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Exact Boundary Synchronization for a Coupled System of Wave Equations with Neumann Boundary Controls*

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(*Dedicated to Philippe G. Ciarlet on the occasion of his* 80*th birthday*)

Abstract In this paper, for a coupled system of wave equations with Neumann boundary controls, the exact boundary synchronization is taken into consideration. Results are then extended to the case of synchronization by groups. Moreover, the determination of the state of synchronization by groups is discussed with details for the synchronization and for the synchronization by 3-groups, respectively.

Keywords Exact boundary synchronization, Exact boundary synchronization by groups, State of synchronization, State of synchronization by groups, Coupled system of wave equations, Neumann boundary control
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1 Introduction

Synchronization is a widespread natural phenomenon. It was first observed by Huygens in 1665 (see [3]). The theoretical research on synchronization phenomenon from the mathematical point of view dates back to Wiener in 1950s (see [13]). However, almost all the previous works focused on systems described by ODEs, and studied the asymptotic synchronization of the states of the system as $t \to +\infty$. For coupled systems governed by PDEs, as shown by Li and Rao, synchronization can be realized in a limited time period by means of proper boundary controls, and after switching off all the controls, the state of synchronization remains. Precisely speaking, Li and Rao considered the exact boundary synchronization for a coupled system of wave equations with Dirichlet boundary controls in any given space dimensions in the framework of weak solutions (see [4, 6]), and acquired related results for the same system in one space dimension with all kinds of boundary controls in the framework of classical solutions (see [2, 9]). Moreover, they got also corresponding results on the exact boundary synchronization by groups in [5, 7].

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In this paper, we consider the following coupled system of wave equations with Neumann boundary controls:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } (0,+\infty)\times\Omega, \\ U = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ \partial_\nu U = DH & \text{on } (0,+\infty)\times\Gamma_1 \end{cases} \tag{1.1}$$

and the corresponding initial data

$$t = 0: \quad U = \widehat{U}_0, \quad U' = \widehat{U}_1, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset$ and $\text{mes}(\Gamma_0) > 0$, ∂_ν denotes the outward normal derivative on the boundary, the coupling matrix $A = (a_{ij})$ is of order N, the boundary control matrix D is a full column-rank matrix of order $N \times M$ $(M \le N)$, both A and D have real constant elements, $U = (u^{(1)}, \cdots, u^{(N)})^{\mathrm{T}}$ and $H = (h^{(1)}, \cdots, h^{(M)})^{\mathrm{T}}$ denote the state variables and the boundary controls, respectively.

Denote

$$\mathcal{H}_0 = L^2(\Omega), \quad \mathcal{H}_1 = H^1_{\Gamma_0}(\Omega), \quad \mathcal{L} = L^2_{\text{loc}}(0,+\infty; L^2(\Gamma_1)), \tag{1.3}$$

where $H^1_{\Gamma_0}(\Omega)$ is the subspace of $H^1(\Omega)$, composed of all the functions with the null trace on Γ_0.

We assume that Ω satisfies the usual multiplier geometric control condition (see [1]). Without loss of generality, we assume that there exists an $x_0 \in \mathbb{R}^n$, such that setting $m = x - x_0$, we have

$$(m,\nu) \le 0, \quad \forall x \in \Gamma_0; \quad (m,\nu) > 0, \quad \forall x \in \Gamma_1, \tag{1.4}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^n$.

Define the linear unbounded operator $-\Delta$ in $\mathcal{H}_0$ by

$$D(-\Delta) = \{\Phi \in H^2(\Omega): \ \Phi|_{\Gamma_0} = 0, \ \partial_\nu \Phi|_{\Gamma_1} = 0\}.$$

Clearly, $-\Delta$ is a positively definite self-adjoint operator with a compact resolvent. Then, for any given $s \in \mathbb{R}$, we can define the operator $(-\Delta)^{\frac{s}{2}}$ with the domain

$$\mathcal{H}_s = D((-\Delta)^{\frac{s}{2}}),$$

which, endowed with the norm $\|\Phi\|_s = \|(-\Delta)^{\frac{s}{2}}\Phi\|_{L^2(\Omega)}$ constitutes a Hilbert space, and its dual space is $\mathcal{H}'_s = \mathcal{H}_{-s}$. In particular, we have (see [11])

$$\mathcal{H}_1 = D(\sqrt{-\Delta}) = \{\Phi \in H^1(\Omega): \ \Phi|_{\Gamma_0} = 0\}.$$

Lemma 1.1 (see [8]) *For any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_{1-s} \times \mathcal{H}_{-s})^N$ with $s > \frac{1}{2}$, and any given boundary function $H \in \mathcal{L}^M$, the mixed initial-boundary value problem* (1.1)–(1.2) *admits a unique weak solution $U \in (C_{\text{loc}}([0,+\infty); \mathcal{H}_{1-s}))^N \cap (C^1_{\text{loc}}([0,+\infty); \mathcal{H}_{-s}))^N$ with continuous dependance.*

Definition 1.1 *System* (1.1) *is exactly null controllable at the time $T > 0$ in the space $(\mathcal{H}_1 \times \mathcal{H}_0)^N$, if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$, there exists a boundary* *control* $H \in \mathcal{L}^M$ *with compact support in* $[0, T]$*, such that the corresponding mixed initial-boundary value problem* (1.1)–(1.2) *admits a unique weak solution* $U \in (C_{\rm loc}([0, +\infty); \mathcal{H}_{1-s}))^N \cap (C^1_{\rm loc}([0, +\infty); \mathcal{H}_{-s}))^N$ *with* $s > \frac{1}{2}$*, satisfying*

$$t \geq T: \quad U = U' \equiv 0. \tag{1.5}$$

Moreover, we have the continuous dependence:

$$\|H\|_{\mathcal{L}^M} \leq c\|(\widehat{U}_0, \widehat{U}_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^N}, \tag{1.6}$$

where c is a positive constant.

For the exact boundary null controllability and the non-exact boundary null controllability of system (1.1), the following results were proved in [8].

Lemma 1.2 *When* $M = N$*, there exists a constant* $T > 0$*, such that system* (1.1) *is exactly null controllable at the time* T *for any given initial data* $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$*.*

However, if there is a lack of boundary controls, we have the following lemma.

Lemma 1.3 *When* $M < N$*, no matter how large* $T > 0$ *is, system* (1.1) *is not exactly null controllable at the time* T *for any given initial data* $(\widehat{U}_0, \widehat{U}_1) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$*.*

Therefore, it is necessary to discuss whether system (1.1) is controllable in some weaker senses when there is a lack of boundary controls, namely, when $M < N$. Although the results are similar to those for the coupled system of wave equations with Dirichlet boundary controls, since the solution to a coupled system of wave equations with Neumann boundary conditions has a relatively weaker regularity, in order to realize the desired result, we need stronger function spaces, and the corresponding adjoint problem is also different.

We first give the following lemma (see [10]).

Lemma 1.4 *For any given* $N \times N$ *matrix* A *and any given full row-rank* $(N-p) \times N$ *matrix* C *with* $1 \leq p < N$*, the following properties are equivalent:*

(1) $\mathrm{Ker}(C)$ *is an invariant subspace of* A:

$$A\mathrm{Ker}(C) \subseteq \mathrm{Ker}(C). \tag{1.7}$$

(2) *There exists a unique matrix* $\overline{A}$ *of order* $(N-p)$*, such that*

$$CA = \overline{A}C. \tag{1.8}$$

Moreover, the matrix $\overline{A}$ *is given by*

$$\overline{A} = CAC^+, \tag{1.9}$$

where C^+ *denotes the Moore-Penrose inverse of* C*:*

$$C^+ = C^{\rm T}(CC^{\rm T})^{-1}. \tag{1.10}$$

Since Lemma 2.1 in [7] is independent of the type of boundary conditions, we still have the following lemma.

Lemma 1.5 *Assume that U is the solution to the mixed problem* (1.1)–(1.2). *Let C be a full row-rank $(N-p)\times N$ matrix* (*where $1\le p<N$*) *such that*

$$t\ge T:\quad CU=0\quad in\ \Omega. \tag{1.11}$$

Then we have either

$$A\mathrm{Ker}(C)\subseteq \mathrm{Ker}(C) \tag{1.12}$$

or there exists a full row-rank $(N-p+1)\times N$ matrix $\widehat{C}$ such that

$$t\ge T:\quad \widehat{C}U=0\quad in\ \Omega. \tag{1.13}$$

2 Exact Boundary Synchronization

Definition 2.1 *System* (1.1) *is exactly synchronizable at the time $T>0$ in the space $(\mathcal{H}_1\times\mathcal{H}_0)^N$, if for any given initial data $(U_0,U_1)\in(\mathcal{H}_1\times\mathcal{H}_0)^N$, there exists a boundary control $H\in\mathcal{L}^M$ with compact support in $[0,T]$, such that the weak solution $U=U(t,x)$ to the mixed initial-boundary value problem* (1.1)–(1.2) *satisfies*

$$t\ge T:\quad u^{(1)}(t,x)\equiv\cdots\equiv u^{(N)}(t,x)\overset{\text{def.}}{=}u(t,x), \tag{2.1}$$

where, $u=u(t,x)$, being unknown a priori, is called the corresponding state of synchronization.

The above definition requires that system (1.1) maintains the state of synchronization even though the boundary control is canceled after the time T.

Theorem 2.1 *Assume that $M<N$. If system* (1.1) *is exactly synchronizable in the space $(\mathcal{H}_1\times\mathcal{H}_0)^N$, then the coupling matrix $A=(a_{ij})$ should satisfy the following condition of compatibility* (*the sums of elements in every row are equal to each other*):

$$\sum_{j=1}^{N}a_{ij}\overset{\text{def.}}{=}a, \tag{2.2}$$

where a is a constant independent of $i=1,\cdots,N$.

Proof By Lemma 1.3, since $M<N$, system (1.1) is not exactly null controllable, then there exists an initial data $(U_0,U_1)\in(\mathcal{H}_1\times\mathcal{H}_0)^N$, such that for any given boundary control H, the corresponding state of synchronization $u(t,x)\not\equiv 0$. Then, noting (2.1), the solution to problem (1.1) corresponding to this initial data satisfies

$$u''-\Delta u+\Big(\sum_{j=1}^{N}a_{ij}\Big)u=0\quad\text{in }\mathcal{D}'((T,+\infty)\times\Omega) \tag{2.3}$$

for all $i=1,\cdots,N$. Then we have

$$\Big(\sum_{j=1}^{N}a_{kj}-\sum_{j=1}^{N}a_{ij}\Big)u=0\quad\text{in }\mathcal{D}'((T,+\infty)\times\Omega) \tag{2.4}$$

for $i, k = 1, \cdots, N$. It follows that

$$\sum_{j=1}^{N} a_{kj} = \sum_{j=1}^{N} a_{ij}, \quad i, k = 1, \cdots, N, \tag{2.5}$$

which is just the required condition of compatibility (2.2).

Now, let

$$C_1 = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}_{(N-1)\times N} \tag{2.6}$$

be the corresponding matrix of synchronization. C_1 is a full row-rank matrix, and $\mathrm{Ker}(C_1) = \mathrm{Span}\{e_1\}$, where $e_1 = (1, 1, \cdots, 1)^{\mathrm{T}}$. Clearly, the synchronization (2.1) can be equivalently written as

$$t \geq T: \quad C_1 U(t, x) \equiv 0 \quad \text{in } \Omega. \tag{2.7}$$

By Lemma 1.4, we have

Lemma 2.1 *The following properties are equivalent:*

(1) *The condition of compatibility* (2.2) *holds*;

(2) $e = (1, 1, \cdots, 1)^{\mathrm{T}}$ *is a right eigenvector of* A, *corresponding to the eigenvalue* a *given by* (2.2);

(3) $\mathrm{Ker}(C_1)$ *is a one-dimensional invariant subspace of* A :

$$A\mathrm{Ker}(C_1) \subseteq \mathrm{Ker}(C_1); \tag{2.8}$$

(4) *There exists a unique matrix* $\overline{A}_1$ *of order* $(N-1)$, *such that*

$$C_1 A = \overline{A}_1 C_1. \tag{2.9}$$

$\overline{A}_1 = (\overline{a}_{ij})$ *is called the reduced matrix of* A *by* C_1, *where*

$$\overline{a}_{ij} = \sum_{p=j+1}^{N} (a_{i+1,p} - a_{ip}) = \sum_{p=1}^{j} (a_{ip} - a_{i+1,p}), \quad i, j = 1, \cdots, N-1. \tag{2.10}$$

Theorem 2.2 *Assume that* $M = N - 1$. *Under the condition of compatibility* (2.2), *if the matrix* $C_1 D$ *is invertible, namely,* $\mathrm{rank}(C_1 D) = N - 1$, *then there exists a constant* $T > 0$ *so large that system* (1.1) *is exactly synchronizable at the time* T *in the space* $(\mathcal{H}_1 \times \mathcal{H}_0)^N$, *moreover, we have the continuous dependence*:

$$\|H\|_{\mathcal{L}^{N-1}} \leq C \|C_1(\widehat{U}_0, \widehat{U}_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^{N-1}}, \tag{2.11}$$

where C *is a positive constant.*

On the other hand, when $\mathrm{rank}(C_1 D) < N - 1$ (*especially, when* $M < N - 1$), *no matter how large* $T > 0$ *is, system* (1.1) *is not exactly synchronizable at the time* T.

Proof Under the condition of compatibility (2.2), let

$$W = C_1 U, \quad \widehat{W}_0 = C_1 \widehat{U}_0, \quad W_1 = C_1 \widehat{U}_1.$$

Noting (2.9), it is easy to see that the original mixed problem (1.1)–(1.2) for U can be reduced to the following self-closed mixed problem for W:

$$\begin{cases} W'' - \Delta W + \overline{A}_1 W = 0 & \text{in } (0,+\infty)\times\Omega, \\ W = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ \partial_\nu W = \overline{D}H & \text{on } (0,+\infty)\times\Gamma_1, \\ t=0:\ W = \widehat{W}_0,\ W' = \widehat{W}_1 & \text{in } \Omega, \end{cases} \tag{2.12}$$

where $\overline{D} = C_1 D$. Noting that C_1 is a surjection from $(\mathcal{H}_1\times\mathcal{H}_0)^N$ onto $(\mathcal{H}_1\times\mathcal{H}_0)^{N-1}$, we easily check that the exact boundary synchronization of system (1.1) for U is equivalent to the exact boundary null controllability of system (2.12) for W. Since $\operatorname{rank}(\overline{D}) = \operatorname{rank}(C_1 D) = N-1$, by Lemma 1.2, for any given initial data $(\widehat{W}_0, \widehat{W}_1) \in (\mathcal{H}_1\times\mathcal{H}_0)^{N-1}$, system (2.12) is exactly null controllable by means of a boundary control $\overline{D}H \in \mathcal{L}^{N-1}$. By (1.6) in Definition 1.1, we get the continuous dependence (2.11). Since $\overline{D}$ is invertible matrix, there exists a corresponding boundary control $H \in \mathcal{L}^{N-1}$, such that system (1.1) is exactly synchronizable.

On the other hand, when $\operatorname{rank}(C_1 D) < N-1$, $\overline{D}H$ can be rewritten as $\widetilde{D}\widetilde{H}$ such that $\widetilde{D}$ is a full column-rank matrix of order $N\times\widetilde{M}$ with $\widetilde{M} < N-1$ and $\widetilde{H}\in\mathcal{L}^{\widetilde{M}}$ with compact support in $[0,T]$, by Lemma 1.3, the reduced system (2.12) is not exactly null controllable, then system (1.1) is not exactly synchronizable either.

3 Exact Boundary Synchronization by p-Groups

When there is a further lack of boundary controls, we consider the exact boundary synchronization by p-groups ($p \geq 1$; when $p=1$, it becomes the exact boundary synchronization). This indicates that the components of U are divided into p groups:

$$(u^{(1)},\cdots,u^{(n_1)}),\ (u^{(n_1+1)},\cdots,u^{(n_2)}),\cdots,(u^{(n_{p-1}+1)},\cdots,u^{(n_p)}), \tag{3.1}$$

where $0 = n_0 < n_1 < n_2 < \cdots < n_p = N$, and each group is required to possess the exact boundary synchronization, respectively.

Definition 3.1 *System* (1.1) *is exactly synchronizable by p-groups at the time $T>0$ in the space $(\mathcal{H}_1\times\mathcal{H}_0)^N$, if for any given initial data $(\widehat{U}_0,\widehat{U}_1)\in(\mathcal{H}_1\times\mathcal{H}_0)^N$, there exists a boundary control $H\in\mathcal{L}^M$ with compact support in $[0,T]$, such that the weak solution $U=U(t,x)$ to the mixed initial-boundary value problem* (1.1)–(1.2) *satisfies*

$$t \geq T: \quad u^{(k)} \overset{\text{def.}}{=} u_s, \quad n_{s-1}+1 \leq k,l \leq n_s,\ 1\leq s\leq p, \tag{3.2}$$

where, $(u_1,\cdots,u_p)^{\mathrm{T}}$, being unknown a priori, is called the corresponding state of synchronization by p-groups.

Let S_s be an $(n_s - n_{s-1} - 1)\times(n_s - n_{s-1})$ full row-rank matrix:

$$S_s = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}, \quad 1\leq s\leq p \tag{3.3}$$

and let C_p be the following $(N-p)\times N$ matrix of synchronization by p-groups:

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}. \tag{3.4}$$

Obviously, we have

$$\text{Ker}(C_p) = \text{Span}\{e_1, \cdots, e_p\}, \tag{3.5}$$

where for $1 \le s \le p$,

$$(e_s)_j = \begin{cases} 1, & n_{s-1}+1 \le j \le n_s, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Thus, (3.2) can be equivalently written as

$$t \ge T: \quad C_pU \equiv 0 \quad \text{or equivalently} \quad U = \sum_{s=1}^{p} u_s e_s \quad \text{in } \Omega. \tag{3.7}$$

Theorem 3.1 *Assume that system* (1.1) *is exactly synchronizable by p-groups. Then we necessarily have $M \ge N-p$. Especially, when $M = N-p$, the coupling matrix $A = (a_{ij})$ should satisfy the following condition of compatibility:*

$$A\text{Ker}(C_p) \subseteq \text{Ker}(C_p). \tag{3.8}$$

Proof By (3.7), we have $C_pU = 0$ in Ω when $t \ge T$. If $A\text{Ker}(C_p) \not\subseteq \text{Ker}(C_p)$, by Lemma 1.5, we can construct a full row-rank $(N-p+1)\times N$ matrix $\widetilde{C}_1$ such that $\widetilde{C}_1U = 0$ in Ω when $t \ge T$. If $A\text{Ker}(\widetilde{C}_1) \not\subseteq \text{Ker}(\widetilde{C}_1)$, still by Lemma 1.5, we can construct another full row-rank $(N-p+2)\times N$ matrix $\widetilde{C}_2$ such that $\widetilde{C}_2U = 0$ in Ω when $t \ge T$, $\cdots$. This procedure should stop at the rth step, where $0 \le r \le p$. Thus, we get a full row-rank $(N-p+r)\times N$ matrix $\widetilde{C}_r$ such that

$$t \ge T: \quad \widetilde{C}_rU = 0 \quad \text{in } \Omega \tag{3.9}$$

and

$$A\text{Ker}(\widetilde{C}_r) \subseteq \text{Ker}(\widetilde{C}_r). \tag{3.10}$$

By Lemma 1.4, there exists a unique matrix $\widetilde{A}$ of order $(N-p+r)$, such that

$$\widetilde{C}_rA = \widetilde{A}\widetilde{C}_r.$$

Setting $W = \widetilde{C}_rU$ in (1.1), we get the following reduced problem:

$$\begin{cases} W'' - \Delta W + \widetilde{A}W = 0 & \text{in } (0,+\infty)\times\Omega, \\ W = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ \partial_\nu W = \widetilde{D}H & \text{on } (0,+\infty)\times\Gamma_1, \\ t=0:\ W = \widetilde{C}_r\widehat{U}_0,\ W' = \widetilde{C}_r\widehat{U}_1 & \text{in } \Omega, \end{cases} \tag{3.11}$$

where $\widetilde{D} = \widetilde{C}_r D$. Moreover, by (3.9) we have

$$t \geq T: \quad W \equiv 0. \tag{3.12}$$

Noting that $\widetilde{C}_r$ is an $(N-p+r) \times N$ full row-rank matrix, the linear mapping

$$(\widehat{U}_0, \widehat{U}_1) \to (\widetilde{C}_r \widehat{U}_0, \widetilde{C}_r \widehat{U}_1) \tag{3.13}$$

is a surjection from $(\mathcal{H}_1 \times \mathcal{H}_0)^N$ onto $(\mathcal{H}_1 \times \mathcal{H}_0)^{N-p+r}$, then, (3.11) is exactly null controllable at the time T in the space $(\mathcal{H}_1 \times \mathcal{H}_0)^{N-p+r}$. By Lemmas 1.2–1.3 (cf. the last paragraph in the proof of Theorem 2.2), we necessarily have

$$\operatorname{rank}(\widetilde{C}_r D) = N - p + r,$$

then

$$M = \operatorname{rank}(D) \geq \operatorname{rank}(\widetilde{C}_r D) = N - p + r \geq N - p. \tag{3.14}$$

In particular, when $M = N - p$, we have $r = 0$, namely, the condition of compatibility (3.8) holds.

Remark 3.1 The condition of compatibility (3.8) is equivalent to the fact that there exist some constants α_{rs} $(1 \leq r, s \leq p)$ such that

$$Ae_s = \sum_{r=1}^{p} \alpha_{rs} e_r, \quad 1 \leq s \leq p, \tag{3.15}$$

or, noting (3.6), A satisfies the following row-sum condition by blocks:

$$\sum_{j=n_{s-1}+1}^{n_s} a_{ij} = \alpha_{rs}, \quad n_{r-1}+1 \leq i \leq n_r, \quad 1 \leq r, s \leq p. \tag{3.16}$$

Especially, this condition of compatibility becomes (2.2) when $p = 1$.

Theorem 3.2 *Let C_p be the $(N-p) \times N$ matrix of synchronization by p-groups defined by* (3.3)–(3.4). *Under the condition of compatibility* (3.8), *assume that the $N \times (N-p)$ boundary control matrix D has full column-rank and satisfies* $\operatorname{rank}(C_p D) = N - p$. *Then system* (1.1) *is exactly synchronizable by p-groups by means of boundary control $H \in \mathcal{L}^{N-p}$, moreover, we have the continuous dependence:*

$$\|H\|_{\mathcal{L}^{N-p}} \leq C \|C_p(\widehat{U}_0, \widehat{U}_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^{N-p}}, \tag{3.17}$$

where C is a positive constant.

On the other hand, when $\operatorname{rank}(C_p D) < N - p$ *(especially, when $M < N - p$), no matter how large $T > 0$ is, system* (1.1) *is not exactly synchronizable by p-groups at the time T.*

Proof Assume that the coupling matrix $A = (a_{ij})$ satisfies the condition of compatibility (3.8). By Lemma 1.4, there exists a unique matrix $\overline{A}_p$ of order $(N-p)$, such that

$$C_p A = \overline{A}_p C_p. \tag{3.18}$$

Setting

$$W = C_p U, \quad \overline{D} = C_p D.$$

We can similarly get the following reduced system for W:

$$\begin{cases} W'' - \Delta W + \overline{A}_p W = 0 & \text{in } (0, +\infty) \times \Omega, \\ W = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu W = \overline{D} H & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ W = C_p \widehat{U}_0, \ W' = C_p \widehat{U}_1 & \text{in } \Omega, \end{cases} \tag{3.19}$$

where W is a vector valued function of $(N-p)$ components. By the assumption that $\text{rank}(\overline{D}) = \text{rank}(C_p D) = N - p$ and Lemma 1.2, system (3.19) is exactly null controllable. Also, by (1.6) in Definition 1.1, we get the continuous dependence (3.17). Then the original system (1.1) for U is exactly synchronizable by p-groups.

On the other hand, when $\text{rank}(C_p D) < N - p$, by Lemma 1.3 (cf. the last paragraph in the proof of Theorem 2.2), the reduced system (2.12) is not exactly null controllable, then system (1.1) is not exactly synchronizable by p-groups.

4 Determination of the State of Synchronization by p-Groups

Now, under the condition of compatibility (3.8), we are going to discuss the determination of the state of synchronization by p-groups for system (1.1). Generally speaking, the state of synchronization should depend on the initial data $(\widehat{U}_0, \widehat{U}_1)$ and the applied boundary control H. However, when the coupling matrix A possesses some good properties, the state of synchronization by p-groups is independent of the applied boundary control, and can be determined entirely by the solution to a system of wave equations with homogeneous boundary condition.

First, by Lemma 1.1 and noting that the space $(\mathcal{H}_1 \times \mathcal{H}_0)^N$ given in Definition 3.1 is included in $(\mathcal{H}_{1-s} \times \mathcal{H}_{-s})^N$ $\left(s > \frac{1}{2}\right)$, differently from the case of Dirichlet boundary controls, the attainable set of states of exact boundary synchronization by p-groups for the system with Neumann boundary controls is not the whole space $(\mathcal{H}_{1-s} \times \mathcal{H}_{-s})^p$. Besides, as in the case of Dirichlet boundary controls (see [6]), the choice of boundary controls is not unique. We have the following theorem.

Theorem 4.1 *Let $\mathbb{H}$ denote the set of all the boundary controls H which can realize the exact boundary synchronization by p-groups at the time T for system* (1.1). *If the condition of compatibility* (3.8) *holds, then for $\varepsilon > 0$ small enough, the value of $H \in \mathbb{H}$ on $(0, \varepsilon) \times \Gamma_1$ can be arbitrarily chosen.*

Proof First of all, there exists a $T_0 > 0$ independent of the initial data, such that, when $T > T_0$, the reduced problem (3.19) is exactly null controllable at the time T. According to the proof of Theorem 3.2, the exact synchronization by p-groups of system (1.1) is equivalent to the exact null controllability of the reduced system (3.19). Therefore, taking an $\varepsilon > 0$ so small that $T - \varepsilon > T_0$, system (1.1) is still exactly synchronizable by p-groups at the time $T - \varepsilon$.

Assuming first that $(\widehat{U}_0, \widehat{U}_1) \in (C_0^\infty(\Omega) \times C_0^\infty(\Omega))^N$, and choosing arbitrarily

$$\widehat{H}_\varepsilon \in (C_0^\infty([0, \varepsilon] \times \Gamma_1))^{N-p},$$

we solve the forward problem (1.1) on the time interval $[0, \varepsilon]$ with $H = \widehat{H}_\varepsilon$, and get the solution $(\widehat{U}_\varepsilon, \widehat{U}'_\varepsilon) \in C^0([0, \varepsilon]; (\mathcal{H}_1 \times \mathcal{H}_0)^N)$. Taking $(\widehat{U}_\varepsilon(\varepsilon, \cdot), \widehat{U}'_\varepsilon(\varepsilon, \cdot)) \in (\mathcal{H}_1 \times \mathcal{H}_0)^N$ as initial data, by

Theorem 3.2, for system (1.1) we can find a boundary control

$$\widetilde{H}_\varepsilon \in (L^2(\varepsilon, T; L^2(\Gamma_1)))^{N-p}$$

such that the corresponding solution $\widetilde{U}_\varepsilon$ satisfies exactly the initial condition

$$t = \varepsilon: \quad \widetilde{U}_\varepsilon = \widehat{U}_\varepsilon(\varepsilon, x), \quad \widetilde{U}_\varepsilon' = \widehat{U}_\varepsilon'(\varepsilon, x)$$

and realizes the synchronization by p-groups at the time $t = T$. Let

$$H = \begin{cases} \widehat{H}_\varepsilon, & t \in (0, \varepsilon), \\ \widetilde{H}_\varepsilon, & t \in (\varepsilon, T), \end{cases} \qquad U = \begin{cases} \widehat{U}_\varepsilon, & t \in (0, \varepsilon), \\ \widetilde{U}_\varepsilon, & t \in (\varepsilon, T). \end{cases}$$

It can be verified that U is the solution to the mixed problem (1.1) with boundary control H, and it is exactly synchronizable by p-groups at the time T. By this way, we get an infinity of boundary controls H, the values of which on $(0, \varepsilon) \times \Gamma_1$ can be taken arbitrarily. Finally, by the denseness of $C_0^\infty(\Omega)$ in $\mathcal{H}_1$ and $\mathcal{H}_0$, we can get the desired result.

The state of synchronization by p-groups is closely related to the properties of the coupling matrix A. Let

$$\mathcal{D}_{N-p} = \{D \in \mathbb{M}^{N\times(N-p)}(\mathbb{R}): \ \mathrm{rank}(D) = \mathrm{rank}(C_p D) = N - p\}.$$

By [7], $D \in \mathcal{D}_{N-p}$ if and only if it can be expressed by

$$D = (C_p^{\mathrm{T}} + (e_1, \cdots, e_p)D_0)\overline{D}, \tag{4.1}$$

where D_0 is a $p \times (N-p)$ matrix, and $\overline{D}$ is a reversible matrix of order $(N-p)$. We have the following theorem.

Theorem 4.2 *Under the condition of compatibility* (3.8), *assume that* A^{T} *possesses an invariant subspace* $\mathrm{Span}\{E_1, E_2, \cdots, E_p\}$ *which is bi-orthonormal to* $\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \cdots, e_p\}$:

$$(E_i, e_j) = \delta_{ij}, \quad 1 \le i, j \le p.$$

Then there exists a boundary control matrix $D \in \mathcal{D}_{N-p}$, *such that the state of synchronization by p-groups* $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ *is independent of the applied boundary controls, and can be determined as follows*:

$$t \ge T: \quad u = \psi, \tag{4.2}$$

where $\psi = (\psi_1, \cdots, \psi_p)^{\mathrm{T}}$ *is the solution to the following problem with homogeneous boundary condition*:

$$\begin{cases} \psi_r'' - \Delta\psi_r + \displaystyle\sum_{s=1}^{p} \alpha_{rs}\psi_s = 0 & \text{in } (0, +\infty) \times \Omega, \\ \psi_r = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \psi_r = 0 & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ \psi_r = (E_r, \widehat{U}_0), \ \psi_r' = (E_r, \widehat{U}_1) & \text{in } \Omega, \end{cases} \tag{4.3}$$

where α_{rs} $(1 \le r, s \le p)$ *are given by* (3.16).

Proof Noting that $\mathrm{Span}\{E_1, E_2, \cdots, E_p\}$ is bi-orthonormal to $\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \cdots, e_p\}$, and taking

$$D_0 = -E^{\mathrm{T}} C_p^{\mathrm{T}}, \quad E = (E_1, E_2, \cdots, E_p) \tag{4.4}$$

in (4.1), we get a boundary control matrix $D \in \mathcal{D}_{N-p}$, such that

$$E_r \in \mathrm{Ker}(D^{\mathrm{T}}), \quad 1 \le r \le p. \tag{4.5}$$

On the other hand, since $\mathrm{Span}\{E_1, E_2, \cdots, E_p\}$ is an invariant subspace of A^{T}, we may denote

$$A^{\mathrm{T}} E_r = \sum_{s=1}^{p} \beta_{sr} E_s, \quad 1 \le r \le p,$$

where β_{sr} are some constants. By

$$(A^{\mathrm{T}} E_r, e_s) = (E_r, A e_s), \quad 1 \le r, s \le p,$$

and noticing (3.15), we have

$$\Big(\sum_{t=1}^{p} \beta_{tr} E_t, e_s\Big) = \Big(E_r, \sum_{t=1}^{p} \alpha_{ts} e_t\Big), \quad 1 \le r, s \le p.$$

Then by bi-orthonormality, we get

$$\beta_{sr} = \alpha_{rs},$$

namely,

$$A^{\mathrm{T}} E_r = \sum_{s=1}^{p} \alpha_{rs} E_s, \quad 1 \le r \le p. \tag{4.6}$$

Let $\psi_r = (E_r, U)$. Taking the inner product with E_r on both sides of (1.1), we get (4.3). Finally, for the state of synchronization by p-groups, by (3.7) we have

$$t \ge T: \quad \psi_r(t) = (E_r, U) = \sum_{s=1}^{p} (E_r, e_s) u_s = u_r, \quad 1 \le r \le p. \tag{4.7}$$

When A^{T} does not possess any invariant subspace $\mathrm{Span}\{E_1, E_2, \cdots, E_p\}$ which is bi-orthonormal to $\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \cdots, e_p\}$, we can use the solution of (4.3) to give an estimate on the state of synchronization by p-groups.

Theorem 4.3 *Under the condition of compatibility* (3.8), *assume that there exists a subspace* $\mathrm{Span}\{E_1, E_2, \cdots, E_p\}$ *that is bi-orthonormal to* $\mathrm{Span}\{e_1, \cdots, e_p\}$. *Then there exist a boundary control matrix* $D \in \mathcal{D}_{N-p}$ *and a constant* c *independent of the initial data, such that the state of synchronization by* p*-groups* $u = (u_1, \cdots, u_p)^{\mathrm{T}}$ *satisfies the following estimate:*

$$t \ge T: \quad \|(u, u')(t) - (\psi, \psi')(t)\|_{(\mathcal{H}_{2-s} \times \mathcal{H}_{1-s})^p} \le c \|C_p(\widehat{U}_0, \widehat{U}_1)\|_{(\mathcal{H}_1 \times \mathcal{H}_0)^{N-p}}, \tag{4.8}$$

where $\psi = (\psi_1, \cdots, \psi_p)^{\mathrm{T}}$ *is the solution to problem* (4.3), *and* $s > \frac{1}{2}$.

Proof Since $\{E_1, E_2, \cdots, E_p\}$ is bi-orthonormal to $\{e_1, \cdots, e_p\}$, similar to (4.4), there exists a boundary control matrix $D \in \mathcal{D}_{N-p}$, such that (4.5) holds. Let $\phi_r = (E_r, U)$. Taking the inner product with E_r on both sides of (1.1)–(1.2), we get

$$\phi_r'' - \Delta\phi_r + (E_r, AU) = 0.$$

Since

$$\begin{aligned}(E_r, AU) = (A^{\mathrm{T}}E_r, U) &= \Big(\sum_{s=1}^{p}\alpha_{rs}E_s + A^{\mathrm{T}}E_r - \sum_{s=1}^{p}\alpha_{rs}E_s, U\Big)\\ &= \sum_{s=1}^{p}\alpha_{rs}(E_s, U) + \Big(A^{\mathrm{T}}E_r - \sum_{s=1}^{p}\alpha_{rs}E_s, U\Big)\\ &= \sum_{s=1}^{p}\alpha_{rs}\phi_s + \Big(A^{\mathrm{T}}E_r - \sum_{s=1}^{p}\alpha_{rs}E_s, U\Big)\end{aligned} \tag{4.9}$$

and for any given $k \in \{1, \cdots, p\}$, we have

$$\begin{aligned}\Big(A^{\mathrm{T}}E_r - \sum_{s=1}^{p}\alpha_{rs}E_s, e_k\Big) &= (E_r, Ae_k) - \sum_{s=1}^{p}\alpha_{rs}(E_s, e_k)\\ &= \Big(E_r, \sum_{s=1}^{p}\alpha_{sk}e_s\Big) - \alpha_{rk}\\ &= \sum_{s=1}^{p}\alpha_{sk}(E_r, e_s) - \alpha_{rk}\\ &= \alpha_{rk} - \alpha_{rk} = 0,\end{aligned} \tag{4.10}$$

we get

$$A^{\mathrm{T}}E_r - \sum_{s=1}^{p}\alpha_{rs}E_s \in \{\mathrm{Ker}(C_p)\}^{\perp} = \mathrm{Im}(C_p^{\mathrm{T}}).$$

Therefore, there exists a vector $R_r \in \mathbb{R}^{N-p}$, such that

$$A^{\mathrm{T}}E_r - \sum_{s=1}^{p}\alpha_{rs}E_s = C_p^{\mathrm{T}}R_r. \tag{4.11}$$

Thus, for $r = 1, \cdots, p$, we have

$$\begin{cases}\phi_r'' - \Delta\phi_r + \sum\limits_{s=1}^{p}\alpha_{rs}\phi_s = -(R_r, C_pU) & \text{in } (0, +\infty)\times\Omega,\\ \phi_r = 0 & \text{on } (0, +\infty)\times\Gamma_0,\\ \partial_\nu\phi_r = 0 & \text{on } (0, +\infty)\times\Gamma_1,\\ t = 0:\ \phi_r = (E_r, \widehat{U}_0),\ \phi_r' = (E_r, \widehat{U}_1) & \text{in } \Omega,\end{cases} \tag{4.12}$$

where α_{rs} $(1 \le r, s \le p)$ are defined by (3.16), and $U = U(t,x) \in C(0,T;(\mathcal{H}_{1-s})^N) \cap C^1(0,T;(\mathcal{H}_{-s})^N)$ is the solution to the mixed initial-boundary value problem (1.1)–(1.2). Moreover, we have

$$t \ge T: \quad \phi_r(t) = (E_r, U) = \sum_{s=1}^{p}(E_r, e_s)u_s = u_r, \quad r = 1, \cdots, p. \tag{4.13}$$

Noting that (4.3) and (4.12) possess the same initial data and the same boundary condition, by the well-posedness for a system of wave equations with Neumann boundary condition, we have (see [12, Chapter III]) that, when $t \geq 0$,

$$\|(\psi, \psi')(t) - (\phi, \phi')(t)\|^2_{(\mathcal{H}_{2-s}\times\mathcal{H}_{1-s})^p} \leq c\int_0^{\mathrm{T}} \|C_pU\|^2_{(\mathcal{H}_{1-s})^{N-p}}\mathrm{d}s, \tag{4.14}$$

where, c is a positive constant. Noting that $W = C_pU$, by well-posedness of the reduced problem (3.19) (see [12, Lemma 1.1]), we have

$$\int_0^{\mathrm{T}} \|C_pU\|^2_{(\mathcal{H}_{1-s})^{N-p}}\mathrm{d}s \leq c(\|C_p(\widehat{U}_0, \widehat{U}_1)\|^2_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-p}} + \|\overline{D}H\|_{\mathcal{L}^{N-p}}). \tag{4.15}$$

Moreover, by (3.17) we have

$$\|\overline{D}H\|_{\mathcal{L}^{N-p}} \leq c\|C_p(\widehat{U}_0, \widehat{U}_1)\|^2_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-p}}. \tag{4.16}$$

Substituting it into (4.15), we have

$$\int_0^{\mathrm{T}} \|C_pU\|^2_{(\mathcal{H}_{1-s})^{N-p}}\mathrm{d}s \leq c\|C_p(\widehat{U}_0, \widehat{U}_1)\|^2_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-p}}, \tag{4.17}$$

then, by (4.14) we get

$$\|(\psi, \psi')(t) - (\phi, \phi')(t)\|^2_{(\mathcal{H}_{2-s}\times\mathcal{H}_{1-s})^p} \leq c\|C_p(\widehat{U}_0, \widehat{U}_1)\|^2_{(\mathcal{H}_1\times\mathcal{H}_0)^{N-p}}. \tag{4.18}$$

Substituting (4.13) into (4.18), we get (4.8).

Remark 4.1 Differently from the case of Dirichlet boundary controls, although the solution to the problem (1.1) with Neumann boundary controls possesses a weaker regularity, the solution to the problem (4.3), which determines the state of synchronization by p-groups, possesses a higher regularity than the original problem (1.1) itself, then, this improved regularity makes it possible to approach the state of synchronization by p-groups by a solution to a relatively smoother problem.

In order to exactly express the state of synchronization by p-groups, we can extend the subspace $\mathrm{Span}\{e_1, \cdots, e_p\}$ to an invariant subspace $\mathrm{Span}\{e_1, \cdots, e_p, \cdots, e_q\}$ of A, such that A^{T} possesses an invariant subspace $\mathrm{Span}\{E_1, \cdots, E_p, \cdots, E_q\}$, which is bi-orthonormal to $\mathrm{Span}\{e_1, \cdots, e_p, \cdots, e_q\}$.

Let

$$P = \sum_{s=1}^{q} e_s \otimes E_s, \tag{4.19}$$

in which the tensor product is defined by

$$(e \otimes E)U = (E, U)e = E^{\mathrm{T}}Ue, \quad \forall U \in \mathbb{R}^N.$$

P can be represented by a matrix of order N. It is easy to see that

$$\mathrm{Im}(P) = \mathrm{Span}\{e_1, e_2, \cdots, e_q\}, \quad \mathrm{Ker}(P) = (\mathrm{Span}\{E_1, E_2, \cdots, E_q\})^{\perp} \tag{4.20}$$

and

$$PA = AP. \tag{4.21}$$

Let $U = U(t,x)$ be the solution to the mixed initial-boundary value problem (1.1)–(1.2). We define its synchronizable part U_s and controllable part U_c, respectively, as follows:

$$U_s := PU, \quad U_c := (I-P)U. \tag{4.22}$$

If system (1.1) is exactly synchronizable by p-groups, then

$$t \geq T: \quad U \in \mathrm{Span}\{e_1, \cdots, e_p\} \subseteq \mathrm{Span}\{e_1, \cdots, e_p, \cdots, e_q\} = \mathrm{Im}(P), \tag{4.23}$$

hence we have

$$t \geq T: \quad U_s = PU = U, \quad U_c = (I-P)U = 0.$$

Noting (4.21), multiplying P and $(I-P)$ from the left on both sides of (1.1) respectively, we see that the synchronizable part U_s of U satisfies the following system:

$$\begin{cases} U_s'' - \Delta U_s + AU_s = 0 & \text{in } (0,+\infty)\times\Omega, \\ U_s = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ \partial_\nu U_s = PDH & \text{on } (0,+\infty)\times\Gamma_1, \\ t=0: \ U_s = P\widehat{U}_0, \ U_s' = P\widehat{U}_1 & \text{in } \Omega, \end{cases} \tag{4.24}$$

while, the controllable part U_c of U satisfies the following system:

$$\begin{cases} U_c'' - \Delta U_c + AU_c = 0 & \text{in } (0,+\infty)\times\Omega, \\ U_c = 0 & \text{on } (0,+\infty)\times\Gamma_0, \\ \partial_\nu U_c = (I-P)DH & \text{on } (0,+\infty)\times\Gamma_1, \\ t=0: \ U_c = (I-P)\widehat{U}_0, \ U_c' = (I-P)\widehat{U}_1 & \text{in } \Omega. \end{cases} \tag{4.25}$$

In fact, under the boundary control H, U_c with the initial data $((I-P)\widehat{U}_0, (I-P)\widehat{U}_1) \in \mathrm{Ker}(P)\times \mathrm{Ker}(P)$ is exactly null controllable, while, U_s with the initial data $(P\widehat{U}_0, P\widehat{U}_1) \in \mathrm{Im}(P)\times\mathrm{Im}(P)$ is exactly synchronizable.

Theorem 4.4 *Assume that the condition of compatibility* (3.8) *holds. Let P be defined by* (4.19)*. If system* (1.1) *is exactly synchronizable by p-groups, and the synchronizable part U_s is independent of the applied boundary control H for $t \geq T$, then we have*

$$p = q \quad \textit{and} \quad PD = 0. \tag{4.26}$$

In particular, if $P\widehat{U}_0 = P\widehat{U}_1 = 0$, then, for such initial data $(\widehat{U}_0, \widehat{U}_1)$, system (1.1) *is exactly null controllable.*

Proof By Theorem 4.1, the value of H on $(0,\varepsilon)\times\Gamma_1$ can be arbitrarily taken. If the synchronizable part U_s is independent of the applied boundary control H for $t \geq T$, then we have

$$PD = 0,$$

hence

$$\mathrm{Im}(D) \subseteq \mathrm{Ker}(P).$$

Noting (4.20), we have

$$\dim \mathrm{Ker}(P) = N - q \quad \text{and} \quad \dim \mathrm{Im}(D) = N - p,$$

then $p = q$.

5 Determination of the State of Exact Boundary Synchronization

In the case of exact boundary synchronization, by Lemma 2.1, $(1, 1, \cdots, 1)^{\mathrm{T}}$ is a right eigenvector of A, corresponding to the eigenvalue a defined by (2.2). Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$ and $E_1, E_2, \cdots, E_r$ with $r \geq 1$ be the Jordan chains of A and A^{T}, respectively, corresponding to the eigenvalue a, and $\mathrm{Span}\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r\}$ is bi-orthonormal to $\mathrm{Span}\{E_1, E_2, \cdots, E_r\}$. Thus we have

$$\begin{cases} A\varepsilon_l = a\varepsilon_l + \varepsilon_{l+1}, & 1 \leq l \leq r, \\ A^{\mathrm{T}} E_k = aE_k + E_{k-1}, & 1 \leq k \leq r, \\ (E_k, \varepsilon_l) = \delta_{kl}, & 1 \leq k, l \leq r \end{cases} \tag{5.1}$$

with

$$\varepsilon_r = (1, 1, \cdots, 1)^{\mathrm{T}}, \quad \varepsilon_{r+1} = 0, \quad E_0 = 0. \tag{5.2}$$

Let $U = U(t, x)$ be the solution to the mixed initial-boundary value problem (1.1)–(1.2). If system (1.1) is exactly synchronizable, then

$$t \geq T: \quad U = u\varepsilon_r, \tag{5.3}$$

where $u = u(t, x)$ is the corresponding state of synchronization. The synchronizable part and the controllable part are, respectively,

$$t \geq T: \quad U_s = u\varepsilon_r, \quad U_c = 0.$$

If the synchronizable part is independent of the applied boundary control H, by Theorem 4.4, we have $r = 1$, then A possesses a left eigenvector E_1 such that

$$(E_1, \varepsilon_1) = 1.$$

Generally speaking, when $r \geq 1$, setting

$$\psi_k = (E_k, U), \quad 1 \leq k \leq r,$$

noting (4.19) and (4.22), we have

$$U_s = \sum_{k=1}^{r} (E_k, U)\varepsilon_k = \sum_{k=1}^{r} \psi_k \varepsilon_k.$$

Thus, $(\psi_1, \cdots, \psi_r)$ are the coordinates of U_s under the basis $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r)$.

Taking the inner product with E_k on both sides of (4.24), we get the following theorem.

Theorem 5.1 *Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$ and $E_1, E_2, \cdots, E_r$ be the Jordan chains of A and A^{T}, respectively, corresponding to the eigenvalue a, in which $\varepsilon_r = (1, \cdots, 1)^{\mathrm{T}}$. Then the synchronizable part $U_s = (\psi_1, \cdots, \psi_r)$ is determined by the following system $(1 \leq k \leq r)$:*

$$\begin{cases} \psi_k'' - \Delta\psi_k + a\psi_k + \psi_{k-1} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \psi_k = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \psi_k = h_k & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ \psi_k = (E_k, \widehat{U}_0), \ \psi_k' = (E_k, \widehat{U}_1) & \text{in } \Omega, \end{cases} \tag{5.4}$$

where

$$\psi_0 = 0, \quad h_k = E_k^{\mathrm{T}} DH. \tag{5.5}$$

Noting (5.3), we have

$$t \geq T: \quad \psi_k = (E_k, U) = (E_k, u\varepsilon_r) = u\delta_{kr}, \quad 1 \leq k \leq r.$$

Thus, the state of synchronization u is determined by

$$t \geq T: \quad u = u(t,x) = \psi_r(t,x).$$

However, in order to get the state of synchronization u, we must solve the whole coupled problem (5.4)–(5.5).

6 Determination of the State of Exact Boundary Synchronization by 3-Groups

In this section, for an example, we will give the details on the determination of the state of exact boundary synchronization by 3-groups for system (1.1). The state of synchronization by p-groups can be discussed in a similar way. We always assume that the condition of compatibility (3.8) is satisfied.

By synchronization by 2-groups, when $t \geq T$, we have

$$u^{(1)}(t,x) \equiv \cdots \equiv u^{(n_1)}(t,x) \overset{\text{def.}}{=} u_1(t,x), \tag{6.1}$$

$$u^{(n_1+1)}(t,x) \equiv \cdots \equiv u^{(n_2)}(t,x) \overset{\text{def.}}{=} u_2(t,x), \tag{6.2}$$

$$u^{(n_2+1)}(t,x) \equiv \cdots \equiv u^{(N)}(t,x) \overset{\text{def.}}{=} u_3(t,x). \tag{6.3}$$

Recall that the matrix C_3 of synchronization by 3-groups is defined by (3.4). Let

$$\begin{cases} e_1 = (\overbrace{1,\cdots,1}^{n_1}, \overbrace{0,\cdots,0}^{n_2-n_1}, \overbrace{0,\cdots,0}^{N-n_2})^{\rm T}, \\ e_2 = (\overbrace{0,\cdots,0}^{n_1}, \overbrace{1,\cdots,1}^{n_2-n_1}, \overbrace{0,\cdots,0}^{N-n_2})^{\rm T}, \\ e_3 = (\overbrace{0,\cdots,0}^{n_1}, \overbrace{0,\cdots,0}^{n_2-n_1}, \overbrace{1,\cdots,1}^{N-n_2})^{\rm T}. \end{cases} \tag{6.4}$$

Obviously, we have that

$$\text{Ker}(C_3) = \text{Span}\{e_1, e_2, e_3\} \tag{6.5}$$

and that the state of synchronization by 3-groups is given by (6.1)–(6.3) means that

$$t \geq T: \quad U = u_1 e_1 + u_2 e_2 + u_3 e_3. \tag{6.6}$$

Since the invariant subspace $\text{Span}\{e_1, e_2, e_3\}$ of A is of dimension 3, it may contain one, two or three eigenvectors of A, thus we can distinguish the followings three cases.

(i) A admits three eigenvectors f_r, g_s and h_t contained in $\text{Span}\{e_1, e_2, e_3\}$, corresponding to eigenvalues λ, μ and ν, respectively. Let $f_1, f_2, \cdots, f_r; g_1, g_2, \cdots, g_s$ and $h_1, \cdots, h_t$ be the Jordan chains corresponding to these right eigenvectors:

$$\begin{cases} Af_i = \lambda f_i + f_{i+1}, & 1 \leq i \leq r, \ e_{r+1} = 0, \\ Ag_j = \mu g_j + g_{j+1}, & 1 \leq j \leq s, \ g_{s+1} = 0, \\ Ah_k = \nu h_k + h_{k+1}, & 1 \leq k \leq t, \ h_{t+1} = 0, \end{cases} \tag{6.7}$$

and let $\xi_1, \xi_2, \cdots, \xi_r$; $\eta_1, \eta_2, \cdots, \eta_s$ and $\zeta_1, \zeta_2, \cdots, \zeta_t$ be the Jordan chains corresponding to the related left eigenvectors:

$$\begin{cases} A^{\mathrm{T}}\xi_i = \lambda\xi_i + \xi_{i-1}, & 1 \le i \le r,\ \xi_0 = 0, \\ A^{\mathrm{T}}\eta_j = \mu\eta_j + \eta_{j-1}, & 1 \le j \le s,\ \eta_0 = 0, \\ A^{\mathrm{T}}\zeta_k = \nu\zeta_k + \zeta_{k-1}, & 1 \le k \le t,\ \zeta_0 = 0, \end{cases} \tag{6.8}$$

such that

$$(f_i, \xi_l) = \delta_{il}, \quad (g_j, \eta_m) = \delta_{jm}, \quad (h_k, \zeta_n) = \delta_{kn} \tag{6.9}$$

for $i, l = 1, \cdots r;\ j, m = 1, \cdots s;\ k, n = 1, \cdots t$ and

$$(f_i, \eta_j) = (f_i, \zeta_k) = (g_j, \xi_i) = (g_j, \zeta_k) = (h_k, \xi_i) = (h_k, \eta_j) = 0 \tag{6.10}$$

for $i = 1, \cdots r;\ j = 1, \cdots s;\ k = 1, \cdots t$.

Taking the inner product with ξ_i, η_j, ζ_k on both sides of (1.1)–(1.2), respectively, and denoting

$$\phi_i = (U, \xi_i), \quad \psi_j = (U, \eta_j), \quad \theta_k = (U, \zeta_k) \tag{6.11}$$

for $i = 1, \cdots r;\ j = 1, \cdots s;\ k = 1, \cdots t$, we get

$$\begin{cases} \phi_i'' - \Delta\phi_i + \lambda\phi_i + \phi_{i-1} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \phi_i = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \phi_i = \xi_i^{\mathrm{T}} DH & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ \phi_i = (\xi_i, \widehat{U}_0),\ \phi_i' = (\xi_i, \widehat{U}_1) & \text{in } \Omega, \end{cases} \tag{6.12}$$

$$\begin{cases} \psi_j'' - \Delta\psi_j + \mu\psi_j + \psi_{j-1} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \psi_j = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \psi_j = \eta_j^{\mathrm{T}} DH & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ \psi_j = (\eta_j, \widehat{U}_0),\ \psi_j' = (\eta_j, \widehat{U}_1) & \text{in } \Omega \end{cases} \tag{6.13}$$

and

$$\begin{cases} \theta_k'' - \Delta\theta_k + \nu\theta_k + \theta_{k-1} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \theta_k = 0 & \text{on } (0, +\infty) \times \Gamma_0, \\ \partial_\nu \theta_k = \zeta_k^{\mathrm{T}} DH & \text{on } (0, +\infty) \times \Gamma_1, \\ t = 0: \ \theta_k = (\zeta_k, \widehat{U}_0),\ \theta_k' = (\zeta_k, \widehat{U}_1) & \text{in } \Omega \end{cases} \tag{6.14}$$

with

$$\phi_0 = \psi_0 = \theta_0 = 0. \tag{6.15}$$

Solving the reduced problems (6.12)–(6.14), we get $\phi_1 \cdots, \phi_r$; $\psi_1, \cdots, \psi_s$ and $\theta_1, \cdots, \theta_t$. Noting that f_r, g_s, h_t are contained in Span$\{e_1, e_2, e_3\}$, we have

$$\begin{cases} e_1 = \alpha_1 f_r + \alpha_2 g_s + \alpha_3 h_t, \\ e_2 = \beta_1 f_r + \beta_2 g_s + \beta_3 h_t, \\ e_3 = \gamma_1 f_r + \gamma_2 g_s + \gamma_3 h_t. \end{cases} \tag{6.16}$$

By (6.6) we have

$$t \geq T: \quad U = (u_1\alpha_1 + u_2\beta_1 + u_3\gamma_1)f_r + (u_1\alpha_2 + u_2\beta_2 + u_3\gamma_2)g_s + (u_1\alpha_3 + u_2\beta_3 + u_3\gamma_3)h_t. \tag{6.17}$$

Noting (6.9)–(6.10), we have

$$t \geq T: \quad \begin{cases} \phi_r = u_1\alpha_1 + u_2\beta_1 + u_3\gamma_1, \\ \psi_s = u_1\alpha_2 + u_2\beta_2 + u_3\gamma_2, \\ \theta_t = u_1\alpha_3 + u_2\beta_3 + u_3\gamma_3. \end{cases} \tag{6.18}$$

Since e_1, e_2, e_3 are linearly independent, the linear system (6.16) is invertible. Then, the state $(u_1, u_2, u_3)^{\mathrm{T}}$ of synchronization by 3-groups can be determined by solving the linear system (6.18).

In particular, when $r = s = t = 1$, the invariant subspace $\mathrm{Span}\{\xi_1, \eta_1, \zeta_1\}$ of A^{T} is bi-orthonormal to $\mathrm{Ker}(C_3) = \mathrm{Span}\{e_1, e_2, e_3\}$. By Theorem 4.2, there exists a boundary control matrix $D \in \mathcal{D}_{N-3}$, such that the state $(u_1, u_2, u_3)^{\mathrm{T}}$ of synchronization by 3-groups is independent of the applied boundary controls.

(ii) A admits two eigenvectors f_r and g_s contained in $\mathrm{Span}\{e_1, e_2, e_3\}$, corresponding to eigenvalues λ and μ, respectively. Let $f_1, f_2, \cdots, f_r$ and $g_1, g_2, \cdots, g_s$ be the Jordan chains corresponding to these right eigenvectors:

$$\begin{cases} Af_i = \lambda f_i + f_{i+1}, & 1 \leq i \leq r, \ e_{r+1} = 0, \\ Ag_j = \mu g_j + g_{j+1}, & 1 \leq j \leq s, \ g_{s+1} = 0 \end{cases} \tag{6.19}$$

and let $\xi_1, \xi_2, \cdots, \xi_r; \eta_1, \eta_2, \cdots, \eta_s$ be the Jordan chains corresponding to the related left eigenvectors:

$$\begin{cases} A^{\mathrm{T}}\xi_i = \lambda\xi_i + \xi_{i-1}, & 1 \leq i \leq r, \ \xi_0 = 0, \\ A^{\mathrm{T}}\eta_j = \mu\eta_j + \eta_{j-1}, & 1 \leq j \leq s, \ \eta_0 = 0, \end{cases} \tag{6.20}$$

such that

$$(f_i, \xi_l) = \delta_{il}, \quad (g_j, \eta_m) = \delta_{jm}, \quad i, l = 1, \cdots, r; \ j, m = 1, \cdots s \tag{6.21}$$

and

$$(f_i, \eta_m) = (g_j, \xi_l) = 0, \quad i, l = 1, \cdots, r; \ j, m = 1, \cdots s. \tag{6.22}$$

Taking the inner product with $\xi_i, \ \eta_j$ on both sides of (1.1)–(1.2), respectively, and denoting

$$\phi_i = (U, \xi_i), \quad \psi_j = (U, \eta_j), \quad i = 1, \cdots r; \ j = 1, \cdots s, \tag{6.23}$$

we get again the reduced problems (6.12)–(6.13).

In this case, since only two eigenvectors f_r, g_s are contained in the invariant subspace $\mathrm{Span}\{e_1, e_2, e_3\}$ of dimension 3, either f_{r-1} or g_{s-1} is contained in $\mathrm{Span}\{e_1, e_2, e_3\}$. For fixing idea, we assume that f_{r-1} is contained in $\mathrm{Span}\{e_1, e_2, \ e_3\}$. Then we have

$$\begin{cases} e_1 = \alpha_1 f_r + \alpha_2 f_{r-1} + \alpha_3 g_s, \\ e_2 = \beta_1 f_r + \beta_2 f_{r-1} + \beta_3 g_s, \\ e_3 = \gamma_1 f_r + \gamma_2 f_{r-1} + \gamma_3 g_s. \end{cases} \tag{6.24}$$

By (6.6) we have

$$t \geq T: \quad U = (u_1\alpha_1 + u_2\beta_1 + u_3\gamma_1)f_r + (u_1\alpha_2 + u_2\beta_2 + u_3\gamma_2)f_{r-1} \\ + (u_1\alpha_3 + u_2\beta_3 + u_3\gamma_3)g_s. \tag{6.25}$$

Noting (6.21)–(6.22), we have

$$t \geq T: \quad \begin{cases} \phi_r = u_1\alpha_1 + u_2\beta_1 + u_3\gamma_1, \\ \phi_{r-1} = u_1\alpha_2 + u_2\beta_2 + u_3\gamma_2, \\ \psi_s = u_1\alpha_3 + u_2\beta_3 + u_3\gamma_3. \end{cases} \tag{6.26}$$

Since e_1, e_2, e_3 are linearly independent, the linear system (6.24) is invertible, then the state $(u_1, u_2, u_3)^{\mathrm{T}}$ of synchronization by 3-groups can be determined by solving the linear system (6.26).

In particular, when $r = 2$, $s = 1$, the invariant subspace $\mathrm{Span}\{\xi_1, \xi_2, \eta_1\}$ of A^{T} is bi-orthonormal to $\mathrm{Span}\{e_1, e_2, e_3\}$. By Theorem 4.2, there exists a boundary control matrix $D \in \mathcal{D}_{N-3}$, such that the state $(u_1, u_2, u_3)^{\mathrm{T}}$ of synchronization by 3-groups is independent of the applied boundary controls.

(iii) A admits only one eigenvector f_r contained in $\mathrm{Span}\{e_1, e_2, e_3\}$, corresponding to the eigenvalue λ. Let $f_1, f_2, \cdots, f_r$ be the Jordan chains corresponding to this right eigenvector:

$$Af_i = \lambda f_i + f_{i+1}, \quad 1 \leq i \leq r, \ f_{r+1} = 0, \tag{6.27}$$

and let $\xi_1, \xi_2, \cdots, \xi_r$ be the Jordan chains corresponding to the related left eigenvector:

$$A^{\mathrm{T}}\xi_i = \lambda\xi_i + \xi_{i-1}, \quad 1 \leq i \leq r, \ \xi_0 = 0 \tag{6.28}$$

such that

$$(f_i, \xi_l) = \delta_{il}, \quad i, l = 1, \cdots, r. \tag{6.29}$$

Taking the inner product with ξ_i on both sides of (1.1)–(1.2), and denoting

$$\phi_i = (U, \xi_i), \quad i = 1, \cdots r, \tag{6.30}$$

we get again the reduced problem (6.12).

In this case, since only one eigenvector f_r is contained in the invariant subspace $\mathrm{Span}\{e_1, e_2, e_3\}$ which is of dimension 3, f_{r-1} and f_{r-2} are necessarily contained in $\mathrm{Span}\{e_1, e_2, e_3\}$, then we have

$$\begin{cases} e_1 = \alpha_1 f_r + \alpha_2 f_{r-1} + \alpha_3 f_{r-2}, \\ e_2 = \beta_1 f_r + \beta_2 f_{r-1} + \beta_3 f_{r-2}, \\ e_3 = \gamma_1 f_r + \gamma_2 f_{r-1} + \gamma_3 f_{r-2}. \end{cases} \tag{6.31}$$

By (6.6) we have

$$t \geq T: \quad U = (u_1\alpha_1 + u_2\beta_1 + u_3\gamma_1)f_r + (u_1\alpha_2 + u_2\beta_2 + u_3\gamma_2)f_{r-1} \\ + (u_1\alpha_3 + u_2\beta_3 + u_3\gamma_3)f_{r-2}. \tag{6.32}$$

Noting (6.29), we have

$$t \geq T: \quad \begin{cases} \phi_r = u_1\alpha_1 + u_2\beta_1 + u_3\gamma_1, \\ \phi_{r-1} = u_1\alpha_2 + u_2\beta_2 + u_3\gamma_2, \\ \phi_{r-2} = u_1\alpha_3 + u_2\beta_3 + u_3\gamma_3. \end{cases} \tag{6.33}$$

Since e_1, e_2, e_3 are linearly independent, the linear system (6.31) is invertible, then the state $(u_1, u_2, u_3)^{\mathrm{T}}$ of synchronization by 3-groups can be determined by solving the linear system (6.33).

In particular, when $r = 3$, the invariant subspace $\{\xi_1, \xi_2, \xi_3\}$ of A^{T} is bi-orthonormal to $\mathrm{Span}\{e_1, e_2, e_3\}$. By Theorem 4.2, there exists a boundary control matrix $D \in \mathcal{D}_{N-3}$, such that the state $(u_1, u_2, u_3)^{\mathrm{T}}$ of synchronization by 3-groups is independent of the applied boundary controls.

References

[1] Bardos, C., Lebeau, G. and Rauch, J., Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, *SIAM J. Control Optim.*, **30**, 1992, 1024–1065.

[2] Hu, L., Li, T.-T. and Rao, B. P., Exact boundary synchronization for a coupled system of 1-D wave equations with coupled boundary conditions of dissipative type, *Communications on Pure and Applied Analysis*, **13**, 2014, 881–901.

[3] Huygens, C., Horologium Oscillatorium Sive de Motu Pendulorum ad Horologia Aptato Demonstrationes Geometricae, Apud F. Muguet, Parisiis, 1673.

[4] Li, T.-T. and Rao, B. P., Exaxt synchronization for a coupled system of wave equation with Dirichlet boundary controls, *Chin. Ann. Math., Ser. B*, **34**(1), 2013, 139–160.

[5] Li, T.-T. and Rao, B. P., A note on the exact synchronization by groups for a coupled system of wave equations, *Math. Meth. Appl. Sci.*, **38**(13), 2015, 2803–2808.

[6] Li, T.-T. and Rao, B. P., On the exactly synchronizable state to a coupled system of wave equations, *Portugaliae Math.*, **72**, 2015, 83–100.

[7] Li, T.-T. and Rao, B. P., Exact synchronization by groups for a coupled system of wave equations with Dirichlet boundary controls, *J. Math. Pures Appl.*, **105**(1), 2016, 86–101.

[8] Li, T.-T. and Rao, B. P., Exact boundary controllability for a coupled system of wave equations with Neumann controls, *Chin. Ann. Math., Ser. B*, **38**(2), 2017, 473–488.

[9] Li, T.-T., Rao, B. P. and Hu, L., Exact boundary synchronization for a coupled system of 1-D wave equations, *ESAIM*: *Control, Optimisation and Calculus of Variations*, **20**, 2014, 339–361.

[10] Li, T.-T., Rao, B. P. and Wei, Y. M., Generalized exact boundary synchronization for a coupled system of wave equations, *Discrete Contin. Dyn. Syst.*, **34**, 2014, 2893–2905.

[11] Lions, J.-L. and Magenes, E., Non-homogeneous Boundary Value Problems and Applications, **I**, Springer-Verlag, New York, Heidelberg, 1972.

[12] Pazy, A., Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, **44**, Springer-Verlag, New York, 1983.

[13] Wiener, N., Cybernetics, or Control and Communication in the Animal and the Machine, 2nd ed., the M.I.T. Press, Cambridge, Mass., John Wiley & Sons, Inc., New York, London, 1961.