be more precise, we have

$$\begin{aligned} C_1 = \frac{1}{\det(H)} \times \Big\{ \Big(\frac{e^{-p_2 L}}{p_2 - p_1} - \frac{e^{-p_3 L}}{p_3 - p_1} \Big) \int_0^L e^{-p_1 x}\psi(x)dx - \frac{e^{-p_1 L}}{p_2 - p_1}\int_0^L e^{-p_2 x}\psi(x)dx \\ + \frac{e^{-p_1 L}}{p_3 - p_1}\int_0^L e^{-p_3 x}\psi(x)dx \Big\}, \end{aligned} \tag{2.16}$$

$$\begin{aligned} C_2 = \frac{1}{\det(H)} \times \Big\{ -\frac{e^{-p_2 L}}{p_2 - p_1}\int_0^L e^{-p_1 x}\psi(x)dx + \Big(\frac{e^{-p_1 L}}{p_2 - p_1} + \frac{e^{-p_3 L}}{p_3 - p_2} \Big) \int_0^L e^{-p_2 x}\psi(x)dx \\ - \frac{e^{-p_2 L}}{p_3 - p_2}\int_0^L e^{-p_3 x}\psi(x)dx \Big\}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} C_3 = \frac{1}{\det(H)} \times \Big\{ \frac{e^{-p_3 L}}{p_3 - p_1}\int_0^L e^{-p_1 x}\psi(x)dx - \frac{e^{-p_3 L}}{p_3 - p_2}\int_0^L e^{-p_2 x}\psi(x)dx \\ + \Big(-\frac{e^{-p_1 L}}{p_3 - p_1} + \frac{e^{-p_2 L}}{p_3 - p_2} \Big) \int_0^L e^{-p_3 x}\psi(x)dx \Big\}. \end{aligned} \tag{2.18}$$

Combining (2.10), (2.12)–(2.14) and (2.16)–(2.18), the result of Lemma 2.3 follows directly.

3 Estimation of the Resolvent

It is usually difficult to identify the Gevrey class regularity for a given strongly continuous semigroup with Definition 1.1. In this paper, we will refer to the following sufficient condition for a strongly continuous semigroup to be of Gevrey class. It is based on the estimation for the resolvent of its infinitesimal generator.

Theorem 3.1 (see [8, Theorem 4, p. 153]) *Let $T(t)$ be a strongly continuous semigroup satisfying $\|T(t)\| \leqslant M e^{\nu t}$. Suppose that, for some $\mu \geqslant \nu$ and α satisfying $0 < \alpha \leqslant 1$,*

$$\lim_{|\omega| \to \infty} \sup |\omega|^{\alpha} \|((\mu + i\omega)I - A)^{-1}\| = C < \infty. \tag{3.1}$$

Then $T(t)$ is of Gevrey class δ for $t > 0$, for every $\delta > \frac{1}{\alpha}$.

From Lemma 2.2, we know that when $|\omega|$ is large enough, $\mathrm{i}\omega \in \rho(A)$. Then for each $\psi \in L^2(0, L)$, the explicit formula of $(\mathrm{i}\omega \mathrm{I} - A)^{-1}\psi$ is obtained by Lemma 2.3. From this formula, we can get the following estimate for the resolvent.

Theorem 3.2 *For any fixed $L \in (0, \infty)$, there exist positive constants $\omega_1 > 0$, $M_1 \geqslant M_2 > 0$, such that for all $\omega \in \mathbb{R}$ with $|\omega| \geqslant \omega_1$, the following inequality holds:*

$$\frac{M_2}{|\omega|^{\frac{2}{3}}} \leqslant \|(\mathrm{i}\omega \mathrm{I} - A)^{-1}\|_{L^2(0,L)} \leqslant \frac{M_1}{|\omega|^{\frac{2}{3}}}. \tag{3.2}$$

In particular, using Theorem 3.1 with $M = 1$, $\mu = \nu = 0$ and $\alpha = \frac{2}{3}$, the linear operator A generates a C_0-semigroup of Gevrey class $\delta \in \left(\frac{3}{2}, \infty\right)$ for $t > 0$.

Remark 3.1 From the left-hand side of estimation (3.2), we get that the linear operator A does not generate an analytic semigroup (see [4, Theorem 5.2]).

Remark 3.2 To our knowledge, no equivalence exists between the estimate of the resolvent (3.2) and the Gevrey class regularity. Thus, we were unable to confirm the optimality of the value $\frac{3}{2}$.

Proof of Theorem 3.2 Without loss of generality, we consider the case where $\lambda = \mathrm{i}\omega$ with $\omega > 0$. Similar estimates can be obtained for $\lambda = \mathrm{i}\omega$ with $\omega < 0$. We still denote by p_1, p_2 and p_3 the three roots of

$$p^3 + p + \mathrm{i}\omega = 0. \tag{3.3}$$

Let, for $j \in \{1, 2, 3\}$ and $\omega > 0$,

$$q_j := \frac{p_j}{\omega^{\frac{1}{3}}}. \tag{3.4}$$

Then q_j is the solution of

$$\mathrm{i} + q_j^3 + \frac{q_j}{\omega^{\frac{2}{3}}} = 0. \tag{3.5}$$

Applying the implicit function theorem to (3.5), we get that there exist $\omega_1 > 0$ and $C(\omega_1) > 0$ such that

$$|q_j - \widetilde{q}_j| \leqslant \frac{C(\omega_1)}{\omega^{\frac{2}{3}}} \quad \text{for all } \omega > \omega_1, \tag{3.6}$$

where $\widetilde{q}_i$ solves $\mathrm{i} + \widetilde{q}_i^3 = 0$ and

$$\widetilde{q}_1 := \mathrm{i}, \quad \widetilde{q}_2 := \mathrm{e}^{\frac{7\mathrm{i}\pi}{6}}, \quad \widetilde{q}_3 := \mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}. \tag{3.7}$$

By (3.4) and (3.6)–(3.7), we get

$$|\operatorname{Re} p_1| \leqslant \frac{C(\omega_1)}{\omega^{\frac{1}{3}}}, \tag{3.8}$$

and for $\omega > \omega_1$ large enough that

$$\operatorname{Re} p_2 < 0, \quad \operatorname{Re} p_3 > 0. \tag{3.9}$$

Moreover, from (3.4) and (3.6)–(3.7), we obtain that for $i \neq j$,

$$\frac{\sqrt{3}}{2}\omega^{\frac{1}{3}} - \frac{2C(\omega_1)}{\omega^{\frac{1}{3}}} \leqslant |p_i - p_j| \leqslant 2\omega^{\frac{1}{3}} + \frac{2C(\omega_1)}{\omega^{\frac{1}{3}}}. \tag{3.10}$$

Next, we estimate the norm of the resolvent $(\mathrm{i}\omega \mathrm{I} - A)^{-1}$. Since for each $\psi \in L^2(0, L)$,

$$(\mathrm{i}\omega \mathrm{I} - A)^{-1}\psi = I_1 + I_2 + I_3 \tag{3.11}$$

with I_1, I_2 and I_3 defined in Lemma 2.3, we have

$$\|(\mathrm{i}\omega \mathrm{I} - A)^{-1}\psi\|_{L^2(0,L)} \leqslant \|I_1\|_{L^2(0,L)} + \|I_2\|_{L^2(0,L)} + \|I_3\|_{L^2(0,L)}. \tag{3.12}$$

By the expression of I_1 in Lemma 2.3, noticing (3.8) and (3.9), we obtain through Hölder's inequality that

$$\begin{aligned}\int_0^L |I_1(x)|^2 \mathrm{d}x = {} & \frac{1}{|\det(H)|^2} \times \int_0^L |\mathrm{e}^{2p_1x}| \cdot \Big| \frac{(p_3 - p_2)\mathrm{e}^{-p_1L}}{(p_2 - p_1)(p_3 - p_1)} \int_0^x \mathrm{e}^{-p_1s}\psi(s)\mathrm{d}s \\ & + \Big(\frac{\mathrm{e}^{-p_2L}}{p_2 - p_1} - \frac{\mathrm{e}^{-p_3L}}{p_3 - p_1}\Big) \int_x^L \mathrm{e}^{-p_1s}\psi(s)\mathrm{d}s - \frac{\mathrm{e}^{-p_1L}}{p_2 - p_1} \int_0^L \mathrm{e}^{-p_2x}\psi(x)\mathrm{d}x \\ & + \frac{\mathrm{e}^{-p_1L}}{p_3 - p_1} \int_0^L \mathrm{e}^{-p_3x}\psi(x)\mathrm{d}x\Big|^2 \mathrm{d}x \\ \leqslant {} & \frac{\Theta\|\psi\|^2_{L^2(0,L)}}{|\det(H)|^2} \times \Big(\Big|\frac{p_3 - p_2}{(p_2 - p_1)(p_3 - p_1)}\Big|^2 + \Big|\frac{\mathrm{e}^{-p_2L}}{p_2 - p_1} - \frac{\mathrm{e}^{-p_3L}}{p_3 - p_1}\Big|^2 \\ & + \Big|\frac{\mathrm{e}^{-p_2L}}{p_2 - p_1}\Big|^2 + \Big|\frac{1}{p_3 - p_1}\Big|^2\Big).\end{aligned} \tag{3.13}$$

Here and hereafter, Θ denotes various positive constants which may depend on L and ω_1, but do not depend on ψ and ω. We obtain from (3.8) to (3.10) and the expression of $\det(H)$ in (2.6) that for ω large enough

$$\int_0^L |I_1(x)|^2 \mathrm{d}x \leqslant \frac{\Theta\|\psi\|^2_{L^2(0,L)}}{\omega^{\frac{4}{3}}}. \tag{3.14}$$

Similarly, by the expression of I_2 in Lemma 2.3, we have

$$\begin{aligned}\int_0^L |I_2(x)|^2 \mathrm{d}x \leqslant {} & \frac{4}{|\det(H)|^2} \times \Big\{\Big| - \frac{(p_1 - p_3)\mathrm{e}^{-p_2L}}{(p_3 - p_2)(p_2 - p_1)}\Big|^2 \int_0^L \Big|\int_0^x \mathrm{e}^{p_2(x-s)}\psi(s)\mathrm{d}s\Big|^2 \mathrm{d}x \\ & + \Big|\frac{\mathrm{e}^{-p_2L}}{p_2 - p_1} \int_0^L \mathrm{e}^{-p_1x}\psi(x)\mathrm{d}x\Big|^2 \int_0^L |\mathrm{e}^{p_2x}|^2 \mathrm{d}x \\ & + \Big|\frac{\mathrm{e}^{-p_1L}}{p_2 - p_1} + \frac{\mathrm{e}^{-p_3L}}{p_3 - p_2}\Big|^2 \int_0^L \Big|\int_x^L \mathrm{e}^{p_2(x-s)}\psi(s)\mathrm{d}s\Big|^2 \mathrm{d}x \\ & + \Big|\frac{\mathrm{e}^{-p_2L}}{p_3 - p_2} \int_0^L \mathrm{e}^{-p_3x}\psi(x)\mathrm{d}x\Big|^2 \int_0^L |\mathrm{e}^{p_2x}|^2 \mathrm{d}x\Big\}.\end{aligned} \tag{3.15}$$

For the estimation of the first term in (3.15), using Hölder's inequality and (3.9), we have

$$\int_0^L \Big|\int_0^x \mathrm{e}^{p_2(x-s)}\psi(s)\mathrm{d}s\Big|^2 \mathrm{d}x \leqslant \int_0^L \Big(\int_0^x \mathrm{e}^{\mathrm{Re}\, p_2(x-s)}|\psi(s)|\mathrm{d}s\Big)^2 \mathrm{d}x$$

$$\begin{aligned}
&\leqslant \int_0^L \Big(\int_0^x \mathrm{e}^{2\operatorname{Re} p_2(x-s)}\mathrm{d}s\Big)\Big(\int_0^x |\psi(s)|^2\mathrm{d}s\Big)\mathrm{d}x \\
&= \int_0^L \frac{\mathrm{e}^{2\operatorname{Re} p_2 x}-1}{2\operatorname{Re} p_2}\Big(\int_0^x |\psi(s)|^2\mathrm{d}s\Big)\mathrm{d}x \\
&\leqslant \frac{L}{-2\operatorname{Re} p_2}\|\psi\|^2_{L^2(0,L)}.
\end{aligned} \tag{3.16}$$

By (3.9), we get

$$\int_0^L |\mathrm{e}^{p_2 x}|^2\mathrm{d}x = \frac{\mathrm{e}^{2\operatorname{Re} p_2 L}-1}{2\operatorname{Re} p_2} < -\frac{1}{2\operatorname{Re} p_2}. \tag{3.17}$$

For the estimation of the third term in (3.15), we obtain through Hölder's inequality and (3.9) that

$$\begin{aligned}
\int_0^L \Big|\int_x^L \mathrm{e}^{p_2(x-s)}\psi(s)\mathrm{d}s\Big|^2\mathrm{d}x &\leqslant \int_0^L \Big(\int_x^L \mathrm{e}^{\operatorname{Re} p_2(x-s)}|\psi(s)|\mathrm{d}s\Big)^2\mathrm{d}x \\
&\leqslant \int_0^L \Big(\int_x^L \mathrm{e}^{2\operatorname{Re} p_2(x-s)}\mathrm{d}s\Big)\Big(\int_x^L |\psi(s)|^2\mathrm{d}s\Big)\mathrm{d}x \\
&= \int_0^L \Big(\frac{\mathrm{e}^{2\operatorname{Re} p_2(x-L)}-1}{-2\operatorname{Re} p_2}\Big)\Big(\int_x^L |\psi(s)|^2\mathrm{d}s\Big)\mathrm{d}x \\
&\leqslant \int_0^L \Big(\frac{\mathrm{e}^{2\operatorname{Re} p_2(x-L)}-1}{-2\operatorname{Re} p_2}\Big)\mathrm{d}x\Big(\int_0^L |\psi(s)|^2\mathrm{d}s\Big) \\
&= \frac{\mathrm{e}^{-2\operatorname{Re} p_2 L}+2\operatorname{Re} p_2 L-1}{(2\operatorname{Re} p_2)^2}\|\psi\|^2_{L^2(0,L)} \\
&< \frac{\mathrm{e}^{-2\operatorname{Re} p_2 L}}{(2\operatorname{Re} p_2)^2}\|\psi\|^2_{L^2(0,L)}.
\end{aligned} \tag{3.18}$$

Combining (3.15) to (3.18), by (3.8)–(3.10) and the expression of $\det(H)$ in (2.6), we deduce that

$$\begin{aligned}
&\int_0^L |I_2(x)|^2\mathrm{d}x \\
&\leqslant \frac{\Theta\|\psi\|^2_{L^2(0,L)}}{|\det(H)|^2}\times\Big\{\Big|-\frac{(p_1-p_3)\mathrm{e}^{-p_2 L}}{(p_3-p_2)(p_2-p_1)}\Big|^2\frac{L}{-2\operatorname{Re} p_2}+\Big|\frac{\mathrm{e}^{-p_2 L}}{p_2-p_1}\Big|^2\frac{L}{-2\operatorname{Re} p_2} \\
&\quad+\Big|\frac{\mathrm{e}^{-p_1 L}}{p_2-p_1}+\frac{\mathrm{e}^{-p_3 L}}{p_3-p_2}\Big|^2\frac{\mathrm{e}^{-2\operatorname{Re} p_2 L}}{(2\operatorname{Re} p_2)^2}+\Big|\frac{\mathrm{e}^{-p_2 L}}{p_3-p_2}\Big|^2\frac{1}{-2\operatorname{Re} p_2}\int_0^L \mathrm{e}^{-2\operatorname{Re} p_3 x}\mathrm{d}x\Big\} \\
&= \frac{\Theta\|\psi\|^2_{L^2(0,L)}\mathrm{e}^{-2\operatorname{Re} p_2 L}}{|\det(H)|^2}\times\Big\{\Big|\frac{p_1-p_3}{(p_3-p_2)(p_2-p_1)}\Big|^2\frac{L}{-2\operatorname{Re} p_2}+\Big|\frac{1}{p_2-p_1}\Big|^2\Big(\frac{L}{-2\operatorname{Re} p_2}\Big) \\
&\quad+\Big|\frac{\mathrm{e}^{-p_1 L}}{p_2-p_1}+\frac{\mathrm{e}^{-p_3 L}}{p_3-p_2}\Big|^2\frac{1}{(2\operatorname{Re} p_2)^2}+\Big|\frac{1}{p_3-p_2}\Big|^2\frac{1}{-2\operatorname{Re} p_2}\frac{\mathrm{e}^{-2\operatorname{Re} p_3 L}-1}{-2\operatorname{Re} p_3}\Big\} \\
&\leqslant \frac{\Theta\|\psi\|^2_{L^2(0,L)}\mathrm{e}^{-2\operatorname{Re} p_2 L}}{|\det(H)|^2}\times\Big\{\Big|\frac{p_1-p_3}{(p_3-p_2)(p_2-p_1)}\Big|^2\frac{L}{-2\operatorname{Re} p_2}+\Big|\frac{1}{p_2-p_1}\Big|^2\Big(\frac{L}{-2\operatorname{Re} p_2}\Big) \\
&\quad+2\Big(\Big|\frac{1}{p_2-p_1}\Big|^2+\Big|\frac{1}{p_3-p_2}\Big|^2\Big)\frac{1}{(2\operatorname{Re} p_2)^2}+\Big|\frac{1}{p_3-p_2}\Big|^2\frac{1}{-2\operatorname{Re} p_2}\frac{1}{2\operatorname{Re} p_3}\Big\} \\
&\leqslant \frac{\Theta\|\psi\|^2_{L^2(0,L)}}{\omega^{\frac{5}{3}}}.
\end{aligned} \tag{3.19}$$

Let us emphasize that, from (3.9), $\operatorname{Re} p_2 < 0$ while $\operatorname{Re} p_3 > 0$. Therefore we cannot deduce the estimates for $\|I_3\|_{L^2(0,L)}$ from the estimates for $\|I_2\|_{L^2(0,L)}$ directly. Thus, we have to directly estimate $\|I_3\|_{L^2(0,L)}$. By the expression of I_3 in Lemma 2.3, we have

$$\begin{aligned}\int_0^L |I_3(x)|^2 \mathrm{d}x \leqslant \frac{4}{|\det(H)|^2} \times \Big\{ &\Big|\frac{(p_2-p_1)\mathrm{e}^{-p_3 L}}{(p_3-p_2)(p_3-p_1)}\Big|^2 \int_0^L \Big|\int_0^x \mathrm{e}^{p_3(x-s)}\psi(s)\mathrm{d}s\Big|^2 \mathrm{d}x \\ &+ \Big|\frac{1}{p_3-p_1}\int_0^L \mathrm{e}^{-p_1 x}\psi(x)\mathrm{d}x\Big|^2 \int_0^L |\mathrm{e}^{-p_3(L-x)}|^2 \mathrm{d}x \\ &+ \Big|-\frac{1}{p_3-p_2}\int_0^L \mathrm{e}^{-p_2 x}\psi(x)\mathrm{d}x\Big|^2 \int_0^L |\mathrm{e}^{-p_3(L-x)}|^2 \mathrm{d}x \\ &+ \Big|\Big(-\frac{\mathrm{e}^{-p_1 L}}{p_3-p_1} + \frac{\mathrm{e}^{-p_2 L}}{p_3-p_2}\Big)\Big|^2 \int_0^L \Big|\int_x^L \mathrm{e}^{p_3(x-s)}\psi(s)\mathrm{d}s\Big|^2 \mathrm{d}x\Big\}. \qquad (3.20)\end{aligned}$$

For the estimation of the first term in (3.20), noticing (3.9), we get through Hölder's inequality that

$$\begin{aligned}\int_0^L \Big|\int_0^x \mathrm{e}^{p_3(x-s)}\psi(s)\mathrm{d}s\Big|^2 \mathrm{d}x &\leqslant \int_0^L \Big(\int_0^x \mathrm{e}^{\operatorname{Re} p_3(x-s)}|\psi(s)|\mathrm{d}s\Big)^2 \mathrm{d}x \\ &\leqslant \int_0^L \Big(\int_0^x \mathrm{e}^{2\operatorname{Re} p_3(x-s)}\mathrm{d}s\Big)\Big(\int_0^x |\psi(s)|^2\mathrm{d}s\Big)\mathrm{d}x \\ &\leqslant \frac{1}{-2\operatorname{Re} p_3}\Big(\int_0^L (1-\mathrm{e}^{2\operatorname{Re} p_3 x})\mathrm{d}x\Big)\Big(\int_0^L |\psi(s)|^2\mathrm{d}s\Big) \\ &= \frac{1}{-2\operatorname{Re} p_3}\Big(L - \frac{1}{2\operatorname{Re} p_3}(\mathrm{e}^{2\operatorname{Re} p_3 L}-1)\Big)\|\psi\|^2_{L^2(0,L)} \\ &< \frac{\mathrm{e}^{2\operatorname{Re} p_3 L}}{(2\operatorname{Re} p_3)^2}\|\psi\|^2_{L^2(0,L)}. \qquad (3.21)\end{aligned}$$

By (3.9), we get

$$\int_0^L |\mathrm{e}^{-p_3(L-x)}|^2 \mathrm{d}x = \frac{1-\mathrm{e}^{-2\operatorname{Re} p_3 L}}{2\operatorname{Re} p_3} < \frac{1}{2\operatorname{Re} p_3}. \qquad (3.22)$$

For the estimation of the last term in (3.20), we get through Hölder's inequality and (3.9) that

$$\begin{aligned}\int_0^L \Big|\int_x^L \mathrm{e}^{p_3(x-s)}\psi(s)\mathrm{d}s\Big|^2 \mathrm{d}x &\leqslant \int_0^L \Big(\int_x^L \mathrm{e}^{\operatorname{Re} p_3(x-s)}|\psi(s)|\mathrm{d}s\Big)^2 \mathrm{d}x \\ &= \int_0^L \Big(\int_x^L \mathrm{e}^{2\operatorname{Re} p_3(x-s)}\mathrm{d}s\Big)\Big(\int_x^L |\psi(s)|^2\mathrm{d}s\Big)\mathrm{d}x \\ &\leqslant \frac{1}{-2\operatorname{Re} p_3}\int_0^L (\mathrm{e}^{2\operatorname{Re} p_3(x-L)}-1)\mathrm{d}x\Big(\int_0^L |\psi(s)|^2\mathrm{d}s\Big) \\ &= \frac{1}{-2\operatorname{Re} p_3}\Big(\frac{1-\mathrm{e}^{-2\operatorname{Re} p_3 L}}{2\operatorname{Re} p_3} - L\Big)\|\psi\|^2_{L^2(0,L)} \\ &\leqslant \frac{L}{2\operatorname{Re} p_3}\|\psi\|^2_{L^2(0,L)}. \qquad (3.23)\end{aligned}$$

Combining (3.20) to (3.23), we obtain through (3.8) to (3.10) that

$$\int_0^L |I_3(x)|^2 \mathrm{d}x$$

$$
\begin{aligned}
&\leqslant \frac{4\|\psi\|_{L^2(0,L)}^2}{|\det(H)|^2}\times\Big\{\Big|\frac{(p_2-p_1)\mathrm{e}^{-p_3L}}{(p_3-p_2)(p_3-p_1)}\Big|^2\frac{\mathrm{e}^{2\operatorname{Re}p_3L}}{(2\operatorname{Re}p_3)^2}+\Big|\frac{1}{p_3-p_1}\Big|^2\Big(\int_0^L\mathrm{e}^{-2\operatorname{Re}p_1x}\mathrm{d}x\Big)\frac{1}{2\operatorname{Re}p_3}\\
&\quad+\Big|-\frac{1}{p_3-p_2}\Big|^2\Big(\int_0^L\mathrm{e}^{-2\operatorname{Re}p_2x}\mathrm{d}x\Big)\frac{1}{2\operatorname{Re}p_3}\\
&\quad+\Big|\Big(-\frac{\mathrm{e}^{-p_1L}}{p_3-p_1}+\frac{\mathrm{e}^{-p_2L}}{p_3-p_2}\Big)\Big|^2\frac{L}{2\operatorname{Re}p_3}\Big\}\\
&\leqslant \frac{\Theta\|\psi\|_{L^2(0,L)}^2}{|\det(H)|^2}\times\Big\{\Big|\frac{(p_2-p_1)\mathrm{e}^{-p_3L}}{(p_3-p_2)(p_3-p_1)}\Big|^2\frac{\mathrm{e}^{2\operatorname{Re}p_3L}}{(2\operatorname{Re}p_3)^2}+\Big|\frac{1}{p_3-p_1}\Big|^2\frac{L}{2\operatorname{Re}p_3}\\
&\quad+\Big|-\frac{1}{p_3-p_2}\Big|^2\frac{\mathrm{e}^{-2\operatorname{Re}p_2L}-1}{-2\operatorname{Re}p_2}\frac{1}{2\operatorname{Re}p_3}+\Big|\Big(-\frac{\mathrm{e}^{-p_1L}}{p_3-p_1}+\frac{\mathrm{e}^{-p_2L}}{p_3-p_2}\Big)\Big|^2\frac{L}{2\operatorname{Re}p_3}\Big\}\\
&\leqslant \frac{\Theta\|\psi\|_{L^2(0,L)}^2}{\omega^{\frac{5}{3}}}.
\end{aligned}
\tag{3.24}
$$

Using (3.12), (3.14), (3.19) and (3.24), the right-hand side of estimation (3.2) follows.

In order to obtain the left-hand side of estimation (3.2), let $\psi(x)=\mathrm{e}^{p_1x}$ and we get

$$
\begin{aligned}
I_1&=\frac{\mathrm{e}^{p_1x}}{\det(H)}\times\Big\{\frac{(p_3-p_2)\mathrm{e}^{-p_1L}}{(p_2-p_1)(p_3-p_1)}x+\Big(\frac{\mathrm{e}^{-p_2L}}{p_2-p_1}-\frac{\mathrm{e}^{-p_3L}}{p_3-p_1}\Big)(L-x)\\
&\quad+\frac{\mathrm{e}^{-p_1L}}{(p_1-p_2)^2}(\mathrm{e}^{(p_1-p_2)L}-1)-\frac{\mathrm{e}^{-p_1L}}{(p_1-p_3)^2}(\mathrm{e}^{(p_1-p_3)L}-1)\Big\}\\
&=\frac{\mathrm{e}^{p_1x}}{\det(H)}\times\Big\{\frac{\mathrm{e}^{-p_2L}}{(p_2-p_1)}(L-x)+\frac{\mathrm{e}^{-p_2L}}{(p_1-p_2)^2}+\frac{(p_3-p_2)\mathrm{e}^{-p_1L}}{(p_2-p_1)(p_3-p_1)}x\\
&\quad-\frac{\mathrm{e}^{-p_3L}}{p_3-p_1}(L-x)-\frac{\mathrm{e}^{-p_1L}}{(p_1-p_2)^2}-\frac{\mathrm{e}^{-p_1L}}{(p_1-p_3)^2}(\mathrm{e}^{(p_1-p_3)L}-1)\Big\}\\
&:=E(x)+D(x),
\end{aligned}
\tag{3.25}
$$

where

$$
E(x):=\frac{\mathrm{e}^{p_1x}}{\det(H)}\Big(\frac{\mathrm{e}^{-p_2L}}{p_2-p_1}(L-x)\Big)
$$

and

$$
\begin{aligned}
D(x)&:=\frac{\mathrm{e}^{p_1x}}{\det(H)}\Big\{\frac{\mathrm{e}^{-p_2L}}{(p_1-p_2)^2}+\frac{(p_3-p_2)\mathrm{e}^{-p_1L}}{(p_2-p_1)(p_3-p_1)}x-\frac{\mathrm{e}^{-p_3L}}{(p_3-p_1)}(L-x)\\
&\quad-\frac{\mathrm{e}^{-p_1L}}{(p_1-p_2)^2}-\frac{\mathrm{e}^{-p_1L}}{(p_1-p_3)^2}(\mathrm{e}^{(p_1-p_3)L}-1)\Big\}.
\end{aligned}
$$

Moreover, by (3.9)–(3.10) and the expression of $\det(H)$ in (2.6), we have

$$
\begin{aligned}
\int_0^L|E(x)|^2\mathrm{d}x&\geqslant\frac{\Theta}{|\det(H)|^2}\times\int_0^L\Big|\frac{\mathrm{e}^{-p_2L}}{p_2-p_1}(L-x)\Big|^2\mathrm{d}x\\
&=\frac{\Theta\mathrm{e}^{-2\operatorname{Re}p_2L}}{|p_1-p_2|^2|\det(H)|^2}\times\int_0^L(L-x)^2\mathrm{d}x\\
&=\frac{\Theta\mathrm{e}^{-2\operatorname{Re}p_2L}}{|p_1-p_2|^2|\det(H)|^2}\times\frac{L^3}{3}\\
&\geqslant\frac{\Theta}{\omega^{\frac{4}{3}}}.
\end{aligned}
\tag{3.26}
$$

Using again (3.8) to (3.10) and the expression of $\det(H)$ in (2.6), we get

$$\begin{aligned}
&\int_0^L |D(x)|^2 \mathrm{d}x \\
&\leqslant \frac{\Theta}{|\det(H)|^2} \times \Big\{ \int_0^L \Big| \frac{\mathrm{e}^{-p_2 L}}{(p_1-p_2)^2} \Big|^2 \mathrm{d}x + \int_0^L \Big| \frac{(p_3-p_2)\mathrm{e}^{-p_1 L}}{(p_2-p_1)(p_3-p_1)} x \Big|^2 \mathrm{d}x \\
&\quad + \int_0^L \Big| -\frac{\mathrm{e}^{-p_3 L}}{(p_3-p_1)} (L-x) \Big|^2 \mathrm{d}x + \int_0^L \Big| -\frac{\mathrm{e}^{-p_1 L}}{(p_1-p_2)^2} \Big|^2 \mathrm{d}x \\
&\quad + \int_0^L \Big| -\frac{\mathrm{e}^{-p_1 L}}{(p_1-p_3)^2} (\mathrm{e}^{(p_1-p_3)L} - 1) \Big|^2 \mathrm{d}x \Big\} \\
&\leqslant \frac{\Theta}{|\det(H)|^2} \times \Big\{ \Big| \frac{\mathrm{e}^{-p_2 L}}{(p_1-p_2)^2} \Big|^2 + \Big| \frac{p_3-p_2}{(p_2-p_1)(p_3-p_1)} \Big|^2 + \Big| \frac{\mathrm{e}^{-p_3 L}}{p_3-p_1} \Big|^2 \\
&\quad + \Big| \frac{1}{(p_1-p_2)^2} \Big|^2 + \Big| \frac{\mathrm{e}^{(p_1-p_3)L} - 1}{(p_1-p_3)^2} \Big|^2 \Big\} \\
&\leqslant \frac{\Theta}{\omega^2}.
\end{aligned} \tag{3.27}$$

Combining (3.11), (3.19), (3.24) to (3.27), and noticing

$$\begin{aligned}
\|(\mathrm{i}\omega \mathrm{I} - A)^{-1}\psi\|_{L^2(0,L)} &= \|E(x) + D(x) + I_2(x) + I_3(x)\|_{L^2(0,L)} \\
&\geqslant \|E(x)\|_{L^2(0,L)} - \|D(x)\|_{L^2(0,L)} - \|I_2(x)\|_{L^2(0,L)} - \|I_3(x)\|_{L^2(0,L)},
\end{aligned}$$

we get that the left-hand side of estimation (3.2) holds. The proof of Theorem 3.2 is completed.

Acknowledgement The authors thank Bingyu Zhang for his interesting comments and many valuable suggestions on this work.

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