Chin. Ann. Math. Ser. B
39(2), 2018, 253–280
DOI: 10.1007/s11401-018-1063-7

Chinese Annals of Mathematics, Series B

Mathematical Analysis of a Chemotaxis-Type Model of Soil Carbon Dynamic*

Alaaeddine HAMMOUDI[1] Oana IOSIFESCU[1]

(*Dedicated to Philippe G. Ciarlet on the occasion of his* 80*th birthday*)

Abstract The goal of this paper is to study the mathematical properties of a new model of soil carbon dynamics which is a reaction-diffusion system with a chemotactic term, with the aim to account for the formation of soil aggregations in the bacterial and microorganism spatial organization (hot spot in soil). This is a spatial and chemotactic version of MOMOS (Modelling Organic changes by Micro-Organisms of Soil), a model recently introduced by M. Pansu and his group. The authors present here two forms of chemotactic terms, first a "classical" one and second a function which prevents the overcrowding of microorganisms. They prove in each case the existence of a nonnegative global solution, and investigate its uniqueness and the existence of a global attractor for all the solutions.

Keywords Soil organic carbon dynamics, Reaction-Diffusion-Advection system, Positive weak solutions, Periodic weak solutions
2000 MR Subject Classification 35B09, 35B10, 35D30, 35K51, 35Q92

1 Introduction

Chemotaxis is the ability of some bacteria to direct their movement according to the gradient of chemicals contained in their environment. In soil, some bacteria microorganisms that degrade organic carbon (SOC for short) are motile and chemotactic. This phenomenon is observed in experiments (see [1]) and on field. Nevertheless to our best knowledge no model of terrestrial carbon cycle adresses this issue. Indeed, these models are essentially compartimental corresponding naturally to systems of ordinary differential equations (e.g. Century, RothC, MOMOS) (see [2]). They are used globally to estimate soil CO_2 emissions in local land management and crop optimization, among other things.

Very few prototypes of spatial soil organic model have been proposed. Some of them use systems of partial differential equations: Balesdent et al. [3] combined vertical directed transport of organic carbon with a degradation phenomenon and diffusion. More recently, Goudjo

Manuscript received June 27, 2017. Revised September 19, 2017.
[1]Institut Montpelliérain Alexander Grothendieck, CNRS, Univ. Montpellier, Place Eugène Bataillon, 34095, Montpellier, France.
E-mail: alaaeddine.hammoudi@univ-montp2.fr iosifescu@math.univ-montp2.fr
*This work was supported by the Laboratories of Excellence (LabEx) NUMEV (solutions Numériques, Matricielles et Modélisation pour l'Environnement et le Vivant), the LabEx CEMEB (Centre Méditerranéen de l'Environnement et de la Biodiversité) and the Ecoles Doctorales SIBAGHE and I2S of Montpellier.

et al. [4] proposed a three dimensional model for dissolved organic matter using also a system of PDEs. Other authors opted for a finite sequence of interconnected systems of ordinary differential equations each localized in a soil layer (see [5]).

We previously studied the model MOMOS proposed by Pansu [6–7], which is a nonlinear system of ordinary differential equations (see [8]) written as

$$\dot{\mathbf{y}} = \mathbf{g}(t, \mathbf{y}),$$

where

$$\mathbf{y} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

and

$$\mathbf{G}(t, \mathbf{y}) = \begin{pmatrix} -k_1(t)u - q(t)u^2 + k_2(t)v + k_3(t)w + f(t) \\ k_1(t)u - (k_2(t) + k_4(t))v \\ k_4(t)v - k_3(t)w. \end{pmatrix}.$$

In these equations the unknown u models the alive microbial biomass, whereas the unknowns v and w are soil organic matters with distinct decomposition rates.

In reality, the nonnegative functions $k_i,\ i \in \{1, 2, 3, 4\}$, q and f depend not only on time but also on space because of the variability in soil clay content. The phenomena described by MOMOS can also be subjected to the influence of transport and sedimentation through transport and diffusion.

In order to test the effect of soil heterogeneity we studied in [9] the following reaction-diffusion-advection initial problem:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \operatorname{div}(\mathbf{A}_i(t, x)\nabla u_i) + \mathbf{B}_i(t, x)\nabla u_i = g_i^+(t, x, \mathbf{u}), \quad (t, x) \in Q_T := (0, T) \times \Omega, \\ \gamma\left(\mathbf{A}_i(t, x)\nabla u_i\right) \cdot \nu + \beta_i(t, x)u_i = 0, \quad (t, x) \in \Sigma_T := (0, T) \times \partial\Omega, \\ u_i(0) = u_{i,0} \quad \text{in } \Omega, \end{cases}$$

where Ω is a domain in $\mathbb{R}^n$ representing the soil, $\mathbf{A}_i$ is a diffusion matrix and $\mathbf{B}_i$ a transport vector, for each $i = 1, 2, 3$.

In [9] the boundary conditions were either of Dirichlet type ($\gamma = 0,\ \beta_i \equiv 1$) or of Neumann-Robin type ($\gamma = 1,\ \beta_i(t, x) \geq 0$). The right hand side term of (1.1) was

$$\mathbf{g}^+(t, x, \mathbf{u}) := \begin{pmatrix} -k_1(t, x)u_1 - q(t, x)|u_1|u_1 + k_2(t, x)u_2 + k_3(t, x)u_3 + f(t, x) \\ k_1(t, x)u_1 - (k_2(t, x) + k_4(t, x))u_2 \\ k_4(t, x)u_2 - k_3(t, x)u_3 \end{pmatrix},$$

where we replaced the term $q(t, x)u_1^2$ with $q(t, x)|u_1|u_1$ for more accuracy, since $q(t, x)u_1$ corresponds to a kinetic coefficient that cannot be negative. We assumed there that the diffusion matrices $\mathbf{A}_i$ were bounded, symmetric and coercive:

$$\begin{aligned} &\mathbf{A}_i \in L^\infty(Q_T)^{n \times n}, \\ &\mathbf{A}_i(t, x)\zeta \cdot \zeta \geq c|\zeta|^2, \quad \forall \zeta \in \mathbb{R}^n, \quad \text{a.e. in } Q_T \text{ with } c > 0 \end{aligned}$$

and the transport vectors $\mathbf{B}_i$ were bounded on Q_T:

$$\mathbf{B}_i \in L^\infty(Q_T)^N, \quad |(\mathbf{B}_i(t,x))_j| \le c_{\max} \quad \text{a.e. in } Q_T \text{ for all } 1 \le j \le n.$$

Also we assumed that the functions k_j, β_i and q were nonnegative and bounded, i.e., for all $j = 1, 2, 3, 4$ and $i = 1, 2, 3$,

$$\begin{aligned} &k_j,\, q \in L^\infty(Q_T), \quad 0 \le k_j(t,x), q(t,x) \le C_{\max} \quad \text{a.e. on } Q_T,\\ &\beta_i \in L^\infty(\Sigma_T), \quad 0 \le \beta_i(t,x) \le C_{\max} \quad \text{a.e. on } \Sigma_T, \end{aligned}$$

where constant $C_{\max} > 0$. Finally we assumed that the initial data and input were nonnegative and bounded:

$$u_{i,0} \in L^2_+(\Omega), \quad f \in L^2_+(Q_T), \quad f(t,x) \le C_{\max} \quad \text{a.e. on } Q_T.$$

In [9] we proved first that this model had a unique weak solution. We were looking for weak solutions, because initial inputs were not regular enough to give rise to more "regular" solutions. Second, for periodic data, we proved the existence of a maximal and a minimal periodic solution of this system. In some particular cases, the minimal and the maximal periodic solutions coincide and this function becomes a global attractor for any bounded solution of the periodic system.

In the present work a new PDEs model is considered to take account of chemotaxis. The chemotactic movement of bacteria to root exudates is well known to play an important role in rhizosphere colonisation. Field studies with tracers and laboratory experiments using soil columns were both used to demonstrate the effect of chemotaxis on microbial movements. So, the model proposed here can represent the spatial heterogeneity of soil microbial biomass, highlighted by recent observations at submicron scale (see [1]).

The new model derived from a simplified MOMOS ODEs model, which comprised only two differential equations instead of the three originally, where the microbial biomass was u and the organic matter was v. As additional simplifying hypothesis soil temperature, soil moisture, soil texture and organic input were considered to be isotropic and constant with time. Hence, the simplified ODEs model can be expressed as

$$\begin{cases} \dot{u} = -k_1 u - q u^2 + k_2 v, \\ \dot{v} = -k_2 v + k_1 u + f \end{cases}$$

with the initial conditions (u_0, v_0), where k_1 is the microbial mortality rate, k_2 is the soil carbon degradation rate, q is the metabolic quotient and f is the soil organic carbon input. It can be proved that the unique positive steady state (u_0^*, v_0^*) is stable (see [8]).

The chemotaxis-type model was finded following the conventional Keller-Segel approach (see [10]), using an advection-diffusion system. This comprised two parabolic equations in a smooth domain with no-flux boundary conditions. The advection term was controlled by the gradient of the chemo-attractant. Applying the same principles to our problem leads to the following reaction-diffusion-chemotaxis system (P_h):

$$\begin{cases} \partial_t u - a\Delta u = -\beta \mathrm{div}(h(u)\nabla v) - k_1 u - q|u|u + k_2 v, \quad (t,x) \in Q_T, \\ \partial_t v - d\Delta v = -k_2 v + k_1 u + f, \quad (t,x) \in \Omega_T, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0, \quad (t,x) \in \Sigma_T, \\ u(0) = u_0 \quad \text{in } \Omega, \\ v(0) = v_0 \quad \text{in } \Omega. \end{cases} \tag{P_h}$$

The parameter β is the chemotaxis sensitivity, a and d are the diffusion coefficients of microbes and soil organic carbon respectively, Ω is a smooth and bounded domain, and $h(\cdot)$ is a continuous function, involved in the modelling of chemotaxis. As bacteria can release exoenzymes to avoid overcrowding, the function h can be selected to limit overcrowding, as required. This new model is, therefore, a new variation of the Keller-Segel approach (see [10]) with the reaction part modified to fit the MOMOS model. In the first equation of (P_h), we change again the term qu^2 by $q|u|u$ (see [9]).

We prove here (see Appendix 1) the existence of Turing patterns that may provide possible explanations for the formation of soil aggregations, for the bacterial (see [21]) and microorganisms spatial organizations (hotspots in soil) or justify the formation of the microscopic patterns observed by Vogel et al. [1]. Although spatial heterogeneity can be verified visually in a numerical simulation (see Appendix 2), formal mathematical analysis is required to confirm its emergence and to provide a mathematical proof of the necessary conditions. The mathematical criteria are based on matrices derived from equations and analysed using conditions on the determinant, trace and eigenvalues.

Keller-Segel model was the earliest mathematical system involving chemotaxis (see [10]). Many other models emerged specially in biology and ecology. Most authors focused their efforts essentially on existence and on asymptotic behaviour of solutions in one or two dimensional domains in order to avoid blow-up of solutions (see [11–15] and references therein).

Unlike the classical Keller-Segel model, where equations are coupled only by the chemotactic term, the system of partial differential equations (P_h) is also coupled through the reaction term. More specifically, the organic matter will not only attract microorganisms, but part of it will be "transformed", under a degradation process, to microorganisms. This mechanism introduces a supplementary linear coupling term in the first equation of this model. Many authors (see [11–13] and references therein) already considered reaction coupling terms, but under some restrictive conditions, which are not verified here. This feed-back in the chemotactic equation is not compatible with mass conservation of microorganisms, unlike in [12, 15]. Furthermore, neither the boundedness of the microorganisms total mass nor the positivity and the boundedness (existence of threshold) of the solution remain immediate, unlike in [11, 13–14].

Our main concern here is to prove the existence of a unique solution to this minimal MOMOS model improved by adding chemotaxis effect. We consider two chemotactic functions h, a "classical" one, $h(u) = u$, and a second one which prevents overcrowding of microorganisms, $h(u) = u(M-u)$ if $0 \le u \le M$ and zero otherwise, proposed by Wrzosek [15].

This paper is organized as follows. Section 2 introduces some notations, results and tools used throughout the paper. Section 3 presents sufficient conditions to get global solutions, and to prove the existence of an exponential attractor, in the case where $h(u) = u$. Section 4 is concerned with the second chemotactic function, where the chemotactic term cancels when u achieves the threshold M, which helps to prove that any local solution is actually global. In Sections 3–4 the domain is two dimensional. In Section 5, still keeping the second form of h and for domains of dimension less than or equal to 3 (the dimension 3 is particularly interesting in applications), we prove the existence of a unique solution, with less restrictions on the initial conditions and forcing term than in Section 4. In Appendix 1 we prove that chemotactic term in system (P_h) is mandatory to obtain Turing patterns and in Appendix 2 we give some numerical simulations.

2 Mathematical Preliminary and Notations

Unless it is explicitly indicated, Ω is a bounded region in $\mathbb{R}^2$ of $\mathcal{C}^3$ class, the constants a, β, q, d, k_1 and k_2 are nonnegative, and f is a nonegative function belonging to an admissible space to be fixed later. In all that follows C denotes a positive constant which may vary from line to line.

We recall here some known results (see [16–17] and references therein) that will help afterwards.

Interpolation space For $0 \le s_0 < s < s_1 < \infty$, $H^s(\Omega)$ is the interpolation space $[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta$ with $s = (1-\theta)s_0 + \theta s_1$ between $H^{s_0}(\Omega)$ and $H^{s_1}(\Omega)$. Furthermore, we have

$$\| \cdot \|_{H^s} \le \| \cdot \|_{H^{s_0}}^{1-\theta} \| \cdot \|_{H^{s_1}}^{\theta}. \tag{2.1}$$

Embedding theorem When $0 < s < 1$, $H^s(\Omega) \subset L^p(\Omega)$ for $\frac{1}{p} = \frac{1-s}{2}$ with the estimate

$$\| \cdot \|_{L^p} \le C_s \| \cdot \|_{H^s}.$$

When $s = 1$, $H^1(\Omega) \subset L^q(\Omega)$ for any $1 \le q < \infty$ and

$$\| \cdot \|_{L^q} \le C_{q,p} \| \cdot \|_{H^1}^{1-\frac{p}{q}} \| \cdot \|_{L^p}^{\frac{p}{q}}, \tag{2.2}$$

where $1 \le p \le q < \infty$.

When $s > 1$, $H^s(\Omega) \subset C(\overline{\Omega})$ with continuous embedding.

Fractional power of the Laplace operator (see [12] and [17, Chapter 2.7]) Let $a_0, a_1 > 0$ be constants and $L = -a_1\Delta + a_0$ be the Laplace operator equipped with the Newman boundary conditions, with the domain $\mathcal{D}(L) = \left\{u \in H^2(\Omega);\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\right\} = H^2_N(\Omega)$. Thus L is a positive definite self-adjoint operator of $L^2(\Omega)$. For $\theta > 0$, the fractional power on L is defined and noted L^θ and L^θ is also a positive definite self-adjoint operator on $L^2(\Omega)$. More

$$\mathcal{D}(L^\theta) = \begin{cases} H^{2\theta}(\Omega), & 0 \le \theta < \dfrac{3}{4}, \\ H^{2\theta}_N(\Omega), & \dfrac{3}{4} < \theta \le \dfrac{3}{2} \end{cases}$$

with the norm equivalence.

There are some useful inequalities.

Biler's lemma (see [18]) Let $0 \leq u \in H^1(\Omega)$ and $N^1_{\log}(u) := \|(u+1)\log(u+1)\|_{L^1}$. For any $\eta > 0$,

$$\|u\|^3_{L^3} \leq \eta \|u\|_{H^1} N^1_{\log}(u) + p(\eta^{-1})\|u\|_{L^1}, \tag{2.3}$$

where $p(\cdot)$ denotes here some increasing function.

Let $\varepsilon \in (0,1]$. It is proved in [13], (2.10)–(2.12) that

$$\|\nabla(u\nabla v)\|_{L^2} \leq C_\varepsilon \|u\|_{H^1}\|v\|_{H^{2+\varepsilon}} \quad \text{for all } u \in H^1(\Omega),\ v \in H^{2+\varepsilon}(\Omega), \tag{2.4}$$

$$\|\nabla(u\nabla v)\|_{L^2} \leq C_\varepsilon \|u\|_{H^{1+\varepsilon}}\|v\|_{H^2} \quad \text{for all } u \in H^{1+\varepsilon}(\Omega),\ v \in H^2(\Omega), \tag{2.5}$$

$$\|\nabla(u\nabla v)\|_{H^1} \leq C \|u\|_{H^2}\|v\|_{H^3} \quad \text{for all } u \in H^2(\Omega),\ v \in H^3(\Omega). \tag{2.6}$$

Local existence We need first to prove the existence of local solution of (P_h). For this purpose, we use the result obtained by Yagi and based on the Galerkin method (see [12, 17]).

Let V and H be seperable Hilbert spaces with dense and compact embedding $V \subset H$. Let V' be the dual space of V and identify H and H' to get

$$V \subset H \subset V'.$$

The duality product between V and V' is denoted by $\langle \cdot, \cdot \rangle$. It coincides with the scalar product on H denoted by $(\cdot, \cdot)$.

Consider the following Cauchy problem of a semilinear abstract differential equation:

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t} + AU = G(U) + F(t), \quad 0 < t \leq T, \\ &U(0) = U_0 \end{aligned} \tag{2.7}$$

in the space V'.

Here, A is the positive definite self-adjoint operator of H defined by a symmetric sesquilinear form $a(U, \widetilde{U})$ on V, with $\langle AU, \widetilde{U} \rangle_{V,V'} = a(U, \widetilde{U})$.

Assumptions on $a(\cdot,\cdot)$

(a.i) $\|a(U,\widetilde{U})\|_H \leq M\|U\|_V \|\widetilde{U}\|_V,\ U, \widetilde{U} \in V,$

(a.ii) $a(U,U) \geq \delta \|U\|^2_V,\ U \in V$

with constants δ, $M > 0$. The operator A is also bounded from V to V'.

Assumptions on $G(\cdot)$ $G(.)$ is a continuous function from V to V', which satisfy

(g.i) For each $\zeta > 0$, there exists an increasing continuous function $\phi_\zeta : [0,\infty) \to [0,\infty)$ such that

$$\|G(U)\|_{V'} \leq \zeta \|U\|_V + \phi_\zeta(\|U\|_H), \quad U \in V.$$

(g.ii) For each $\zeta > 0$, there exists an increasing continuous function $\psi_\zeta : [0,\infty) \to [0,\infty)$ such that

$$\|G(U) - G(\widetilde{U})\|_{V'} \leq \zeta \|U - \widetilde{U}\|_V + \psi_\zeta(\|U\|_H + \|\widetilde{U}\|_H)$$

$$\times \|U - \widetilde{U}\|_H (\|U\|_V + \|\widetilde{U}\|_V + 1), \quad U, \widetilde{U} \in V.$$

Finally $F(\cdot) \in L^2(0,T;V')$ is a given function and $U_0 \in H$ is an initial value. Then, we have the following result (see [12]).

Theorem 2.1 *Under Assumptions* (a.i), (a.ii), (g.i) *and* (g.ii) *and for every* $F(\cdot) \in L^2(0,T;V')$ *and* $U_0 \in H$, *there exists a unique local solution* U *of* (2.7) *such that*

$$U \in H^1(0,T(U_0,F);V') \cap C([0,T(U_0,F)];H) \cap L^2(0,T(U_0,F),V).$$

Here $T(U_0,F)$ *is determined by the norm* $\|U_0\|_H$ *and* $\|F\|_{L^2(0,T;V')}$.

3 First Case: $h(u) = u$

3.1 Local existence and positivity

Let ε_0 arbitrarily fixed, $\varepsilon_0 \in (0,1)$.

Theorem 3.1 *Let* $u_0 \in L^2(\Omega)$, $v_0 \in H^{1+\varepsilon_0}(\Omega)$ *and* $f \in L^2(0,T;H^{\varepsilon_0}(\Omega))$ *be nonnegative functions. Then* (P_h) *has a unique nonnegative local solution on an interval* $[0,T_0]$ *such that*

$$\begin{aligned} &u \in H^1(0,T_0;H^1(\Omega)') \cap C([0,T_0];L^2(\Omega)) \cap L^2(0,T_0,H^1(\Omega)), \\ &v \in H^1(0,T_0;H^{\varepsilon_0}(\Omega)) \cap C([0,T_0];H^{1+\varepsilon_0}(\Omega)) \cap L^2(0,T_0,H_N^{2+\varepsilon_0}(\Omega)), \end{aligned}$$

where T_0 *depends only on* $\|f\|_{L^2(0,T;H^{\varepsilon_0}(\Omega))}$, $\|u_0\|_{L^2(\Omega)}$ *and* $\|v_0\|_{H^{1+\varepsilon_0}(\Omega)}$.

Proof **First Step** Construction of a unique local solution.

Let $A_1 = -a\Delta + k_1$ and $A_2 = -d\Delta + k_2$ be two operators with the same domain $H_N^2(\Omega)$. A_1 and A_2 are positive self-adjoint operators on $L^2(\Omega)$. We can then define their corresponding fractional power operators (see [17]), as described in the previous section.

Let $V = H^1(\Omega) \times H_N^{2+\varepsilon_0}(\Omega)$ and $H = L^2(\Omega) \times H^{1+\varepsilon_0}(\Omega)$. Identifying H with its dual space gives $V \subset H = H' \subset V'$ and $V' = (H^1(\Omega))' \times H^{\varepsilon_0}(\Omega)$ with the duality product:

$$\langle U, \widetilde{U} \rangle_{V,V'} = \langle u, \widetilde{u} \rangle_{H^1,(H^1)'} + \langle A_2^{1+\frac{\varepsilon_0}{2}} v, A_2^{\frac{\varepsilon_0}{2}} \widetilde{v}) \rangle_{L^2,L^2},$$

where $U = (u,v) \in V$ and $\widetilde{U} = (\widetilde{u}, \widetilde{v}) \in V'$.

We also set a symmetric sesquilinear form on $V \times V$,

$$a(U,\widetilde{U}) = \int_\Omega \{a\nabla u \cdot \nabla \widetilde{u} + k_1 u \widetilde{u}\} \mathrm{d}x + (A_2^{1+\frac{\varepsilon_0}{2}} v, A_2^{1+\frac{\varepsilon_0}{2}} \widetilde{v})_{L^2}$$

for $U = (u,v)$ and $\widetilde{U} = (\widetilde{u}, \widetilde{v}) \in V$.

This form is in fact a linear isomorphism A from V to V':

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

and A becomes a positive definite self-adjoint operator in H.

Finally let $f(\cdot) \in L^2(0,T,H^{\varepsilon_0}(\Omega))$ and let $G : V \to V'$ be the mapping

$$G(U) := \begin{pmatrix} \beta \nabla (u \nabla v) - q|u|u + k_2 v \\ k_1 u \end{pmatrix}$$

with $U=(u,v)\in V$.

Then (P_h) is the following semilinear differential equation:

$$\begin{aligned}&\frac{\mathrm{d}U}{\mathrm{d}t}+AU=G(U)+F(t)\quad \text{in } V',\quad 0<t\le T,\\ &U(0)=U_0,\end{aligned}\tag{3.1}$$

where $F(t)=\binom{0}{f(t)}$.

In order to apply the existence result of Theorem 2.1 to problem (3.1), let us verify the assumptions on $a(\cdot,\cdot)$ and $G(\cdot)$.

The assumptions on $a(\cdot,\cdot)$ are classically satisfied (see for example [12]).

For the conditions on G we have that for an arbitrary $U=(u,v)\in V$ and $\delta>0$,

$$\begin{aligned}\|\nabla\cdot(u\nabla v)\|_{(H^1)'}&\le C\|u\|_{L^4}\|\nabla v\|_{L^4}\le \|u\|_{L^2}^{\frac12}\|u\|_{H^1}^{\frac12}\|v\|_{H^1}^{\frac12}\|v\|_{H^2}^{\frac12}\\ &\le \|u\|_{L^2}^{\frac12}\|u\|_{H^1}^{\frac12}\|v\|_{H^{1+\varepsilon_0}}^{\frac{1+\varepsilon_0}{2}}\|v\|_{H^{2+\varepsilon_0}}^{\frac{1-\varepsilon_0}{2}}\\ &\le C\|U\|_H^{1+\frac{\varepsilon_0}{2}}\|U\|_V^{1-\frac{\varepsilon_0}{2}}\le \zeta\|U\|_V+\phi_\zeta(\|U\|_H)\end{aligned}$$

and

$$\|v\|_{(H^1)'}\le C\|U\|_H,\quad \|u^2\|_{(H^1)'}\le C\|u\|_{L^3}^2\le \zeta\|u\|_{H^1}+\phi_\zeta(\|u\|_{L^2}).$$

Finally it is clear that

$$\|u\|_{H^{\varepsilon_0}}\le \zeta\|u\|_{H^1}+C_\zeta(\|u\|_{L^2}).$$

All these inequalities show that the condition (g.i) is fullfiled.

From the embedding theorem, we have

$$\begin{aligned}\Big|\int_\Omega(\widetilde{u}-u)\nabla v\cdot\nabla\rho\,\mathrm{d}x\Big|&\le C\|\widetilde{u}-u\|_{L^2}\|\widetilde{v}\|_{H^{2+\varepsilon_0}}\|\rho\|_{H^1},\\ \Big|\int_\Omega\nabla(\widetilde{v}-v)u\cdot\nabla\rho\,\mathrm{d}x\Big|&\le C\|u\|_{H^1}\|\widetilde{v}-v\|_{H^{1+\varepsilon_0}}\|\rho\|_{H^1},\end{aligned}$$

and using the interpolation theorem and Young inequality we obtain

$$\begin{aligned}\|u-\widetilde{u}\|_{H^{\varepsilon_0}}&\le C\|U-\widetilde{U}\|_V^{\varepsilon_0}\|U-\widetilde{U}\|_H^{1-\varepsilon_0}\\ &\le \zeta\|U-\widetilde{U}\|_V+C_\zeta\|U-\widetilde{U}\|_H\end{aligned}$$

for an arbitrary $\zeta>0$. On the other hand we have that

$$\begin{aligned}\|u|u|-\widetilde{u}|\widetilde{u}|\|_{(H^1)'}&\le C(\|(|u|-|\widetilde{u}|)u\|_{L^{\frac32}}+\|(u-\widetilde{u})|\widetilde{u}|\|_{L^{\frac32}})\\ &\le C\|u-\widetilde{u}\|_{L^2}(\|u\|_{L^6}+\|\widetilde{u}\|_{L^6}).\end{aligned}$$

All these inequalities permit to show that condition (g.ii) is fullfiled too.

Second Step Positivity of the solution.

Now let us take the following semilinear system:

$$\begin{aligned}&\frac{\mathrm{d}U}{\mathrm{d}t}+AU=\widetilde{G}(U)+F(t),\quad 0<t\le T,\\ &U(0)=U_0,\end{aligned}\tag{3.2}$$

where A, F and Y_0 are defined as previously, and the mapping $\widetilde{G} : V \to V'$ is defined by

$$\widetilde{G}(U) := \begin{pmatrix} \beta\nabla(u\nabla v) - q|u|u + k_2|v| \\ k_1 u \end{pmatrix}.$$

By Theorem 2.1. there exists a local solution $U = (u, v)$ on $[0, T_0] \times \Omega$ with T_0 depending only on U_0 and F. Let us define $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$.

We multiply the first equation by $-u^-$ and we integrate in space. So

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^-\|_{L^2}^2 + a\|\nabla u^-\|_{L^2}^2 + k_1\|u^-\|_{L^2}^2 &\le \int_\Omega \beta u^-\nabla v\nabla u^- \mathrm{d}x \\ &\le \beta\|\nabla v\|_{L^\infty}\int_\Omega u^-|\nabla u^-|\mathrm{d}x\end{aligned}$$

for $0 < t \le T_0$.

Using Young inequality we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^-\|_{L^2}^2 + a\|\nabla u^-\|_{L^2}^2 \le \|\nabla v\|_{L^\infty}(C_\varepsilon\|u^-\|_{L^2}^2 + \varepsilon\|\nabla u^-\|_{L^2}^2)$$

with $\varepsilon > 0$ small enough and $C_\varepsilon > 0$.

Taking $\varepsilon = \frac{a}{\|\nabla v\|_{L^\infty}}$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u^-\|_{L^2}^2 &\le C\|\nabla v\|_{L^\infty}^2\|u^-\|_{L^2}^2 \\ &\le C\|v\|_{H^{2+\varepsilon_0}}^2\|u^-\|_{L^2}^2.\end{aligned}$$

Since $v \in L^2(0, T_0; H^{2+\varepsilon_0}(\Omega))$ and $\|u_0^-\|_{L^2}^2 = 0$ by Gronwall lemma we deduce that u is nonnegative on $[0, T_0]$. By classical results on linear parabolic equations v is nonnegative on $[0, T_0]$ too. So, the nonnegative solution U of (3.2) is also a solution of (3.1).

Remark 3.1 If initial conditions U_0 and data f are not positive, this theorem proves anyway the existence of a local solution. However, as this is an ecology model, only nonnegative solutions make sense.

With minor changes due to our different problem (P_h), we prove as in [13] the following theorems.

Theorem 3.2 *Let $U_0 = (u_0, v_0) \in H^1(\Omega) \times H_N^2(\Omega)$ and $f \in L^2(0, T; H^1(\Omega))$. Then there exists a unique local solution $U = (u, v)$ of (3.1) on an interval $[0, T_{U_0,f}]$ such that*

$$\begin{aligned}&u \in H^1(0, T_{U_0,f}; L^2(\Omega)) \cap C([0, T_{U_0,f}]; H^1(\Omega)) \cap L^2(0, T_{U_0,f}, H_N^2(\Omega)),\\ &v \in H^1(0, T_{U_0,f}; H^1(\Omega)) \cap C([0, T_{U_0,f}]; H_N^2(\Omega)) \cap L^2(0, T_{U_0,f}, H_N^3(\Omega)),\end{aligned}$$

where $T_{U_0,f}$ is determined by $\|f\|_{L^2(0,T;H^1(\Omega))}$, $\|u_0\|_{H^1(\Omega)}$ and $\|v_0\|_{H^2(\Omega)}$.

Theorem 3.3 *Let $U_0 = (u_0, v_0) \in H_N^2(\Omega) \times H_N^3(\Omega)$ and $f \in L^2(0, T; H_N^2(\Omega))$. Then there exists a unique local solution $U = (u, v)$ to (3.1) on an interval $[0, T_{U_0,f}]$ such that*

$$\begin{aligned}&u \in H^1(0, T_{U_0,f}; H^1(\Omega)) \cap C([0, T_{U_0,f}]; H_N^2(\Omega)) \cap L^2(0, T_{U_0,f}, H_N^3(\Omega)),\\ &v \in H^1(0, T_{U_0,f}; H_N^2(\Omega)) \cap C([0, T_{U_0,f}]; H_N^3(\Omega)) \cap L^2(0, T_{U_0,f}, D(A_2^2)(\Omega)),\end{aligned}$$

where $T_{U_0,f}$ is determined by $\|f\|_{L^2(0,T;H^2(\Omega))}$, $\|u_0\|_{H^2(\Omega)}$ and $\|v_0\|_{H^3(\Omega)}$.

3.2 Global existence

This section is devoted to proving the following result.

Theorem 3.4 *Let $\varepsilon_0 \in (0,1)$ and let $u_0 \in L^2(\Omega)$, $v_0 \in H^{1+\varepsilon_0}(\Omega)$ and $f \in L^2(0,T,H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be nonnegative functions. Then there exists a unique global and nonnegative solution (u,v) for the system (P_h) with $h(u)=u$ such that*

$$u \in H^1(0,T;(H^1(\Omega))') \cap C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$
$$v \in H^1(0,T;H^{\varepsilon_0}(\Omega)) \cap C([0,T];H^{1+\varepsilon_0}(\Omega)) \cap L^2(0,T;H_N^{2+\varepsilon_0}(\Omega)).$$

Proof We proceed in two steps.

First Step We show that $\|v\|_{H^1(\Omega)}$ and $N^1_{\log}(u) = \|(u+1)\log(u+1)\|_{L^1(\Omega)}$ are bounded for all $t \in [0,T_0]$.

We consider the function $\log(u+1)$. Since $\nabla\log(u+1) = \frac{\nabla u}{u+1}$, it follows that $\log(u+1) \in L^2(0,T_0;H^1(\Omega))$. Noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \{(u+1)\log(u+1) - u\}\,\mathrm{d}x = \Big\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \log(u+1)\Big\rangle_{H^1,(H^1)'},$$

we obtain from the first equation of (P_h) multiplied by $\log(u+1)$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \{(u+1)\log(u+1) - u\}\mathrm{d}x + 4a\int_\Omega |\nabla\sqrt{u+1}|^2\,\mathrm{d}x$$
$$= \beta\int_\Omega \frac{u}{u+1}\nabla u\nabla v\,\mathrm{d}x + \int_\Omega (-k_1u - qu^2 + k_2v)\log(u+1)\mathrm{d}x.$$

So using Stokes theorem we deduce

$$\int_\Omega \frac{u}{u+1}\nabla u\nabla v\,\mathrm{d}x = \int_\Omega (\log(u+1) - u)\Delta v\,\mathrm{d}x \le \frac{\eta}{2}\|\Delta v\|^2_{L^2(\Omega)} + \frac{1}{2\eta}\|u\|^2_{L^2(\Omega)}.$$

Since

$$(k_1u + qu^2)\log(u+1) \ge k_1((u+1)\log(u+1) - u),$$

if we denote $\Psi(t) = \|(u+1)\log(u+1) - u\|_{L^1(\Omega)}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) + k_1\Psi(t) \le \frac{\eta}{2}\|\Delta v\|^2_{L^2(\Omega)} + \Big(\frac{k_2^2}{2\varepsilon} + \frac{\beta^2}{2\eta}\Big)\|u\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|v\|^2_{L^2(\Omega)}$$

with arbitary ε, $\eta > 0$.

From the second equation of (P_h) multiplied respectively by v and Δv, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^2 + d\int_\Omega |\nabla v|^2\mathrm{d}x + k_2\int_\Omega v^2\mathrm{d}x = k_1\int_\Omega u\,v\,\mathrm{d}x + \int_\Omega vf\,\mathrm{d}x$$
$$\le \Big(\frac{k_1A}{2} + \frac{B}{2}\Big)\|v\|^2_{L^2(\Omega)} + \frac{k_1}{2A}\|u\|^2_{L^2(\Omega)} + \frac{1}{2B}\|f\|^2_{L^2(\Omega)}$$

with arbitrary $A, B > 0$ and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^2 + d\int_\Omega |\Delta v|^2\mathrm{d}x + k_2\int_\Omega |\nabla v|^2\mathrm{d}x = k_1\int_\Omega u\Delta v\,\mathrm{d}x + \int_\Omega f\Delta v\,\mathrm{d}x$$

$$\leq \Big(\frac{k_1 C}{2} + \frac{D}{2}\Big)\|\Delta v\|^2_{L^2(\Omega)} + \frac{k_1}{2C}\|u\|^2_{L^2(\Omega)} + \frac{1}{2D}\|f\|^2_{L^2(\Omega)}$$

with arbitrary $C, D > 0$.

Choosing $\varepsilon = d$, $\eta = k_2$, $A = \frac{k_2}{2k_1}$, $B = \frac{k_2}{2}$, $C = \frac{d}{2k_1}$, $D = \frac{d}{2}$, we deduce

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\Psi(t) + \|v\|^2_{H^1(\Omega)}) + \frac{d}{2}\|\Delta v\|^2_{L^2(\Omega)} + k_2\|v\|^2_{H^1(\Omega)} + k_1\Psi(t)\\ &\leq \Big(\frac{k_1^2}{k_2} + \frac{\beta^2}{2k_2} + \frac{k_2^2}{2d} + \frac{k_1^2}{d}\Big)\|u\|^2_{L^2(\Omega)} + \Big(\frac{1}{k_2} + \frac{1}{d}\Big)\|f\|^2_{L^2(\Omega)}. \qquad (3.3)\end{aligned}$$

By addition of the first two equation of (P_h) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^1(\Omega)} + \|v\|_{L^1(\Omega)}) + q\|u\|^2_{L^2(\Omega)} = \|f\|_{L^1(\Omega)}, \qquad (3.4)$$

which implies that for all $t \in [0, T_0]$ we have the inequality

$$\begin{aligned}&\|u(t)\|_{L^1(\Omega)} + \|v(t)\|_{L^1(\Omega)}\\ &\leq \|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)} + \int_0^t \|f(s)\|_{L^1(\Omega)}\mathrm{d}s. \qquad (3.5)\end{aligned}$$

As $N^1_{\log}(u) = \int_\Omega (u+1)\log(u+1)\,\mathrm{d}x$ we have $\Psi(t) = N^1_{\log}(u(t)) - \|u(t)\|_{L^1(\Omega)}$. Denote $\delta := \max\big(1, \frac{1}{q}\big(\frac{k_1^2}{k_2} + \frac{\beta^2}{2k_2} + \frac{k_2^2}{2d} + \frac{k_1^2}{d}\big)\big)$ and $\sigma := \min(k_1, k_2) > 0$. Therefore from (3.3)–(3.5) we obtain the following inequality:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(N^1_{\log}(u(t)) + \|v(t)\|^2_{H^1(\Omega)} + (\delta - 1)\|u(t)\|_{L^1(\Omega)} + \delta\|v(t)\|_{L^1(\Omega)})\\ &\quad + \sigma(N^1_{\log}(u(t)) + \|v(t)\|^2_{H^1(\Omega)} + (\delta - 1)\|u(t)\|_{L^1(\Omega)} + \delta\|v(t)\|_{L^1(\Omega)})\\ &\leq \sigma(\delta + 1)(\|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)})\\ &\quad + \delta\|f(t)\|_{L^1(\Omega)} + \Big(\frac{1}{k_2} + \frac{1}{d}\Big)\|f(t)\|^2_{L^2(\Omega)} + \sigma\delta\int_0^t \|f(s)\|_{L^1(\Omega)}\mathrm{d}s.\end{aligned}$$

We denote $g(t) = N^1_{\log}(u(t)) + \|v(t)\|^2_{H^1(\Omega)} + (\delta - 1)\|u(t)\|_{L^1(\Omega)} + \delta\|v(t)\|_{L^1(\Omega)}$.

Since $g(t)$ satisfies the following ordinary differential inequality:

$$\begin{aligned}g'(t) + \sigma g(t) &\leq \sigma(\delta + 1)(\|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)}) + \delta\|f(t)\|_{L^1(\Omega)}\\ &\quad + \Big(\frac{1}{k_2} + \frac{1}{d}\Big)\|f(t)\|^2_{L^2(\Omega)} + \sigma\delta\int_0^t \|f(s)\|_{L^1(\Omega)}\mathrm{d}s = C\end{aligned}$$

with $C > 0$ depending only on $\|f\|_{L^\infty(0,T;L^2(\Omega))}$, $\|u_0\|_{L^2(\Omega)}$ and $\|v_0\|_{H^{1+\varepsilon_0}(\Omega)}$, we get

$$g(t) \leq \mathrm{e}^{-\sigma t} g(0) + C \quad \text{for all } t \geq 0. \qquad (3.6)$$

Thus the inequality

$$\begin{aligned}&N^1_{\log}(u(t)) + \|v(t)\|^2_{H^1(\Omega)}\\ &\leq N^1_{\log}(u_0) + \|v_0\|^2_{H^1(\Omega)} + (\delta - 1)\|u_0\|_{L^1(\Omega)} + \delta\|v_0\|_{L^1(\Omega)} + C \qquad (3.7)\end{aligned}$$

holds for all $t \in [0, T_0]$, where the last constant $C > 0$ is independent of T_0 and depends only on $\|f\|_{L^\infty(0,T;L^2(\Omega))}$, $\|u_0\|_{L^2(\Omega)}$ and $\|v_0\|_{H^{1+\varepsilon_0}(\Omega)}$.

Second Step We take $t_1 \in (0, T_0)$ so that $v(t_1) \in H^2_N(\Omega)$ and $u(t_1) \in H^1(\Omega)$ and we set $u(t_1) = u_1$ and $v(t_1) = v_1$. From Theorem 3.1 we already know that such a time t_1 exists, arbitrary small. In this step t varies in $[t_1, T_0]$. From the first equation of (P_h) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2} + a\|\nabla u\|^2_{L^2} + k_1\|u\|^2_{L^2} + q\|u\|^3_{L^3} = \int_\Omega uv\,\mathrm{d}x + \frac{\beta}{2}\int_\Omega u^2\Delta v\,\mathrm{d}x.$$

From Young inequality and interpolation inequality (2.1) we get

$$\begin{aligned}\int_\Omega u^2\Delta v\,\mathrm{d}x &\le \eta\|\Delta v\|^3_{L^3} + \eta^{-\frac{1}{2}}\|u\|^3_{L^3(\Omega)}\\ &\le \eta C\|v\|^2_{H^3}\|v\|^1_{H^1} + \eta^{-\frac{1}{2}}\|u\|^3_{L^3}\end{aligned}$$

with $\eta > 0$ arbitrary.

Therefore (3.7) together with this yields that

$$\int_\Omega u^2\Delta v\,\mathrm{d}x \le \eta C\|v\|^2_{H^3} + \eta^{-\frac{1}{2}}\|u\|^3_{L^3}.$$

In addition

$$\int_\Omega uv\,\mathrm{d}x \le \chi\|u\|^3_{L^3(\Omega)} + \chi^{-\frac{1}{2}}\|v\|^{\frac{3}{2}}_{H^1(\Omega)}$$

with $\chi > 0$ arbitrary.

Using Biler's lemma (2.3) we verify from (3.7) that

$$\|u\|^3_{L^3(\Omega)} \le \eta C\|u\|^2_{H^1} + p(\eta^{-1})$$

with p a positive increasing function, depending on $\|f\|_{L^\infty(0,T;L^2(\Omega))}$, $\|u_0\|_{L^2(\Omega)}$ and $\|v_0\|_{H^{1+\varepsilon_0}(\Omega)}$ as well as the constant $C > 0$.

Thus we deduce the following inequality:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2} + a\|\nabla u\|^2_{L^2} + k_1\|u(t)\|^2_{L^2} + q\|u\|^3_{L^3}\\ &\le \xi(\|v\|^2_{H^3} + \|u\|^2_{H^1}) + p(\xi^{-1})\end{aligned} \tag{3.8}$$

with p a positive increasing function depending on $\|f\|_{L^\infty(0,T;L^2(\Omega))}$, $\|u_0\|_{L^2(\Omega)}$ and $\|v_0\|_{H^{1+\varepsilon_0}(\Omega)}$, $\xi > 0$ an arbitrary constant.

On the other hand, we consider v as a solution of the Cauchy problem

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}v + A_2 v = k_1 u + f, \quad t_1 \le t \le T_0,\\ &v(t_1) \in H^1(\Omega)\end{aligned}$$

in the space $H^1(\Omega)$. Since $k_1 u + f \in L^2(t_1, T_0; H^1(\Omega))$ and $v_1 \in \mathcal{D}(A_2) = H^2_N(\Omega)$ it follows that $v \in L^2(t_1, T_0; \mathcal{D}(A_2^{\frac{3}{2}}) \cap H^1(t_1, T_0; \mathcal{D}(A_2^{\frac{1}{2}})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}A_2^{\frac{1}{2}}v = -A_2^{\frac{3}{2}}v + k_1 A_2^{\frac{1}{2}}u + A_2^{\frac{1}{2}}f, \quad t_1 \le t \le T_0.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A_2 v\|_{L^2}^2 + \|A_2^{\frac{3}{2}} v\|_{L^2}^2 \le C\{\|A_2^{\frac{1}{2}} u\|_{L^2}^2 + \|A_2^{\frac{1}{2}} f\|_{L^2}^2\}.$$

As $D(A^{\frac{3}{2}}) \subset H^3(\Omega)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A_2 v\|_{L^2}^2 + \delta\|v\|_{H^3}^2 \le C\{\|u\|_{H^1}^2 + \|f\|_{H^1}^2\} \tag{3.9}$$

with some $\delta > 0$. Let $a_1 = \min(a, k_1) > 0$. We now sum up (3.9) and (3.8) multiplied by $\frac{2C}{a_1}$, where $C > 0$ is the constant appearing in (3.9). Then it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{C}{a_1}\|u\|_{L^2}^2 + \|A_2 v\|_{L^2}^2\Big\} + C\Big(1 - \xi\frac{2C}{a_1}\Big)\|u\|_{H^1}^2 + \Big(\delta - \xi\frac{2C}{a_1}\Big)\|v\|_{H^3}^2 \\
&\le C_1\{\|f(t)\|_{H^1}^2 + p(\xi^{-1})\}
\end{aligned} \tag{3.10}$$

with some constant $C_1 > 0$ independent of T_0. Choosing ξ small enough we conclude that

$$\int_{t_1}^{s} (\|v(t)\|_{H^3}^2 + \|u(t)\|_{H^1}^2)\,\mathrm{d}t \le C_2\Big\{\|u_1\|_{L^2}^2 + \|v_1\|_{H^2}^2 + \int_{t_1}^{T} (\|f(t)\|_{H^1}^2 + 1)\,\mathrm{d}t\Big\}$$

with some constant $C_2 > 0$ dependent on $\|f\|_{L^\infty(0,T;L^2(\Omega))}$ and the initial condition U_0 through $\|u_0\|_{L^2}$ and $\|v_0\|_{H^{1+\varepsilon_0}}$, but independent of T_0. The norms $\|u\|_{L^2(t_1,T_0;H^1(\Omega))}$ and $\|v\|_{L^2(t_1,T_0;H^3(\Omega))}$ do not depend on T_0 and hence those of $\|u\|_{C([t_1,T_0];L^2(\Omega))}$ and $\|v\|_{C([t_1,T_0];H^2(\Omega))}$ do not depend either.

In particular this shows that the solution (u, v) can be extended as a weak solution beyond the T_0.

3.3 Exponential attractor

Suppose that f is a positive constant function. Then we have the following result.

Proposition 3.1 *Let $u_0 \in H_N^2(\Omega)$ and $v_0 \in H_N^3(\Omega)$ be nonnegative functions. Let u, v be the global solution of (P_h). Then, with some continuous increasing function $p(\cdot)$ the following estimate holds:*

$$\|u(t)\|_{H^2(\Omega)} + \|v(t)\|_{H^3(\Omega)} \le p(\|u_0\|_{H^2(\Omega)} + \|v_0\|_{H^3(\Omega)} + f)$$

for $0 < t < \infty$.

Proof Using (3.10) we deduce the existence of two constants $\sigma > 0$ and $C > 0$ such that

$$\begin{aligned}
\|u(t)\|_{L^2}^2 + \|v(t)\|_{H^2}^2 \le{}& C\mathrm{e}^{-\sigma t}(\|u_0\|_{L^2}^2 + \|v_0\|_{H^2}^2) \\
&+ p(f + N_{\log}^1(u_0) + \|v_0\|_{H^1}).
\end{aligned} \tag{3.11}$$

Multiplying the first equation of (P_h) by Δu and integrating over Ω gives

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 + a\|\Delta u\|_{L^2}^2 + k_1\|\nabla u\|_{L^2}^2 \\
&\le \beta\Big(\varepsilon\|\Delta u\|_{L^2}^2 + \frac{1}{2\varepsilon}\int_\Omega |\nabla u|^2|\nabla v|^2\,\mathrm{d}x + \frac{1}{2\varepsilon}\int_\Omega |u|^2|\Delta v|^2\,\mathrm{d}x\Big)
\end{aligned}$$

$$+\,\varepsilon'\|\nabla u\|_{L^2}^2 + C_{\varepsilon'}\|\nabla v\|_{L^2}^2, \tag{3.12}$$

where $\varepsilon, \varepsilon'$ and $C_{\varepsilon'}$ are positive constants derived from Young inequality. Using technical inequalities proved in [13, Proposition 4.1], we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 + (a-\beta\varepsilon)\|\Delta u\|_{L^2}^2 + (k_1-\varepsilon')\|\nabla u\|_{L^2}^2\\
\le{}& \frac{\beta}{2\varepsilon}\Big(\int_\Omega |\nabla u|^2|\nabla v|^2\,\mathrm{d}x + \int_\Omega |u|^2|\Delta v|^2\,\mathrm{d}x\Big) + C_{\varepsilon'}\|\nabla v(t)\|_{L^2}^2\\
\le{}& \frac{\beta}{2\varepsilon}(\eta\|\Delta u\|_{L^2}^2 + p(\|u\|_{L^2} + \|v\|_{H^2} + \eta^{-1})) + C_{\varepsilon'}\|\nabla v\|_{L^2}^2.
\end{aligned}$$

Taking $\eta = \varepsilon^2,\ \varepsilon = \frac{a}{2\beta}$ leads to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 + a\|\Delta u\|_{L^2}^2 + 2(k_1-\varepsilon')\|\nabla u\|_{L^2}^2\\
\le{}& \frac{\beta^2}{a}p(\|u\|_{L^2} + \|v\|_{H^2(\Omega)}) + C_{\varepsilon'}\|\nabla v\|_{L^2}^2.
\end{aligned} \tag{3.13}$$

Take the second equation of (P_h) operated by Δ, choose $\Delta^2 v$ as a test function and integrate the product in Ω. After some calculations as in [13], we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta v\|_{L^2}^2 + d\|\Delta^2 v\|_{L^2)}^2 + 2k_2\|\nabla\Delta v\|_{L^2}^2 \le \frac{k_1^2}{d}\|\Delta u\|_{L^2}^2. \tag{3.14}$$

We sum (3.14) multiplied by γ and (3.13). Thus we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \gamma\|\nabla\Delta v\|_{L^2}^2) + \gamma d\|\Delta^2 v\|_{L^2}^2 + \Big(a - \frac{\gamma k_1^2}{d}\Big)\|\Delta u\|_{L^2}^2\\
&+ 2(k_1-\varepsilon')(\|\nabla u\|_{L^2}^2 + \frac{k_2\gamma}{k_1-\varepsilon'}\|\nabla\Delta v\|_{L^2}^2)\\
\le{}& p(\|u\|_{L^2} + \|v\|_{H^2}).
\end{aligned}$$

Then for γ and ε' small enough, there exists a positive constant σ' such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \gamma\|\nabla\Delta v\|_{L^2}^2) + \sigma'(\|\nabla u\|_{L^2}^2 + \gamma\|\nabla\Delta v(t)\|_{L^2}^2)\\
\le{}& p(\|u\|_{L^2} + \|v\|_{H^2}).
\end{aligned} \tag{3.15}$$

So, we can find $\chi > 0$ such as (3.11) is valid when $\sigma = \chi$ and

$$\begin{aligned}
&\|u(t)\|_{H^1(\Omega)}^2 + \|v(t)\|_{H^3}^2\\
\le{}& \mathrm{e}^{-\chi t}(\|u_0\|_{H^1}^2 + \|v_0\|_{H^3}^2) + p(f + \|u_0\|_{L^2} + \|v_0\|_{H^2}).
\end{aligned} \tag{3.16}$$

We verify also that

$$\int_0^t (\|\Delta^2 v(s)\|_{L^2}^2 + \|u(s)\|_{H^2}^2)\mathrm{d}s \le C(\|v_0\|_{H^3}^2 + \|u_0\|_{H^1}^2) + tp(f + \|u_0\|_{L^2} + \|v_0\|_{H^2}).$$

Finally, taking the first equation of (P_h) operated by ∇ and multiplied by $\nabla\Delta u$, as in [13], gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|_{L^2}^2 + a\|\nabla\Delta u\|_{L^2}^2 = \beta\int_\Omega \nabla(\nabla\cdot u\nabla v)\cdot\nabla\Delta u\mathrm{d}x$$

$$+ k_1 \int_\Omega \nabla u \cdot \nabla \Delta u \mathrm{d}x + 2q \int_\Omega u \nabla u \cdot \nabla \Delta u \mathrm{d}x$$
$$- k_2 \int_\Omega \nabla v \cdot \nabla \Delta u \mathrm{d}x, \tag{3.17}$$

that is,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2_{L^2(\Omega)} + a\|\nabla\Delta u\|^2_{L^2(\Omega)} \le \frac{a}{2}\|\nabla\Delta u\|^2_{L^2} + C\int_\Omega |\nabla(\nabla\cdot(u\nabla v)|^2 \mathrm{d}x$$
$$+ C\Big(\int_\Omega |u\nabla u|^2 \mathrm{d}x + \|\nabla v\|^2_{L^2}\Big). \tag{3.18}$$

The terms $\int_\Omega |\nabla(\nabla\cdot(u\nabla v)|^2\mathrm{d}x$ and $\int_\Omega |u\nabla u|^2\mathrm{d}x$ of (3.18) can be estimated (see [13, Proof of Proposition 4.1, Step 6]) by

$$\eta\|\nabla\Delta u\|^2_{L^2} + p(\|u\|_{H^1} + \|v\|_{H^3}, +\eta^{-1})$$

with an arbitrary $\eta > 0$. Thus we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2_{L^2} + \frac{a}{2}\|\nabla\Delta u\|^2_{L^2} + \zeta\|\Delta u\|^2_{L^2}$$
$$\le \eta\|\nabla\Delta u\|^2_{L^2} + p(\|u\|_{H^1} + \|v\|_{H^3} + \eta^{-1}). \tag{3.19}$$

Hence we can find a constant $\chi > 0$ such that (3.16) is valid and

$$\|u(t)\|^2_{H^2} \le \mathrm{e}^{-\chi t}\|u_0\|^2_{H^2} + p(f + \|u_0\|_{H^1} + \|v_0\|_{H^3}). \tag{3.20}$$

To prove the existence of an exponential attractor, we will use the following result.

Proposition 3.2 *Let $u_0 \in L^2(\Omega)$, $v_0 \in H^{1+\varepsilon_0}(\Omega)$ be nonnegative functions. Then there exists a continuous increasing function $p(\cdot)$ independent of u_0 and v_0, such that*

$$\|u(t)\|^2_{H^2} + \|v(t)\|^2_{H^3} \le p(f + N^1_{\log}(u_0) + \|v_0\|_{H^1(\Omega)} + t^{-1}).$$

Proof Since the proof follows exactly the same ideas and technical difficulties as in the proof of [13, Theorem 4.6], we skip it here.

We can now prove the existence of an exponential attractor: Let $H = L^2(\Omega) \times H^1(\Omega)$ and consider the initial value problem

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t} + AU = G(U), \\ &U(0) = U_0 \end{aligned} \tag{E}$$

in H, with A as in Subsection 3.1 and $D(A) = H^2_n(\Omega) \times H^3_n(\Omega)$ and

$$G(U) := \begin{pmatrix} \beta\nabla(u\nabla v) - q|u|u + k_2 v \\ k_1 u + f \end{pmatrix}.$$

Let $K = \{(u, v) \in L^2_+(\Omega) \times H^{1+\varepsilon_0}_+(\Omega)\}$ be the space of initial values and $U_0 \in K$.

We have proved already the existence of a unique global solution $U = (u, v)$ continuous with respect to the initial condition U_0. We define then a continuous semigroup $\{S(t)_{t\ge 0}\}$ on K by $S(t)U_0 = U(t)$. For a fixed $t > 0$, $S(t)$ maps K into $K \cap D(A)$.

Denote $B_r := \{(u, v) \in K; \|u_0\|_{L^2} + \|v_0\|_{H^{1+\varepsilon_0}} \le r\}$ a bounded ball of K with radius $r > 0$.

Proposition 3.3 *There exists a universal constant $C > 0$ such that the following statement holds: For each $r > 0$ there exists a time $t_r > 0$ such that*

$$\sup_{t \geq t_r} \sup_{U_0 \in B_r} \|S(t)U_0\|_{H^2(\Omega)\times H^3(\Omega)} \leq C.$$

Proof Fix $0 < r < \infty$. By t_r and C_r we denote some time and positive constant which depend on r but are uniform in $U_0 \in B_r$, respectively. By Proposition 3.2, there exist a time t_r and a constant C_r such that for $t \geq t_r$,

$$\|u(t)\|^2_{H^2} + \|v(t)\|^2_{H^3} \leq C_r. \tag{3.21}$$

The desired estimate will be established step by step.

Let us add the first equation of (P_h) and the second one multiplied by 2 and let us integrate in space the result. If $\Phi(t) := \|u(t)\|_{L^1} + 2\|v(t)\|_{L^1}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + \frac{k_2}{2}\Phi(t) = \int_\Omega \Big(- qu^2 + k_1 u + \frac{k_2}{2}u \Big)\,\mathrm{d}x + f|\Omega| \leq \Big\{ \frac{1}{4q}\Big(k_1 + \frac{k_2}{2}\Big)^2 + f \Big\}|\Omega|.$$

Thus

$$\Phi(t) \leq \Big\{ \Phi(0) - \frac{2}{k_2}\Big(\frac{1}{4q}\Big(k_1 + \frac{k_2}{2}\Big)^2 + f\Big)|\Omega| \Big\} \mathrm{e}^{-\frac{k_2}{2}t} + \frac{2}{k_2}\Big(\frac{1}{4q}\Big(k_1 + \frac{k_2}{2}\Big)^2 + f\Big)|\Omega|$$

and we deduce

$$\|u(t)\|_{L^1} + 2\|v(t)\|_{L^1} \leq C(C_r \mathrm{e}^{-ct} + 1)$$

with $C,\ c > 0$ universal constants and $C_r > 0$ a constant depending on r. This shows that there exists a time denoted by t_r such that for all $t \geq t_r$,

$$\|u(t)\|_{L^1} + \|v(t)\|_{L^1} \leq C \tag{3.22}$$

with $C > 0$ a universal constant.

From (3.6) and (3.22) it follows that

$$g(t) \leq (g(0) - C)\mathrm{e}^{-\sigma(t-t_r)} + C \quad \text{for } t \geq t_r.$$

Then there exists another time t_r and another universal constant $C > 0$ such that

$$\|v(t)\|_{H^1} \leq C, \quad N^1_{\log}(u(t)) \leq C \quad \text{for } t \geq t_r.$$

From (3.11) and (3.21) we deduce that

$$\|v(t)\|_{H^2(\Omega)} + \|u(t)\|_{L^2(\Omega)} \leq C_r \mathrm{e}^{-\sigma(t-t_r)} + C \quad \text{for } t \geq t_r,$$

and that there exist another time t_r and another constant $C > 0$, such that

$$\|v(t)\|_{H^2(\Omega)} + \|u(t)\|_{L^2(\Omega)} \leq C \quad \text{for } t \geq t_r.$$

Finally using (3.16), (3.20) and repeat the argument we finish the proof.

Let $\mathscr{B} = \{(u,v) \in H^2_N(\Omega) \times H^3_N(\Omega) \;/\; \|u\|_{H^2(\Omega)} + \|v\|_{H^3(\Omega)} \le C\} \cap K$ with C the constant appearing in Proposition 3.3. We proved that $\mathscr{B}$ is a compact absorbing set for $(\{S(t)\}_{t\ge 0}, K)$. Hence by Temam [19], there exists a global attractor $\mathscr{A} \subset K$, where $\mathscr{A}$ is a compact and connected subset of K.

Let $\mathscr{H} = \overline{\underset{t\ge t_{\mathscr{B}}}{\cup} S(t)\mathscr{B}}$ where $t_{\mathscr{B}}$ is such that $S(t)\mathscr{B} \subset \mathscr{B}$. Then $\mathscr{H}$ is a compact set of K with $\mathscr{A} \subset \mathscr{H} \subset K$. Since $\mathscr{H}$ is absorbing and positively invariant for $\{S(t)_{t\ge 0}\}$, we apply to the dynamical system $(\{S(t)\}_{t\ge 0}, \mathscr{H})$ as follows.

Theorem 3.5 (see [20, Theorem 3.1]) *Let $\Gamma(t, U_0) = S(t)U_0$ be a mapping from $[0,T] \times \mathscr{H}$ into $\mathscr{H}$. If G satisfies*

$$\|G(U) - G(V)\| \le \|A^{\frac{1}{2}}(U - V)\|, \quad U, V \in \mathscr{H} \tag{C$_1$}$$

and Γ is such that

$$\|\Gamma(t, U_0) - \Gamma(s, V_0)\| \le C_T(|t - s| + \|U_0 - V_0\|_H, \quad t, s \in [0,T],\ U_0, V_0 \in \mathscr{H} \tag{C$_2$}$$

for each $T > 0$, then there is an exponential attractor $\mathscr{M}$ for $(\{S(t)\}, \mathscr{H})$.

Thus we obtain the following result.

Theorem 3.6 *There exists an exponential attractor $\mathscr{M}$ of the dynamical system $(\{S(t)\}_{t\ge 0}, \mathscr{H})$ in H*

Proof Since the forcing term f is constant and the reaction coupling of the first equation of (E) is linear in U: $k_2 v$, the proof is the same as provided in [13, Theorem 5.1].

4 Second Case: $h(u) = u(M - u)$

Let M be a positive constant and consider a continuous function $\widetilde{h}$ of h such as

$$\begin{cases} \widetilde{h}(u) = u(M - u) & \text{if } 0 \le u \le M, \\ \widetilde{h}(u) = 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Then we have the following result.

Proposition 4.1 *Let $\varepsilon_0 > 0$ and f be a nonnegative function in $L^2(0,T; H^1(\Omega)) \cap L^\infty(\Omega_T)$. For each nonnegative initial condition (u_0, v_0) in $L^2(\Omega) \times H^{1+\varepsilon_0}(\Omega)$ there exists a constant T_0 such that $0 < T_0 \le T$ and a unique nonnegative solution (u, v) of $(P_{\widetilde{h}})$ such that*

$$u \in H^1(0, T_0; (H^1(\Omega))') \cap C([0, T_0]; L^2(\Omega)) \cap L^2(0, T_0; H^1(\Omega)),$$
$$v \in H^1(0, T_0; H^{\varepsilon_0}(\Omega)) \cap C([0, T_0]; H^{1+\varepsilon_0}(\Omega)) \cap L^2(0, T_0; H^{2+\varepsilon_0}_\nu(\Omega)).$$

Proof The proof is essentially the same as in Subsection 3.2.

Moreover we can prove the following result.

Lemma 4.1 *Suppose that $M \ge \big(\frac{\|f\|_{L^\infty(\Omega_T)}}{q}\big)^{\frac{1}{2}}$ and $M' = \frac{qM^2 + k_1 M}{k_2} > 0$. If the initial condition (u_0, v_0) satisfies almost everywhere in Ω the following inequalities:*

$$0 \le u_0(x) \le M, \quad 0 \le v_0(x) \le M',$$

then the solution $(u,\ v)$ *of* $(P_{\widetilde{h}})$ *satisfies*

$$0 \le u(t,x) \le M, \quad 0 \le v(t,x) \le M'$$

almost everywhere in Ω_T*.*

Proof Define $\widetilde{u} = M - u$ and $\widetilde{v} = M' - v$. Thus we get

$$\begin{aligned}
\widetilde{u}_t &= a\Delta\widetilde{u} - \beta \mathrm{div}(\widetilde{h}(\widetilde{u})\nabla\widetilde{v}) - (2qM + k_1)\widetilde{u} + q\widetilde{u}^2 + k_2\widetilde{v} + qM^2 + k_1M - k_2M', \\
\widetilde{v}_t &= d\Delta\widetilde{v} - k_2\widetilde{v} + k_1\widetilde{u} + k_2M' - k_1M - f.
\end{aligned}$$

As $M \ge \big(\frac{\|f\|_{L^\infty(\Omega_T)}}{q}\big)^{\frac{1}{2}}$ and $M' = \frac{qM^2+k_1M}{k_2}$, we obtain

$$qM^2 + k_1M - k_2M' = 0$$

and

$$k_2M' - k_1M - f \ge 0.$$

We multiply the first equation by $-\widetilde{u}^-$ and the second by $-\widetilde{v}^-$ and we integrate in space. Thank to the identity: $\int_\Omega \widetilde{h}(\widetilde{u})\nabla\widetilde{v}\nabla\widetilde{u}^-\mathrm{d}x = 0$ and since $q\widetilde{u}^2\widetilde{u}^-$, $k_2\widetilde{u}^-\widetilde{v}^+$, $k_1\widetilde{v}^-\widetilde{u}^+ \ge 0$ almost everywhere in Ω, we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{u}^-\|^2_{L^2(\Omega)} \le (2qM + k_1)\|\widetilde{u}^-\|^2_{L^2(\Omega)} + k_2\int_\Omega \widetilde{v}^-\widetilde{u}^-\,\mathrm{d}x$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{v}^-\|^2_{L^2(\Omega)} \le k_2\|\widetilde{v}^-\|^2_{L^2(\Omega)} + k_1\int_\Omega \widetilde{v}^-\widetilde{u}^-\,\mathrm{d}x.$$

Taking the sum of the two previous inequalities and using Young inequality, it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{u}^-\|^2_{L^2(\Omega)} + \|\widetilde{v}^-\|^2_{L^2(\Omega)}) \le C(\|\widetilde{u}^-\|^2_{L^2(\Omega)} + \|\widetilde{v}^-\|^2_{L^2(\Omega)})$$

for some constant $C > 0$. By Gronwall lemma we get

$$\|\widetilde{u}^-\|^2_{L^2(\Omega)} + \|\widetilde{v}^-\|^2_{L^2(\Omega)} = 0,$$

which completes the proof.

Remark 4.1 (i) If the hypothesis of Lemma 4.1 are fulfilled, thanks to this lemma, the solution obtained in Proposition 4.1 is global in Ω_T,

(ii) By Proposition 4.1 and Lemma 4.1 it follows that (u, v) is also a solution of (P_h) with $h(u) = u(M - u)$.

The uniqueness of the solution is obtained in the following.

Theorem 4.1 *Let* $f \in L^\infty(\Omega_T) \cap L^2(0, T; H^1(\Omega))$ *be a nonnegative function. Let* $h(u) = u(M - u)$ *and suppose that* $M \ge \big(\frac{\|f\|_{L^\infty(\Omega_T)}}{q}\big)^{\frac{1}{2}}$*. Let* $(u_0, v_0) \in L^2(\Omega) \times H^{1+\varepsilon_0}(\Omega)$ *such that*

$0 \le u_0 \le M$ and $0 \le v_0 \le M'$ with $M' = \frac{qM^2 + k_1 M}{k_2}$. Then there exists a unique global solution for (P_h) which is nonnegative and such that

$$u \in L^\infty(\Omega_T) \cap H^1(0,T;(H^1(\Omega))') \cap C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$
$$v \in H^1(0,T;H^{\varepsilon_0}(\Omega)) \cap C([0,T];H^{1+\varepsilon_0}(\Omega)) \cap L^2(0,T;H^{2+\varepsilon_0}(\Omega))$$

and

$$0 \le u \le M, \quad 0 \le v \le M'.$$

Proof We skip here the proof of uniqueness since there is rigorously the same as in Theorem 5.1.

5 A Three Dimensional Domain

In order to prove the global existence of a solution of system (P_h), we supposed in the previous sections that Ω was a two dimensional domain and the initial conditions $(u_0, v_0) \in L^\infty(\Omega) \times H^{1+\varepsilon_0}(\Omega)$ were nonnegative and verifying some regularity conditions. These conditions are quite restrictive for a model of soil organic carbon and three dimensional domains are obviously more relevant in applications than bidimensional ones.

In this section we prove that if Ω is of dimension less than or equal to 3, if $h = \widetilde{h}$ (4.1) and if both initial conditions and forcing term are nonnegative and less regular that in the previous section: $(u_0, v_0) \in (L^2(\Omega))^2$ and $f \in L^2(0,T;L^2(\Omega))$, then the system (P_h) has a global nonnegative solution. Furthermore, if $(u_0, v_0) \in (L^\infty(\Omega))^2$ and $f \in L^\infty(\Omega_T)$, then the solution is unique.

Here we use the following setting:

$$V = H^1(\Omega) \times H^1(\Omega),$$
$$H = L^2(\Omega) \times L^2(\Omega),$$
$$V' = (H^1(\Omega))' \times (H^1(\Omega))'.$$

We let $\widetilde{h}$ be the continuous function defined by (4.1). Let us consider the following system:

$$\begin{cases} \partial_t u - a\Delta u = -\beta \operatorname{div}(\widetilde{h}(\overline{u})\nabla v) - k_1 u - q|u|u + k_2 v & \text{in } \Omega_T, \\ \partial_t v - d\Delta v = -k_2 v + k_1 u + f & \text{in } \Omega_T, \\ \nabla u \cdot \eta(x) = \nabla v \cdot \eta(x) = 0 & \text{on } \Sigma_T, \\ u(0,\cdot) = u_0, v(0,\cdot) = v_0 & \text{in } \Omega, \end{cases} \tag{P-S}$$

where $(u_0, v_0) \in (L^2(\Omega))^2$, $f \in L^2(0,T;L^2(\Omega))$ and $\overline{u}$ is a function in $X = L^2(\Omega_T)$.

For the sake of simplicity we take $\dim(\Omega) = 3$, since all results remain the same if $\dim(\Omega) < 3$.

We will apply the Schauder fixed point theorem but let us first gather some more information.

First Step Invariant ball.

For any function $\overline{u} \in X$ the existence of a unique local solution of (P-S) $(u_{\overline{u}}, v_{\overline{u}})$ follows by direct application of Theorem 2.1. Additionally we have the following result.

Proposition 5.1 *Let* $(u_0, v_0) \in (L^2(\Omega))^2$ *and* $f \in L^2(0,T;L^2(\Omega))$.

(1) *For any* $\overline{u} \in X$ *the unique local solution* $(u_{\overline{u}}, v_{\overline{u}})$ *of* (P-S) *is global and satisfies:*

$$u_{\overline{u}} \in H^1(0,T;(H^1(\Omega))') \cap C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$
$$v_{\overline{u}} \in H^1(0,T;(H^1(\Omega))') \cap C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

(2) *Furthermore, for all* $\overline{u} \in X$, *there exist two constant* $R > 0$ *and* $C > 0$ *such that*

$$\|u_{\overline{u}}\|_{L^2(\Omega_T)} \le R, \quad \|u\|_W \le C, \tag{5.1}$$

where

$$W = \{u \in L^2(0,T;H^1(\Omega)),\ u_t \in L^2(0,T;(H^1(\Omega))')\}.$$

Proof To prove that$(u_{\overline{u}}, v_{\overline{u}})$ is global in time, we multiply the first equation by $u_{\overline{u}}$ and the second by $v_{\overline{u}}$ and use Young inequality to get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{\overline{u}}\|^2_{L^2(\Omega)} + a\|\nabla u_{\overline{u}}\|^2_{L^2(\Omega)} + \int_\Omega \{k_1|u_{\overline{u}}|^2 + q|u_{\overline{u}}|^3\}\mathrm{d}x \\ &\le k_2\int_\Omega \Big\{u_{\overline{u}}^2 + \frac{1}{4}v_{\overline{u}}^2\Big\}\mathrm{d}x + \frac{M_{\overline{u}}^2}{4}\Big(\frac{M^2}{8a}\|\nabla v_{\overline{u}}\|^2_{L^2(\Omega)} + \frac{a}{2}\frac{4}{M^2}\|\nabla u_{\overline{u}}\|^2_{L^2(\Omega)}\Big)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{\overline{u}}\|^2_{L^2(\Omega)} + d\|\nabla v_{\overline{u}}\|^2_{L^2(\Omega)} + k_2\|v_{\overline{u}}\|^2_{L^2(\Omega)} \\ &\le \frac{k_2}{2}\|v_{\overline{u}}\|^2_{L^2(\Omega)} + \int_\Omega \Big\{\frac{k_1^2}{2k_2}u_{\overline{u}}^2\Big\}\mathrm{d}x + \frac{k_2}{4}\|v_{\overline{u}}\|^2_{L^2(\Omega)} + \frac{1}{k_2}\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

Multiplying by $\rho > 0$ the first inequality and adding to the second one gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\rho\|u_{\overline{u}}\|^2_{L^2(\Omega)} + \|v_{\overline{u}}\|^2_{L^2(\Omega)}) + \frac{\rho a}{2}\|\nabla u_{\overline{u}}\|^2_{L^2(\Omega)} + d\|\nabla v_{\overline{u}}\|^2_{L^2(\Omega)} + \int_\Omega \{\rho k_1|u_{\overline{u}}|^2 + \rho q|u_{\overline{u}}|^3\}\mathrm{d}x \\ &\le \rho\frac{M^4}{32a}\|\nabla v_{\overline{u}}\|^2_{L^2(\Omega)} + C\int_\Omega |u_{\overline{u}}|^2\mathrm{d}x + \frac{1}{k_2}\|f\|^2_{L^2(\Omega)},\end{aligned}$$

where $C = \frac{k_1^2}{2k_2} + k_2$. For $\rho = \frac{16ad}{M^4}$ we obtain the following inequality:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\rho\|u_{\overline{u}}\|^2_{L^2} + \|v_{\overline{u}}\|^2_{L^2(\Omega)}) + \frac{\rho a}{2}\|\nabla u_{\overline{u}}\|^2_{L^2(\Omega)} + \frac{d}{2}\|\nabla v_{\overline{u}}\|^2_{L^2(\Omega)} \\ &+ \int_\Omega \{(\rho k_1 - C)|u_{\overline{u}}|^2 + \rho q|u_{\overline{u}}|^3\}\mathrm{d}x \le \frac{1}{k_2}\|f\|^2_{L^2(\Omega)}.\end{aligned} \tag{5.2}$$

If $(\rho k_1 - C) \ge 0$ we finish the proof of part 1. If $(\rho k_1 - C) < 0$ then, for any $0 < \lambda < \rho q$, note $K_\lambda = \frac{4}{27}\frac{(\rho k_1 - C)^3}{(\rho q - \lambda)^2} < 0$. By a simple real analysis argument, we deduce

$$(\rho k_1 - C)|u|^2 + \rho q|u|^3 \ge \lambda|u|^3 + K_\lambda$$

for any $u \in X$. Hence the inequality (5.2) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\rho\|u_{\overline{u}}\|^2_{L^2(\Omega)} + \|v_{\overline{u}}\|^2_{L^2(\Omega)}) + \frac{\rho a}{2}\|\nabla u_{\overline{u}}\|^2_{L^2(\Omega)} + \frac{d}{2}\|\nabla v_{\overline{u}}\|^2_{L^2(\Omega)} + \int_\Omega \{\lambda|u_{\overline{u}}|^3 + K_\lambda\}$$

$$\le \frac{1}{k_2}\|f\|^2_{L^2(\Omega)}.$$

Since $u_0,\ v_0 \in L^2(\Omega)$ we deduce that $u_{\overline{u}},\ v_{\overline{u}}$ are bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T,H^1(\Omega))$, and this bound does not depend on $\overline{u}$. Using interpolation technique we obtain that $u_{\overline{u}}$ is bounded in $L^4(0,T;(L^3(\Omega))$ and consequently $|u_{\overline{u}}|u_{\overline{u}}$ is bounded in $L^2(0,T;(L^{\frac{3}{2}}(\Omega))$, independent of $\overline{u}$.

Combining Hölder inequality, the boundedness of $u_{\overline{u}},\ v_{\overline{u}}$ in $L^2(0,T,H^1(\Omega))$ and $L^4(0,T;(L^3(\Omega))$ and the continous injection of $L^2(0,T;H^1(\Omega))$ into $L^2(0,T;(L^3(\Omega))$, we obtain that $\partial_t u_{\overline{u}},\ \partial_t v_{\overline{u}}$ are bounded in $L^2(0,T;(H^1(\Omega))')$, independent of $\overline{u}$. So we finish the proof.

We can then define the mapping $\Pi : X \to X$ such that $u_{\overline{u}} = \Pi(\overline{u})$ is the unique solution of (P-S). From (5.1) the ball $B_R \subset X$ is invariant by Π.

Second Step Compactness of $\Pi(B_R)$. The second statement of the previous proposition implies that $\Pi(B_R) \subset \{u \in W,\ \|u\|_W \le C\}$. But the embedding of W into $L^2(0,T,L^2(\Omega))$ is compact thanks to the Aubin-Lions lemma.

Third Step Π is a continuous mapping. Let $z_n \in B_R$ such that $z_n \to z$ in $L^2(\Omega_T)$ strong and let $u_n = \Pi(z_n)$. Then $U_n = (u_n, v_n)$ satisfies the system (P-S)$_n$:

$$\begin{cases} \partial_t u_n - a\Delta u_n = -\beta \mathrm{div}(\widetilde{h}(z_n)\nabla v_n) - k_1 u_n - q|u_n|u_n + k_2 v_n & \text{in } \Omega_T, \\ \partial_t v_n - d\Delta v_n = -k_2 v_n + k_1 u_n + f & \text{in } \Omega_T, \\ \nabla u_n \cdot \nu = \nabla v_n \cdot \nu = 0 & \text{on } \Sigma_T, \\ u_n(0,\cdot) = u_0, v_n(0,\cdot) = v_0 & \text{in } \Omega. \end{cases}$$

Since the sequence $(u_n, v_n)_{n\ge 1}$ is bounded in W^2 and $(L^\infty([0,T];L^2(\Omega)))^2$, there exists by the Aubin-Lions lemma a subsequence (not relabeled) such that

$$\begin{cases} u_n \to u \text{ in } L^2(\Omega_T), \quad u_n \to u \quad \text{a.e.} & \text{in } (\Omega_T), \\ \nabla u_n \rightharpoonup \boldsymbol{\xi} & \text{in } (L^2(\Omega_T))^3, \\ \partial_t u_n \rightharpoonup \psi & \text{in } L^2(0,T,(H^1(\Omega))'). \end{cases}$$

To prove that $\nabla u = \boldsymbol{\xi}$, we take a test function $\varphi \in (\mathcal{D}(\Omega_T))^3$, so that

$$\int_0^T \int_\Omega \nabla u_i^n \varphi \ \mathrm{d}x\mathrm{d}t = -\int_0^T \int_\Omega u_i^n \nabla\varphi \ \mathrm{d}x\mathrm{d}t.$$

Taking the limit when $n \to \infty$ of both sides of this equation, we obtain

$$\int_0^T \int_\Omega \boldsymbol{\xi}_i \varphi \ \mathrm{d}x\mathrm{d}t = -\int_0^T \int_\Omega u_i \nabla\varphi \ \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \nabla u_i \varphi \ \mathrm{d}x\mathrm{d}t$$

and we conclude by a density argument. To prove that $\partial_t u = \psi$, we use a similar computation for the derivative with respect to time, with test function $\varphi \in C_c^1(0,T,H^1(\Omega))$. Thus we have

$$\begin{aligned} &u_n \rightharpoonup u && \text{in } L^2(0,T,H^1(\Omega)), \\ &\partial u_n \rightharpoonup \partial u && \text{in } L^2(0,T,(H^1(\Omega))'), \\ &|u_n|u_n \rightharpoonup |u|u && \text{in } L^2(0,T,L^{\frac{3}{2}}(\Omega)), \end{aligned} \tag{5.3}$$

where the last assertion is a straightforward consequence of the upper bound of sequence $|u_n|u_n$ in $L^2(0,T,L^{\frac{3}{2}}(\Omega))$ and the a.e. convergence of the sequence $(u_n)_{n\geq 1}$ in Ω_T. We obtain also a similar convergence for v_n towards v as in (5.3).

Finally, thanks to suitable choices of test functions it follows that the limit function v is a solution of the following problem:

$$\begin{cases} \partial_t v - d\Delta v = -k_2 v + k_1 u + f & \text{in } \Omega_T, \\ \nabla v \cdot \nu = 0 & \text{on } \Sigma_T, \\ v(0,\cdot) = v_0 & \text{in } \Omega. \end{cases} \tag{5.4}$$

Obviously we prove that $\nabla v_n \to \nabla v$ in $(L^2(\Omega_T)^3$, when $n \to \infty$ and thereby v_n strongly converges to v in $L^2(0,T,H^1(\Omega))$. Since h is continous, $z_n \to z$ in $L^2(\Omega_T)$ and $\nabla v_n \to \nabla v$ in $(L^2(\Omega_T)^3$, there exists a subsequnce (not relabeled) such that $\widetilde{h}(z_{n_k})\nabla v_{n_k} \to \widetilde{h}(z)\nabla v$ a.e. in Ω_T. As $\widetilde{h}(z_{n_k})\nabla v_{n_k}$ is bounded in $L^2(\Omega_T)$, we obtain by the dominated convergence theorem:

$$\widetilde{h}(z_{n_k})\nabla v_{n_k} \to \widetilde{h}(z)\nabla v \quad \text{in } L^2(\Omega_T)$$

and we can pass to the limit in the (P-S)$_n$ system. Thus

$$\begin{cases} \partial_t u - a\Delta u = -\beta \operatorname{div}(\widetilde{h}(z)\nabla v) - k_1 u - q|u|u + k_2 v & \text{in } \Omega_T, \\ \partial_t v - d\Delta v = -k_2 v + k_1 u + f & \text{in } \Omega_T, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0 & \text{on } \Sigma_T, \\ u(0,\cdot) = u_0, v(0,\cdot) = v_0 & \text{in } \Omega \end{cases} \tag{5.5}$$

and we get $u = \Pi(z)$.

By the uniqueness of the solution (u,v) of (5.5), we deduce that all the sequence converges. We conclude that Π is a continuous mapping.

We can now apply the Schauder fixed point theorem to prove the existence statement of the following result.

Proposition 5.2 *Let f be a nonnegative function in $L^2(0,T;L^2(\Omega))$. For each couple of nonnegative functions $(u_0,v_0) \in (L^2(\Omega))^2$, there exists a nonnegative solution for the problem (P_h) with $h = \widetilde{h}$.*

To prove the positivity of the solution, we proceed as in Section 4: We multiply the first equation by $-u^-$ and the second by $-v^-$, we integrate in space and we add the two equations. Thanks to the identity $\int_\Omega h(u)\nabla v \nabla u^- = 0$, a straightforward calculation gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u^-\|^2_{L^2(\Omega)} + \|v^-\|^2_{L^2(\Omega)}) \leq C(\|u^-\|^2_{L^2(\Omega)} + \|v^-\|_{L^2(\Omega)})$$

with $C > 0$. We finish the proof by applying the Gronwall lemma.

For the uniqueness of solution of problem (P_h) we have the following result.

Theorem 5.1 *Let $f \in L^\infty(\Omega_T)$ be a nonnegative function. Consider $(u_0,v_0) \in (L^\infty(\Omega))^2$ such that $0 \leq u_0 \leq M$ and $0 \leq v_0(x) \leq v_M$ almost everywhere in Ω, where v_M is a positive constant. Then there exists a constant $\alpha \geq 0$ such that*

$$0 \leq u(t,x) \leq M\mathrm{e}^{\alpha t}, \quad 0 \leq v(t,x) \leq v_M \mathrm{e}^{\alpha t} \tag{5.6}$$

and the solution of problem (P_h) *is unique, when* $h = \widetilde{h}$.

Proof Let $\widetilde{u} = u - M\mathrm{e}^{\alpha t}$ and $\widetilde{v} = v - v_M\mathrm{e}^{\alpha t}$. Then we have

$$\begin{aligned}\widetilde{u}_t &= a\Delta\widetilde{u} - \beta\nabla(h(u)\nabla\widetilde{v}) - k_1\widetilde{u} - q\widetilde{u}^2 + k_2\widetilde{v} \\ &\quad - (\alpha M + k_1 M + 2quM - k_2 v_M)\mathrm{e}^{\alpha t} - qM^2\mathrm{e}^{2\alpha t}\end{aligned}$$

and

$$\widetilde{v}_t = a\Delta\widetilde{v} + k_1\widetilde{u} + \{f + \mathrm{e}^{\alpha t}((-k_2 - \alpha)v_M + k_1 M)\}.$$

We take α large enough such that

$$f + \mathrm{e}^{\alpha t}((-k_2 - \alpha)v_M + k_1 M) \leq 0$$

and

$$\alpha M + k_1 M - k_2 v_M \geq 0.$$

Multiplying the first equation by $\widetilde{u}^+$ and the second by $\widetilde{v}^+$ and then adding the two equations gives

$$\begin{aligned}&\frac{1}{2}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{u}^+\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{v}^+\|^2_{L^2(\Omega)}\Big) \\ &\leq \beta\int_\Omega h(u)\nabla v\nabla\widetilde{u}^+ + C(\|\widetilde{u}^+\|^2_{L^2(\Omega)} + \|\widetilde{v}^+\|^2_{L^2(\Omega)}).\end{aligned}$$

Thanks to (4.1) $\beta\int_\Omega h(u)\nabla v\nabla\widetilde{u}^+ = 0$ and we obtain (5.6) by using Gronwall lemma.

To prove uniqueness, suppose that there exist two solutions (u_1, v_1) and (u_2, v_2). Then $\overline{u} = u_1 - u_2$ and $\overline{v} = v_1 - v_2$ verify

$$\begin{aligned}&\overline{u}_t = a\Delta\overline{u} - \beta\nabla(h(u_1)\nabla v_1 - (h(u_2)\nabla v_2) - k_1\overline{u} - q{u_1}^2 + qu_2^2 + k_2\overline{v}, \\ &\overline{v}_t = d\Delta\overline{v} - k_2\overline{v} + k_1\overline{u}, \\ &\overline{u}_0 = \overline{v}_0 = 0 \quad \text{a.e in } \Omega.\end{aligned} \tag{5.7}$$

Multiplying the first equation by $\overline{u}$, the second by $\overline{v}$ and integrating over Ω leads to

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{u}\|^2_{L^2(\Omega)} + a\|\nabla\overline{u}\|^2_{L^2(\Omega)} \\ &\leq \beta\int_\Omega |(h(u_1)\nabla v_1 - h(u_2)\nabla v_2)\nabla\overline{u}|\,\mathrm{d}x + C(\|\overline{u}\|^2_{L^2(\Omega)} + \|\overline{v}\|^2_{L^2(\Omega)})\end{aligned}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{v}\|^2_{L^2(\Omega)} + d\|\nabla\overline{v}\|^2_{L^2(\Omega)} + k_2\|\overline{v}\|^2_{L^2(\Omega)} = k_1\int_\Omega \overline{uv}\,\mathrm{d}x.$$

It follows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{u}\|^2_{L^2(\Omega)} + a\|\nabla\overline{u}\|^2_{L^2(\Omega)} \\ &\leq \beta\int_\Omega (|h(u_1) - h(u_2)|\,|\nabla v_1| + h(u_2)\,|\nabla\overline{v}|)|\nabla\overline{u}|\,\mathrm{d}x + C(\|\overline{u}\|^2_{L^2(\Omega)} + \|\overline{v}\|^2_{L^2(\Omega)})\end{aligned} \tag{5.8}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{v}\|^2_{L^2(\Omega)} + d\|\nabla\overline{v}\|^2_{L^2(\Omega)} + k_2\|\overline{v}\|^2_{L^2(\Omega)} \leq C(\|\overline{u}\|^2_{L^2} + \|\overline{v}\|^2_{L^2(\Omega)}). \tag{5.9}$$

Recalling (5.6) $\overline{u}$ and $\overline{v}$ are bounded in Ω_T. Classical parabolic regularity results and (5.4) imply that $v \in L^p(0,T,W^{2,p}(\Omega))$ for each $p \in (1,\infty)$. By Sobolev embedding, there is $p > 3$ such that $\nabla v_1 \in L^2(0,T;L^\infty(\Omega))$. Hence

$$\begin{aligned}
&\int_\Omega |h(u_1) - h(u_2)|\,|\nabla v_1|\,|\nabla\overline{u}|\,\mathrm{d}x \\
&\leq M\|\nabla v_1\|_{L^\infty(\Omega)}\|\overline{u}\|_{L^2(\Omega)}\|\nabla\overline{u}\|_{L^2(\Omega)} \\
&\leq \varepsilon\|\nabla\overline{u}\|^2_{L^2(\Omega)} + C_\varepsilon\|\nabla v_1\|^2_{L^\infty(\Omega)}\|\overline{u}\|^2_{L^2(\Omega)}
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
\int_\Omega h(u_2)\,|(\nabla\overline{v})|\,|\nabla\overline{u}| &\leq \frac{M}{2}\|\nabla\overline{v}\|_{L^2(\Omega)}\|\nabla\overline{u}\|_{L^2(\Omega)} \\
&\leq \varepsilon'\|\nabla\overline{u}\|^2_{L^2(\Omega)} + C'_\varepsilon\|\nabla\overline{v}\|^2_{L^2(\Omega)}.
\end{aligned} \tag{5.11}$$

We sum up (5.8)–(5.9) multiplied by a constant $\sigma > 0$ small enough, and we use (5.10) and (5.11) with a wise choise of $\varepsilon, \varepsilon'$ and σ such that $\varepsilon + \varepsilon' \leq a$ and $\sigma C'_\varepsilon \leq d$. Thereby we prove the existence of a constant $C > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sigma\|\overline{u}\|^2_{L^2(\Omega)} + \|\overline{v}\|^2_{L^2(\Omega)}) \leq C(\|\nabla v_1\|_{L^\infty(\Omega)} + 1)(\sigma\|\overline{u}\|^2_{L^2(\Omega)} + \|\overline{v}\|^2_{L^2(\Omega)}).$$

The Gronwall lemma entail that $\|\overline{u}(t)\|_{L^2(\Omega)} = \|\overline{v}(t)\|_{L^2(\Omega)} = 0$ for every $t \in [0,T]$, which completes the proof.

6 Appendices

6.1 Appendix 1

Non-emergence of spatial patterns in (P_h) model without chemotaxis term ($\beta = 0$)

Firstly we considere the PDEs system (P_h) without chemotaxis term ($\beta = 0$). As in Lotka-Volterra systems (see [20]), also known as the predator-prey equations, diffusion alone cannot disturb a constant equilibrium, and so spatial heterogeneity cannot emerge. Using the following notation:

$$\widetilde{x} = \sqrt{\frac{k_1}{a}}x, \quad \widetilde{t} = k_1 t, \quad \alpha = \frac{q}{k_1}, \quad \zeta = \frac{k_2}{k_1}, \quad c = \frac{f}{k_1}, \quad D = \frac{d}{a},$$

we obtain the following non-dimensional equations (we revoke the notation):

$$\begin{cases} \partial_t u = \Delta u - u - \alpha u^2 + \zeta v, \\ \partial_t v = D\Delta v + u - \zeta v + c, \end{cases} \qquad (x,t) \in \Omega \times (0;T) \tag{6.1}$$

with the same initial conditions and boundary conditions as (P_h) system . Without diffusion, the system (6.1) has a unique positive steady state:

$$u^* = \sqrt{\frac{c}{\alpha}}, \quad v^* = \frac{u^* + c}{\zeta}. \tag{6.2}$$

To assess the steady state stability, the system is linearised around (u^*, v^*). Setting

$$\varepsilon w_1 = u - u^*, \quad \varepsilon w_2 = v - v^*,$$

where $0 < \varepsilon \ll 1$, gives the following linear system:

$$\begin{cases} \partial_t w_1 = \Delta w_1 - w_1 - 2\alpha u^* w_1 + \zeta w_2, \\ \partial_t w_2 = D\Delta w_2 + w_1 - \zeta w_2, \end{cases} \qquad (x,t) \in \Omega \times (0,T) \tag{6.3}$$

with no-flux boundary conditions.

As in [20–21], we look for a solution of the form:

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \propto \mathrm{e}^{(\mathbf{i}\mathbf{k}\cdot\mathbf{x} + \rho t)}. \tag{6.4}$$

Let $k = |\mathbf{k}|$ be the Euclidean norm of the wave vector. We obtain the following eigenvalue problem:

$$\mathbf{A}\mathbf{w} = \rho\mathbf{w},$$

where $\mathbf{A}$ is the two by two matrix

$$\mathbf{A} = \begin{pmatrix} -1 - 2\alpha u^* - k^2 & \zeta \\ 1 & -\zeta - k^2 D \end{pmatrix}.$$

The eigenvalue ρ depends on k.

Turing instability occurs (which means that spatial patterns appear) when $\rho(k^2) > 0$, for a given value of k. But the matrix $\mathbf{A}$ has a strictly negative trace and a positive determinant, and so $\rho(k^2) < 0$ for all values of k . Hence no patterns will emerge in this case.

Emergence of spatial patterns in (P_h) model with $\beta > 0$

Finally, for the model with both diffusion and chemotaxis, it can be proven that the equilibrium solutions of the equation system (P_h) can be rendered non-stable under certain conditions, and thus produce patterns and spatial heterogeneity. As in the previous section, the system (P_h) was linearised around the steady state (u^*, v^*). We obtain the following system:

$$\begin{cases} \partial_t w_1 = \Delta w_1 - e\Delta w_2 - w_1 - 2\alpha u^* w_1 + \zeta w_2, \\ \partial_t w_2 = D\Delta w_2 + w_1 - \zeta w_2, \end{cases} \qquad (x,t) \in \Omega \times (0,T), \tag{6.5}$$

where

$$e = \beta h(u^*) \frac{k_1}{a}.$$

Looking for solutions like in (6.4), the following eigenvalue problem must be solved:

$$\mathbf{B}\mathbf{w} = \rho\mathbf{w}, \tag{6.6}$$

where $\mathbf{B}$ is the two by two matrix

$$\mathbf{B} = \begin{pmatrix} -1 - 2\alpha u^* - k^2 & \zeta + ek^2 \\ 1 & -\zeta - k^2 D \end{pmatrix}.$$

In this case, the trace of matrix $\mathbf{B}$ is strictly negative while its determinant can be strictly negative for some values of k. Thus, taking chemotaxis into account in the system may lead to the emergence of spatial patterns.

6.2 Appendix 2

Numerical simulations

A set of validated parameters derived from studies published [7] was used to run numerical simulations. The data used came from an Andean Pramo site near Gavidia, Venezuela. As pattern geometries depend on the shape of the spatial domain (see [20]), two different forms of spatial domain were tested. Figures below show the numerical simulations of the soil microbial biomass compartment for the nearly rectangular and circular domains, using either $h(u) = h_1(u) = u$ which does not prevent explicitly any overcrowding (Figures 1–2), or $h(u) = h_2(u) = u(M - u)$ which explicitly does prevent overcrowding (Figures 3–4). These figures show the spatial variability and patterns obtained for soil microbial biomass after 60 days and for the two spatial domain shapes. The soil microbial biomass pattern agrees with the distribution within the soil matrix of the microbial hot spots at micron scale. Numerical simulations were performed using COMSOL Multiphysics 5.0.

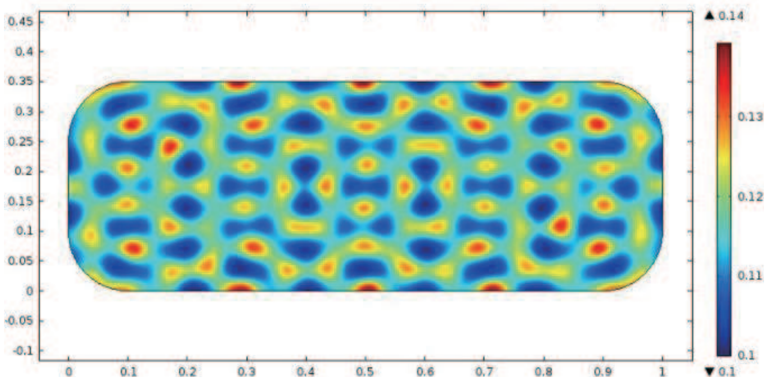

Figure 1 Spatial microbial biomass distribution when $h = h_1$ after 60 days.

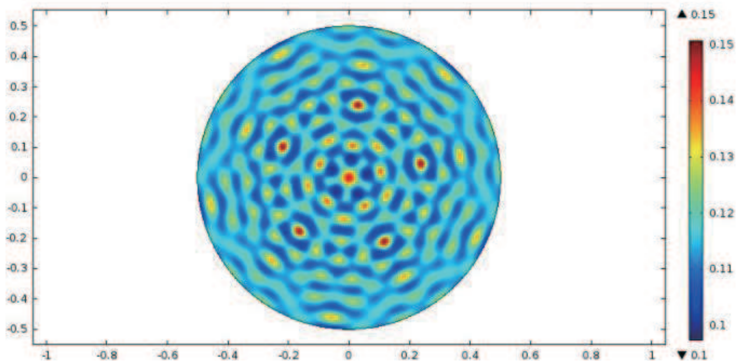

Figure 2 Spatial microbial biomass distribution when $h = h_1$ after 60 days.

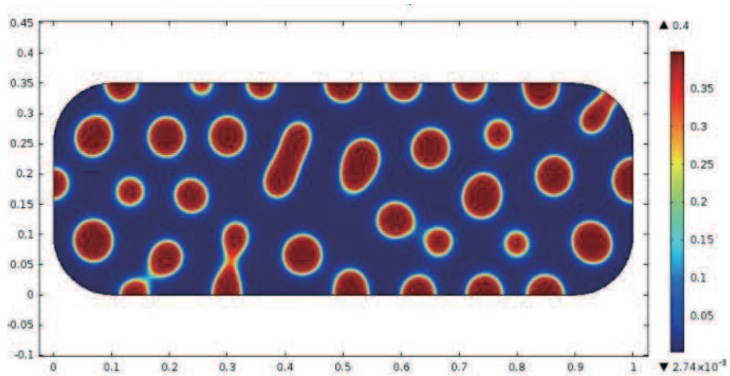

Figure 3 Spatial microbial biomass distribution when $h = h_2$ after 60 days.

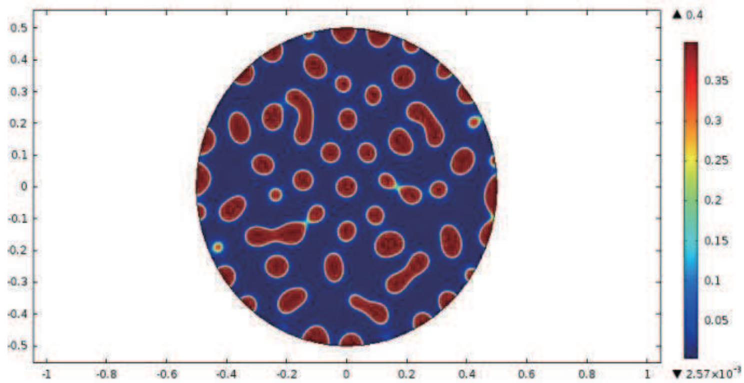

Figure 4 Spatial microbial biomass distribution when $h = h_2$ after 60 days.

Acknowledgement We thank DR Martial Bernoux-UMR Eco&Sols (INRA, SupAgro, CIRAD, IRD) for his expertise and advice on soil carbon dynamics.

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