

Internal Controllability for Parabolic Systems Involving Analytic Non-local Terms

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*(This article is dedicated to Phillippe G. Ciarlet in the occasion of his
80th birthday, with gratitude and admiration for his mastery
and continuous support. Merci Philippe!)*

Abstract This paper deals with the problem of internal controllability of a system of heat equations posed on a bounded domain with Dirichlet boundary conditions and perturbed with analytic non-local coupling terms. Each component of the system may be controlled in a different subdomain. Assuming that the unperturbed system is controllable—a property that has been recently characterized in terms of a Kalman-like rank condition—the authors give a necessary and sufficient condition for the controllability of the coupled system under the form of a unique continuation property for the corresponding elliptic eigenvalue system. The proof relies on a compactness-uniqueness argument, which is quite unusual in the context of parabolic systems, previously developed for scalar parabolic equations. The general result is illustrated by two simple examples.

Keywords Parabolic systems, Non-local potentials, Analyticity, Null controllability, Kalman rank condition, Spectral unique continuation

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1 Introduction

1.1 Motivation

Nonlocal parabolic systems are relevant in a variety of applications to Biology and Physics (see [24]). They have been analyzed exhaustively in the recent past, in particular in the context of the non-local fractional Laplacian, and significant progress has been achieved. But controllability issues for these models remain very much unexplored. Here we analyse parabolic systems coupled by non-local lower order perturbations, the principal part being a classical constant coefficient parabolic system.

The content of this paper is a natural combination of the methods developed in [21] to achieve sharp results for parabolic systems coupled through constant coefficient matrices and

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those in [15] devoted to scalar equations perturbed by non-local lower order potentials. Our goal here is to derive a simple and exploitable spectral necessary and sufficient condition of controllability and the corresponding dual observability one.

This paper is very much inspired in the pioneering ideas introduced by J. L. Lions in his famous SIAM Review article [19] that stimulated a significant step forward on the state of the art. The early developments in this field were summarized with mastery in the celebrated survey article by D. L. Russell [25]. The presentation in this paper is concise, relying significantly on various tools of functional analysis that are developed and presented in a self-contained manner in the more recent book by Phillippe G. Ciarlet [7].

1.2 Problem formulation and main result

Let us now present the problem under consideration into more details.

Let Ω be a smooth domain of \mathbb{R}^N ($N \in \mathbb{N}^*$), $T > 0$, $n \in \mathbb{N}^*$ and $m \in \mathbb{N}^*$ (with possibly $m < n$). Let ω_i ($i \in [1, m]$) be some open subsets of Ω that can be chosen arbitrarily (in particular all the ω_i 's may be disjoint).

We are interested in the controllability of the following system of heat equations with Dirichlet boundary conditions:

$$\begin{cases} \partial_t Y = D\Delta Y + \int_{\Omega} A(x, \xi)Y(t, \xi)d\xi + \sum_{i=1}^m B_i u_i 1_{\omega_i} & \text{in } (0, T) \times \Omega, \\ Y(t, x) = 0 & \text{in } (0, T) \times \partial\Omega, \\ Y(0) = Y^0 \end{cases} \quad (1.1)$$

with $Y^0 \in [L^2(\Omega)]^n$, $u = (u_1, \dots, u_m) \in [L^2(\Omega)]^m$ (which play the role of distributed controls), $A \in \mathcal{M}_n(\mathbf{H}) \subset \mathcal{M}_n(L^2(\Omega \times \Omega))$ (where \mathbf{H} is a space of admissible potentials that will be introduced afterwards in (1.12)), B_i being the i -th column of $B \in \mathcal{M}_{n,m}(\mathbb{R})$.

The coupling matrix $D \in \mathcal{M}_n(\mathbb{R})$ is assumed to satisfy the ellipticity condition

$$\langle DX, X \rangle \geq C \|X\|^2, \quad \forall X \in \mathbb{R}^n \quad (1.2)$$

(here and hereafter, $\|\cdot\|$ will always denote the euclidean norm). Condition (1.2) is sufficient to ensure the well-posedness of (1.13), since the principal part $D\Delta$ in (1.13) is strongly parabolic in the sense of [17, Chapter 7, Definition 7].

More precisely, we consider the so-called null controllability problem, the goal being to drive the system to the null final target $Y(T) \equiv 0$ by a suitable choice of the controls $u = (u_1, \dots, u_m) \in [L^2(\Omega)]^m$.

The scalar case (i.e., $n = 1$) has been analyzed in [15] for a scalar potential $a \in \mathbf{H}$. Our goal here is to extend those results to coupled systems, obtaining a simple and exploitable spectral necessary and sufficient condition of controllability and the corresponding dual observability one.

The controllability and observability of systems of partial differential equations have been intensively studied in the last decade, leading to important progress. We shall refer to some of

the existing literature in the end of this introduction. But, as indicated above, the number of articles devoted to non-local problems is very limited.

Our analysis will follow a combination of the methods developed in [21] for the analysis of parabolic systems and in [15] to handle non-local coupling terms. Accordingly, we shall use in an essential manner the spectral decomposition of the Laplacian.

Let $\{\lambda_k\}_{k \geq 1}$ be the eigenvalues of $-\Delta$ with Dirichlet boundary conditions and $e_k \in H_0^1(\Omega)$ be the corresponding eigenfunctions, constituting an orthonormal basis of $L^2(\Omega)$.

Before considering the non-locally perturbed case, let us first recall some recent results on models involving constant coefficient coupling terms:

$$\begin{cases} \partial_t Z = D^* \Delta Z + A^* Z & \text{in } (0, T) \times \Omega, \\ Z = 0 & \text{in } (0, T) \times \partial\Omega, \\ Z(0) = Z^0, \end{cases} \quad (1.3)$$

where $Z^0 \in (L^2(\Omega))^n$ and $A^* \in \mathcal{M}_n(\mathbb{R})$ is a constant coupling matrix.

Here, rather than dealing with the controllability problem we consider the dual observability one. It concerns the obtention of full estimates on the state Z at time $t = T$ out of partial measurements on the control subsets ω_i .

In [21] it was proved that system (1.3) is observable on $(0, T)$ in the sense that there exists $C = C(T) > 0$ such that for every $Z^0 \in [L^2(\Omega)]^n$, the solution Z of (1.3) verifies

$$\|Z(T)\|^2 \leq C(T) \sum_{i=1}^m \int_0^T \int_{\omega_i} |B_i^* Z(t, x)|^2 dx dt \quad (1.4)$$

if and only if

$$\text{rank } K(\lambda_p) = n, \quad \forall p \geq 1, \quad (1.5)$$

where

$$K(\lambda) := [B|(-\lambda D + A)B| \cdots |(-\lambda D + A)^{n-1}B]. \quad (1.6)$$

Moreover, following [21, Proof of Theorem 3] and [23, Proof of Theorem 2.2], a precise upper bound on the observability constant $C(T)$ in (1.4) can be given for $T > 0$ small enough, getting

$$\|Z(T)\|^2 \leq C e^{\frac{C}{T}} \sum_{i=1}^m \int_0^T \int_{\omega_i} |B_i^* Z(t, x)|^2 dx dt. \quad (1.7)$$

If $A^* = 0$, it is easy to prove that (1.6) is equivalent to the following Kalman rank condition:

$$\text{rank } K_D = n, \quad (1.8)$$

where, by definition,

$$K_D := [B|DB| \cdots |D^{n-1}B] \in \mathcal{M}_{n, nm}(\mathbb{R}), \quad (1.9)$$

that only concerns the coupling matrix D and the control one B . When $A^* \neq 0$ though, we get a sequence of spectral conditions, depending on the eigenvalues of the Laplacian.

In all what follows, we decompose the initial condition as

$$Z^0(x) = \sum_{k=1}^{\infty} Z_k e_k(x), \quad (1.10)$$

where $(Z_k)_{k \in \mathbb{N}^*} \in (l^2(\mathbb{N}^*))^n$.

The observability inequality (1.7), as pointed out in [16, Remark 6.1] (see also [23, Lemma 3.3] with $\beta = 1$ and $\alpha = \frac{1}{2}$), can be rewritten, in terms of the Fourier series expansion of the initial datum Z^0 given in (1.10), as

$$\sum_{k=1}^{\infty} e^{-R\sqrt{\lambda_k}} \|Z_k\|^2 \leq C(T) \sum_{i=1}^m \int_0^T \int_{\omega_i} |B_i^* Z(t, x)|^2 dx dt \quad (1.11)$$

for some $R > 0$ and $C(T) > 0$ independent of $Z^0 \in [L^2(\Omega)]^n$.

Note that this kind of observability inequality (which is related to reachability issues, see e.g. [12]), introduced in [16], has also been used in [15, Lemma 2], for instance, to deal with non-local perturbations. Note also that estimating R in (1.11) and, more precisely, finding explicit lower bounds on R (in terms for instance of the geometries of Ω , ω_i and the coupling matrices D and A) is an open problem, related to the optimal weights that can be considered in a Carleman estimate for the solutions of (2.1) (see [16] and Lemma 2.1 below), which are not known in general. This constitutes a challenging problem, also related to the cost of controllability and its dependence with respect to the geometry, which is still unknown in dimension greater than 1. Summarizing, the constant $R > 0$ so that (1.11) holds is known to exist, but very little is known on its actual value and its dependence on the parameters of the system under consideration.

This spectral observability inequality motivates the introduction of the following Hilbert space of non-local potentials (that was mentioned before when describing the class of models under consideration)

$$\begin{aligned} \mathbf{H} := \left\{ f(x, \xi) = \sum_{m,j \geq 1} f_{mj} e_m(x) e_j(\xi) \right. \\ \left. \in L^2(\Omega \times \Omega) \mid \sum_{m=1}^{\infty} \left(\sum_{j=1}^{\infty} \frac{1}{\lambda_j} |f_{mj}|^2 \right) \frac{1}{\lambda_m} e^{R\sqrt{\lambda_m}} < \infty \right\}, \end{aligned} \quad (1.12)$$

$R > 0$ being as in (1.11). Let us emphasize that kernels $A \in \mathcal{M}_n(\mathbf{H})$ enjoy the following property (see for instance [15, Remark 5]):

$$x \in \Omega \mapsto \int_{\Omega} A(x, \xi) f(\xi) d\xi \text{ is analytic, } \quad \forall f \in H_0^1(\Omega)^n.$$

Let us now consider the following (forward) adjoint system of (1.1) involving also the non-local coupling terms:

$$\begin{cases} \partial_t Z = D^* \Delta Z + \int_{\Omega} A^*(\xi, x) Z(t, \xi) d\xi & \text{in } (0, T) \times \Omega, \\ Z = 0 & \text{in } (0, T) \times \partial\Omega, \\ Z(0) = Z^0 \end{cases} \quad (1.13)$$

for some $Z^0 \in (L^2(\Omega))^n$.

Our goal is to extend the observability inequalities above for this complete model involving the non-local perturbations. We are able to reduce the observability problem under consideration to a unique continuation property for an elliptic problem, usually called Fattorini's Criterion (see [13]). This condition is much easier to be verified in practice, as illustrated by two examples in Section 3. Note however that, due to the presence of the non-local term, this property is not a consequence of the existing wide literature on the unique continuation for elliptic problems and that analyticity assumptions are imposed on the kernel. As a consequence of the spectral observability inequality, by duality, we shall also derive the controllability property for the original control system involving the non-local terms.

The main result of this paper is the following.

Theorem 1.1 *Consider any $T > 0$ and assume that $A(x, \xi) \in \mathcal{M}_n(\mathbf{H})$, where \mathbf{H} is defined in (1.12), and that K_D verifies the Kalman rank condition (1.8).*

Then, there exists $C(T) > 0$ such that any solution Z of (1.13) (involving the non-local perturbation terms) verifies

$$\|Z(T)\|^2 \leq C(T) \sum_{i=1}^m \int_0^T \int_{\omega_i} \|B_i^* Z(t, x)\|^2 dx dt \quad (1.14)$$

if and only if the following unique continuation property is verified for every $\lambda \in \mathbb{R}$:

$$\begin{cases} -D^* \Delta \varphi - \int_{\Omega} A^*(\xi, x) \varphi(\xi) d\xi = \lambda \varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \\ B^* \varphi = 0 & \text{in } \Omega, \\ \Rightarrow \varphi \equiv 0. \end{cases} \quad (1.15)$$

Equivalently, under condition (1.15), system (1.1) is null-controllable on $(0, T)$, in the sense that for any $Y^0 \in [L^2(\Omega)]^n$, there exists $u \in [L^2((0, T) \times \Omega)]^m$ such that the corresponding solution to (1.1) verifies $Y(T, \cdot) = 0$.

The proof of the main result consists in obtaining the inequality (1.12) for the complete system (1.13) on the basis of the same inequality for the system in the absence of non-local perturbations (1.3). This is done applying a compactness-uniqueness argument, and reduces the issue to the fulfillment of the unique continuation property above (1.15) for the spectral problem. Once (1.12) is proved for the complete adjoint system (1.13), the null controllability result for (1.1) is a direct consequence of a classical duality principle.

Compactness-uniqueness arguments have rarely been applied in the context of heat equations because of the strong time irreversibility. In [15] this principle was applied in a satisfactory manner for scalar parabolic equations involving non-local potentials, provided they belong to the space \mathbf{H} . This compactness-uniqueness technique, which applies in the context of non-local perturbation terms, cannot be used for pointwise space-varying coupling terms. The main novelty of the present article is to extend this analysis to parabolic systems involving non-local terms.

Several other remarks are in order.

Remark 1.1 (1) We are unable to derive an explicit estimate on the cost of controllability in small time, similar to the one given in (1.7), because we use a contradiction argument.

(2) Remark that in (1.15), $B^*\varphi = 0$ is assumed on all Ω and not only ω . This is a consequence of the analyticity properties of the kernel A^* . This fact facilitates the needed unique continuation property, which becomes a problem of an algebraic nature since localisation (in the space variable) issues do not arise.

(3) The hypothesis that $A(x, \xi)$ belongs to $\mathcal{M}_n(\mathbf{H})$ is necessary in our study to develop the compactness-uniqueness argument. However, it is likely that this hypothesis to be of purely technical nature. In fact, there is no reason that A should be analytic, and it is likely that one might obtain the same result for any kernel that is regular enough to ensure that equation (1.1) is well-posed, for example $A(x, \xi) \in \mathcal{M}_n(L^2(\Omega \times \Omega))$. Hence, a natural conjecture would be that the main result of Theorem 1.1 holds under the assumption that $A(x, \xi) \in \mathcal{M}_n(L^2(\Omega \times \Omega))$ (instead of $A(x, \xi) \in \mathcal{M}_n(\mathbf{H})$) and provided the unique continuation property (1.15) holds.

Note that in [22] a 1-d scalar equation is considered and that the analyticity assumption is avoided within the particular class of kernels in separated variables $A(x, \xi) = A_x(x)A_\xi(\xi)$, under the assumption that $A_x(x)$ does not vanish in the subset where the control is being applied. On the other hand, as indicated by Patrick Gérard in a private communication, unique continuation may fail for the spectral problem with smooth kernels in separated variables of compact support even in the scalar case $n = 1$. Accordingly it fails for time-dependent parabolic problems too. This example shows the necessity of some additional assumption on the kernel, such as analyticity, for the unique continuation property to hold even for scalar equations. Finding sharp conditions on the non-local kernel for unique continuation in the context of systems is an interesting open problem.

1.3 Bibliographical comments

As indicated above, there is an extensive literature devoted to the controllability properties of PDE systems but problems involving non-local terms are rarely considered. Apart from references [15] and [22], we would like to mention [20], where a Carleman estimate for a scalar non-local parabolic equation with an integral term involving the solution and its first order derivatives is proved, with applications to unique continuation and inverse problems.

Concerning parabolic systems without non-local terms, some of the existing results concern the following topics and techniques (see also the survey [1] for earlier results). For a more detailed presentation, concerning also the hyperbolic and dispersive case, we refer to [21].

(1) One-dimensional results (i.e., $d = 1$) were obtained in [2–4, 6].

(2) Multi-dimensional results were obtained in [10] for constant or time-dependent coupling terms, and partial results in the case of space-dependent coupling terms were obtained in [2–3, 5–6, 11, 18].

(3) The nonlinear case was notably studied in [8–9, 14].

(4) Observability properties for systems involving a superposition of different dynamics (notably coupled systems of heat and wave equations) were studied in [26].

2 Proof of the Main Result

Assuming that the spectral unique continuation property (1.15) is verified, the proof consists in showing that the null-controllability of (1.1) holds. To do this, using the equivalence between null controllability and observability, it suffices to show that the observability inequality (1.14) holds for the complete system (1.13).

The proof of this inequality for the complete system involving the non-local terms relies on a compactness-uniqueness argument similar to the one in [15, Proof of (16)]. We proceed in several steps.

Step 1 Splitting of the solution. To get (1.14), first of all, we decompose the solution Z of (1.13) into two parts $Z = \zeta + p$, where p verifies

$$\begin{cases} \partial_t p = D^* \Delta p & \text{in } (0, T) \times \Omega, \\ p = 0 & \text{in } (0, T) \times \partial\Omega, \\ p(0) = Z^0 \end{cases} \quad (2.1)$$

and ζ verifies

$$\begin{cases} \partial_t \zeta = D^* \Delta \zeta + \int_{\Omega} A^*(\xi, x) \zeta(t, \xi) d\xi + \int_{\Omega} A^*(\xi, x) p(t, \xi) d\xi & \text{in } (0, T) \times \Omega, \\ \zeta = 0 & \text{in } (0, T) \times \partial\Omega, \\ \zeta(0) = 0. \end{cases} \quad (2.2)$$

From (1.8) and (1.11), we already know that

$$\|p(T)\|^2 \leq C \sum_{i=1}^m \int_0^T \int_{\omega_i} \|B_i^* p(t, x)\|^2 dx dt. \quad (2.3)$$

Step 2 An auxiliary Carleman estimate. Let us prove the following useful Carleman estimate.

Lemma 2.1 *There exist two constants $C_0 > 0$ (not depending on T) and $C(T) > 0$ such that for any $Z^0 \in [L^2(\Omega)]^n$, the solution p of (2.1) verifies*

$$\int_0^T \int_{\Omega} e^{-\frac{C_0}{t}} \|p(t, x)\|^2 dx dt \leq C(T) \sum_{k=1}^{\infty} \int_0^T \int_{\Omega} \sum_{i=1}^m \|B_i^* p(t, x) 1_{\omega_i}(x)\|^2 dx dt. \quad (2.4)$$

Proof We follow the computations of [16, Remark 6.1]. First of all, we decompose Z^0 in the Hilbert basis $\{e_k\}$ as

$$p(x) = \sum_{k=1}^{\infty} p_k e_k(x).$$

For $C_0 > 0$ (to be determined later on) we remark that

$$\int_0^T \int_{\Omega} e^{-\frac{C_0}{t}} \|p(t, x)\|^2 dx dt = \sum_{k=1}^{\infty} \int_0^T e^{-\frac{C_0}{t}} \|\tilde{p}_k(t)\|^2 dt, \quad (2.5)$$

where \tilde{p}_k is the unique solution of the ordinary differential equation

$$\begin{cases} \tilde{p}_k'(t) = -\lambda_k D^* \tilde{p}_k(t) & \text{in } (0, T) \times \Omega, \\ \tilde{p}_k(0) = p_k. \end{cases} \quad (2.6)$$

Using the ellipticity condition (1.2), there exists $C_1 > 0$ (independent of C_1) such that for any $t > 0$, one has

$$\|\tilde{p}_k(t)\|^2 \leq \|p_k\|^2 e^{-C_1 \lambda_k t}. \quad (2.7)$$

Hence, from (2.5) and (2.7) we deduce that

$$\int_0^T \int_{\Omega} e^{-\frac{C_0}{t}} \|p(t, x)\|^2 dx dt \leq \sum_{k=1}^{\infty} \|p_k\|^2 \left(\int_0^T e^{-\frac{C_0}{t} - C_1 \lambda_k t} dt \right). \quad (2.8)$$

Besides, it is well-known that, as $\lambda \rightarrow \infty$,

$$\int_0^T e^{-\frac{C_0}{t} - C_1 \lambda_k t} dt \simeq \left(\frac{\pi^2 C_0}{C_1^3} \right)^{\frac{1}{4}} e^{-2\sqrt{C_1 C_0 \lambda}}.$$

Hence, there exists some $C_2 > 0$ such that for any $k > 0$, one has

$$\int_0^T e^{-\frac{C_0}{t} - C_1 \lambda_k t} dt \leq C_2 e^{-2\sqrt{C_0 C_1 \lambda_k}}.$$

We deduce from (2.8) that

$$\int_0^T \int_{\Omega} e^{-\frac{C_0}{t}} \|p(t, x)\|^2 dx dt \leq C_2 \sum_{k=1}^{\infty} \|p_k\|^2 e^{-2\sqrt{C_0 C_1 \lambda_k}}. \quad (2.9)$$

Inequality (2.4) then follows by using (1.11) together with (2.9) and taking C_0 large enough such that $2\sqrt{C_0 C_1} > R$.

Step 3 Reduction to the proof of two inequalities. We remark that in order to obtain (1.14), it is enough to prove the two following key inequalities:

$$\sum_{i=1}^m \int_0^T \int_{\omega_i} \|B_i^* p(t, x)\|^2 dx dt \leq C \sum_{i=1}^m \int_0^T \int_{\omega_i} \|B_i^* Z(t, x)\|^2 dx dt \quad (2.10)$$

and

$$\|Z(T)\|^2 \leq C \sum_{i=1}^m \int_0^T \int_{\omega_i} \|B_i^* p(t, x)\|^2 dx dt. \quad (2.11)$$

Step 4 Proof of (2.10). Assume that (2.10) is not verified whereas (1.15) is verified. Then, there exists a sequence $(Z_n^0)_{n \in \mathbb{N}}$ such that the corresponding solution p_n of (2.1) with initial condition Z_n^0 verifies

$$\sum_{i=1}^m \int_0^T \int_{\omega_i} \|B_i^* p(t, x)\|^2 dx dt = 1 \quad (2.12)$$

and the corresponding solution Z_n of (1.13) with initial condition Z_n^0 is such that

$$\sum_{i=1}^m \int_0^T \int_{\omega_i} \|B_i^* Z_n(t, x)\|^2 dx dt < \frac{1}{n}. \quad (2.13)$$

We also call ζ_n the solution to (2.2) where p is replaced by p_n , so that we have the relation

$$Z_n = p_n + \zeta_n. \quad (2.14)$$

We are going to prove that $\zeta_n \rightarrow 0$ (up to a subsequence) strongly in $L^2((0, T) \times \Omega)$, which is obviously in contradiction with (2.12)–(2.13) since these estimates together with (2.14) imply

$$1 \leq 2 \left(\frac{1}{n} + \sum_{i=1}^m \int_0^T \int_{\omega_i} \|B_i^* \zeta_n(t, x)\|^2 dx dt \right).$$

First of all, let us remark that there exists $C > 0$ such that

$$\left\| \int_{\Omega} A^*(\xi, x) Z_n(t, \xi) d\xi \right\|_{L^2((0, T), H^{-1}(\Omega))} \leq C. \quad (2.15)$$

It is an easy consequence of the computation given in [15, (21)] applied on each component of A^* . Hence, by classical energy estimates and compactness arguments, one may assume that ζ_n converges strongly in $L^2((0, T) \times \Omega)$ to some $\zeta \in L^2((0, T) \times \Omega)$. This implies, together with (2.4) and (2.14), that if we fix $\delta \in (0, T)$, $(Z_n)_{n \in \mathbb{N}}$ is bounded in $L^2((\delta, T), \Omega)$. Hence, $(Z_n)_{n \in \mathbb{N}}$ can be assumed to converge weakly in $L^2((\delta, T), \Omega)$ to some $Z \in L^2((\delta, T), \Omega)$. Then, one can prove that Z solves the following PDE:

$$\partial_t Z = D^* \Delta Z + \int_{\Omega} A^*(\xi, x) Z(t, \xi) d\xi \quad \text{in } (0, T) \times \Omega.$$

Moreover, we also know, thanks to (2.13), that $B_i^* Z(t, x) = 0$ on $(0, T) \times \omega_i$, $\forall i \in [1, m]$. Using the well-known Fattorini criterion for approximate controllability (see [13]), proving that $Z \equiv 0$ is equivalent to proving the following assertion:

$$\begin{cases} -D^* \Delta \varphi - \int_{\Omega} A^*(\xi, x) \varphi(\xi) d\xi = \lambda \varphi & \text{on } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \\ B_i^* \varphi = 0 & \text{on } (0, T) \times \omega_i, \forall i \in [1, m], \\ \Rightarrow \varphi \equiv 0. \end{cases} \quad (2.16)$$

Hence, we consider any $\varphi \in L^2(\Omega)$ verifying

$$\begin{cases} -D^* \Delta \varphi - \int_{\Omega} A^*(\xi, x) \varphi(\xi) d\xi = \lambda \varphi & \text{on } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \\ B_i^* \varphi(x) = 0 & \text{on } (0, T) \times \omega_i, \quad \forall i \in [1, m]. \end{cases} \quad (2.17)$$

We will prove the following analyticity property on φ .

Lemma 2.2 *Any $\varphi \in L^2(\Omega)$ verifying the first two lines of (2.17) for some $\lambda \in \mathbb{R}$ is analytic on Ω .*

Proof From (2.17) and taking into account that $x \mapsto \int_{\Omega} A^*(\xi, x) \varphi(\xi) d\xi$ is analytic on Ω (hence C^∞ on Ω) since $\varphi \in H_0^1(\Omega)$, an easy induction argument gives that $\varphi \in C^\infty(\Omega)$.

Now, consider any component of A^* that we call a^* and that we decompose as

$$a^*(\xi, x) = \sum_{m, j \geq 1} a_{m,j}^* e_m(\xi) e_j(x).$$

Using condition (1.12) and since $A^* \in \mathcal{M}_n(\mathbf{H})$, we obtain that for any $j \in \mathbb{N}$, one has

$$\sum_{m=1}^{\infty} \frac{1}{\lambda_m} e^{R\sqrt{\lambda_m}} |a_{m,j}^*|^2 < \infty,$$

implying thanks to (2.17) that for any $\varphi \in H_0^1(\Omega)^n$, one has $\mathcal{K}(\varphi) = 0$ on $\partial\Omega$, where \mathcal{K} is given by

$$\mathcal{K} : \varphi \in L^2(\Omega) \mapsto \int_{\Omega} A^*(\xi, x) \varphi(\xi) d\xi.$$

Hence, another easy induction argument enables us to conclude that

$$\varphi \in \bigcap_{n=0}^{\infty} \mathcal{D}(\Delta^n), \quad (2.18)$$

where Δ represents here the Dirichlet Laplace operator with domain $H_0^1(\Omega) \cap H^2(\Omega)$. Let us now prove that φ is moreover analytic. Let $k \in \mathbb{N}$. In what follows, C is a constant that may vary from inequality to inequality and is independent of k . We consider the scalar product of the first line of (2.17) by the vector $\Delta^{2k+1} \varphi$ and we integrate on Ω . Taking into account (2.18), we obtain after some integrations by parts that

$$\|D^* \Delta^{k+1} \varphi\|_{L^2(\Omega)}^2 - \langle \varphi, \Delta^{2k+1}(\mathcal{K}\varphi) \rangle_{L^2(\Omega)} = \lambda \|\nabla \Delta^k \varphi\|_{L^2(\Omega)}^2,$$

so that notably

$$\|D^* \Delta^{k+1} \varphi\|_{L^2(\Omega)}^2 \leq \lambda \|\nabla \Delta^k \varphi\|_{L^2(\Omega)}^2 + |\langle \varphi, \Delta^{2k+1}(\mathcal{K}\varphi) \rangle_{L^2(\Omega)}|. \quad (2.19)$$

Let us focus on $\|\Delta^{2k+1}(\mathcal{K}\varphi)\|_{L^2(\Omega)}$. Following the computations of [15, Remark 5], one easily infers that for any component of A^* that we call a^* and that we decompose as $a^*(\xi, x) =$

$\sum_{m,j \geq 1} a_{mj}^* e_m(\xi) e_j(x)$, we have

$$\left\| \Delta^{2k+1} \left(\int_{\Omega} a^*(\xi, x) \varphi(\xi) d\xi \right) \right\|_{L^2(\Omega)}^2 \leq C \|\varphi\|_{H_0^1(\Omega)} \sum_{m \geq 1} \lambda_m^{2k+1} \sum_{j \geq 1} \frac{|a_{mj}^*|^2}{\lambda_j}. \quad (2.20)$$

By looking at the power series expansion of the exponential function, we deduce that $\frac{\lambda_m^{2k+2}}{R^{4k+4}(4k+4)!} \leq e^{R\sqrt{\lambda_m}}$, so that

$$\lambda_m^{2k+1} \leq R^{4k+4} (4k+4)! \frac{e^{R\sqrt{\lambda_m}}}{\lambda_m}.$$

Using the definition of \mathbf{H} given in (1.12) together with (2.20), we deduce that

$$\|\Delta^{2k+1}(\mathcal{K}(\varphi))\|_{L^2(\Omega)}^2 \leq C R^{4k+4} (4k+4)! \|\varphi\|_{H_0^1(\Omega)}^2. \quad (2.21)$$

Using Young's inequality and plugging (2.21) into (2.19) we obtain that

$$\begin{aligned} \|D^* \Delta^{k+1} \varphi\|_{L^2(\Omega)}^2 &\leq C \lambda \|\varphi\|_{H^{2k+1}}^2 + \|\varphi\|_{L^2(\Omega)}^2 + C R^{4k+4} (4k+4)! \|\varphi\|_{H_0^1(\Omega)}^2 \\ &\leq C (\lambda \|\varphi\|_{H^{2k+1}}^2 + R^{4k+4} (4k+4)! \|\varphi\|_{H_0^1(\Omega)}^2). \end{aligned} \quad (2.22)$$

Using the ellipticity condition (1.2) and taking into account that

$$\|\varphi\|_{H^{2k+2}(\Omega)} \leq C^{k+1} \|D^* \Delta^{k+1} \varphi\|_{L^2(\Omega)},$$

we obtain from (2.22) that

$$\|\varphi\|_{H^{2k+2}(\Omega)}^2 \leq C^k (\lambda \|\varphi\|_{H^{2k+1}(\Omega)}^2 + R^{4k+4} (4k+4)! \|\varphi\|_{H_0^1(\Omega)}^2). \quad (2.23)$$

Easy interpolation arguments together with an induction enable us to obtain from (2.23) that for any $m \in \mathbb{N}^*$, one has

$$\begin{aligned} \|\varphi\|_{H^m(\Omega)}^2 &\leq C(1 + C^m \lambda + \dots + C^{m(m-1)} \lambda^{m-1} + m R^{2m} (2m)!) \|\varphi\|_{H_0^1(\Omega)}^2 \\ &\leq C(C^{m^2} + C^m (2m)!) \|\varphi\|_{H_0^1(\Omega)}^2 \\ &\leq C^{m+1} (2m)! \|\varphi\|_{H_0^1(\Omega)}^2, \end{aligned}$$

where from now on C is a constant that might depend on λ or R but not on m . Now, using Sobolev embedding theorems together with the inequality $\sqrt{(2m)!} \leq C^m m!$, we deduce that for any $m \in \mathbb{N}$, one has

$$\sum_{|\alpha|=m} \|\partial^\alpha \varphi\|_{L^\infty(\Omega)} \leq C^{m+1} m! \|\varphi\|_{H_0^1(\Omega)}, \quad (2.24)$$

where for any multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$, we write for simplicity $|\alpha| = \alpha_1 + \dots + \alpha_N$ and $\partial^\alpha \varphi = \frac{\partial^{\alpha_1 + \dots + \alpha_N}}{\partial x_{\alpha_1} \dots \partial x_{\alpha_N}} \varphi$. It is well-known that inequality (2.24) implies the analyticity of φ on Ω , which finishes the proof.

We are now ready to complete the proof of Step 4. Let us consider any $\varphi \in L^2(\Omega)^N$ verifying (2.17). Using Lemma 2.2, we deduce that φ is analytic on Ω , which implies that $B^*\varphi$ is also analytic on Ω . Hence, using the last line of (2.17), we deduce that $B^*\varphi = 0$ in Ω .

Now, using assumption (1.15), (2.16) is verified and hence $Z \equiv 0$ on $(0, T) \times \Omega$. We deduce that p_n converges weakly to $-\zeta$ in $L^2((0, T) \times \Omega)$, which implies that $\zeta_n \rightarrow 0 = \zeta$ in $L^2((0, T) \times \Omega)$ because of (2.2). This leads to the desired contradiction.

Step 5 Proof of (2.11). This inequality is a consequence of (2.3) and easy energy estimates on ξ using equation (2.2) and arguing as in the proof of [15, (21)].

Finally, we have proved that (1.15) implies (1.14). The fact that the null-controllability of (1.1) (i.e., (1.14)) implies (1.15) is standard and is omitted.

3 Two Simple Examples of Application

3.1 Indirect controllability of cascade systems of two equations

In what follows, we consider the case of two coupled equations with cascade structure and control on the first component.

More precisely, we consider the following system:

$$\begin{cases} \partial_t Y^1 = d_{11}\Delta Y^1 + d_{12}\Delta Y^2 + \int_{\Omega} (a_{11}(x, \xi)Y^1(t, \xi)d\xi \\ \quad + a_{12}(x, \xi)Y^2(t, \xi)d\xi) + u1_{\omega} & \text{in } (0, T) \times \Omega, \\ \partial_t Y^2 = d_{21}\Delta Y^1 + d_{22}\Delta Y^2 + \int_{\Omega} a_{21}(x, \xi)Y^1(t, \xi)d\xi & \text{in } (0, T) \times \Omega, \\ Y^1(t, x) = Y^2(t, x) = 0 & \text{in } (0, T) \times \partial\Omega, \\ (Y^1(0), Y^2(0)) = (Y_0^1, Y_0^2) & \text{in } \Omega. \end{cases} \quad (3.1)$$

Here D is given by

$$D := \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

and is assumed to verify (1.2). The non-local potential A is given by

$$A(x, \xi) := \begin{pmatrix} a_{11}(x, \xi) & a_{12}(x, \xi) \\ a_{21}(x, \xi) & 0 \end{pmatrix}.$$

We consider the control operator B given by

$$B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The control acts on some open subset $\omega \subset \Omega$. We are going to prove the following sufficient condition for the controllability of (3.1).

Theorem 3.1 *Consider any $T > 0$ and assume that $a_{ij}(x) \in \mathbf{H}$ for $(i, j) \in \{1, 2\}^2$, $d_{21} \neq 0$ and $d_{22} \neq 0$. Then, (3.1) is null-controllable.*

Proof First, observe that condition (1.8) is equivalent to $d_{21} \neq 0$. Then, applying Theorem 1.1, the null-controllability of (3.1) is equivalent to the following unique continuation property:

$$\begin{cases} -d_{21}\Delta\varphi - \int_{\Omega} a_{21}(\xi, x)\varphi(\xi)d\xi = 0 & \text{in } \times \Omega, \\ -d_{22}\Delta\varphi = \lambda\varphi & \text{in } \times \Omega, \\ \varphi = 0 & \text{in } \partial\Omega, \\ \Rightarrow \varphi \equiv 0. \end{cases} \quad (3.2)$$

By contradiction, assume that there exists some $\varphi \not\equiv 0$ verifying the first three equations of (3.2). Let us decompose a_{21} as follows: $a_{21}(\xi, x) := \sum_{k,l} c_{kl}e_k(x)e_l(\xi)$. Since $d_{22} \neq 0$, it is clear from the second equation of (3.2) that there exists $m \in \mathbb{N}^*$ such that $\lambda = d_{22}\lambda_m$. In this case, without loss of generality we may assume that $\varphi(x) = e_m(x)$. Using the spectral decomposition of a_{21} , we obtain that

$$\int_{\Omega} a_{21}(\xi, x)\varphi(\xi)d\xi = \sum_k c_{km}e_k(x).$$

Moreover, one has $-d_{21}\Delta\varphi(x) = d_{21}\lambda_me_m(x)$. Hence, we deduce that a_{21} is necessarily such that the two following conditions are verified:

- (1) $c_{km} = 0$ if $k \neq m$.
- (2) $c_{mm} = d_{21}\lambda_m$.

The conclusion follows since such an a_{21} cannot be in \mathbf{H} in view of (1.12).

3.2 Simultaneous controllability of two equations with diagonal principal part

In what follows, we consider the case of two coupled equations with simultaneous control:

$$\begin{cases} \partial_t Y^1 = d_{11}\Delta Y^1 + \int_{\Omega} a_{11}(x, \xi)Y^1(t, \xi)d\xi \\ \quad + \int_{\Omega} a_{12}(x, \xi)Y^2(t, \xi)d\xi + 1_{\omega}u & \text{in } (0, T) \times \Omega, \\ \partial_t Y^2 = d_{22}\Delta Y^2 + \int_{\Omega} a_{21}(x, \xi)Y^1(t, \xi)d\xi \\ \quad + \int_{\Omega} a_{22}(x, \xi)Y^2(t, \xi)d\xi + 1_{\omega}u & \text{in } (0, T) \times \Omega, \\ Y^1(t, x) = Y^2(t, x) = 0 & \text{in } (0, T) \times \partial\Omega, \\ (Y^1(0), Y^2(0)) = (Y_0^1, Y_0^2) & \text{in } \Omega. \end{cases} \quad (3.3)$$

Here D is given by

$$D := \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix},$$

where $d_{11} > 0$ and $d_{22} > 0$. A is given by

$$A(x, \xi) := \begin{pmatrix} a_{11}(x, \xi) & a_{12}(x, \xi) \\ a_{21}(x, \xi) & a_{22}(x, \xi) \end{pmatrix},$$

where $a_{ij}(x) \in \mathbf{H}$ for $i, j = 1, 2$. We consider the control operator B given by

$$B := \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The control acts on some open subset $\omega \subset \Omega$.

We are going to prove the following sufficient condition for the controllability of (3.3).

Theorem 3.2 *Consider any $T > 0$ and assume that $a_{ij}(x) \in \mathbf{H}$ for $(i, j) \in \{1, 2\}^2$, $(i, j) \neq (2, 2)$ and $d_{11} \neq d_{22}$. Then, (3.3) is null-controllable if the following conditions (for instance) are verified:*

- (1) $a_{11} = a_{21}$.
- (2) a_{12} and a_{22} are symmetric in the variables (x, ξ) .

Proof Remark that the Kalman rank condition (1.8) is verified here since $d_{11} \neq d_{22}$ and each component of B is nonzero. Hence, we can apply Theorem 1.1 and we obtain that the null-controllability of (3.3) is equivalent to the following unique continuation property:

$$\begin{cases} -d_{11}\Delta\varphi^1 - \int_{\Omega} a_{11}(\xi, x)\varphi^1(\xi)d\xi - \int_{\Omega} a_{21}(\xi, x)\varphi^2(\xi)d\xi = \lambda\varphi^1 & \text{in } \Omega, \\ -d_{22}\Delta\varphi^2 - \int_{\Omega} a_{12}(\xi, x)\varphi^1(\xi)d\xi - \int_{\Omega} a_{22}(\xi, x)\varphi^2(\xi)d\xi = \lambda\varphi^2 & \text{in } (0, T) \times \Omega, \\ \varphi^1 + \varphi^2 = 0 & \text{in } \Omega, \\ \varphi^1 = \varphi^2 = 0 & \text{on } \partial\Omega, \\ \Rightarrow \varphi^1 = \varphi^2 = 0 & \text{in } \Omega. \end{cases} \quad (3.4)$$

Substituting φ^2 in the first two equations of (3.4) and using the hypothesis $a_{11} = a_{21}$, we obtain that (3.4) is equivalent to

$$\begin{cases} -d_{11}\Delta\varphi^1 = \lambda\varphi^1 & \text{in } \Omega, \\ -d_{22}\Delta\varphi^1 - \int_{\Omega} a_{12}(\xi, x)\varphi^1(\xi)d\xi + \int_{\Omega} a_{22}(\xi, x)\varphi^1(\xi)d\xi = \lambda\varphi^1 & \text{in } (0, T) \times \Omega, \\ \varphi^1(x) = 0 & \text{on } \partial\Omega, \\ \Rightarrow \varphi^1 = 0 & \text{in } \Omega. \end{cases} \quad (3.5)$$

From the first line of (3.5) we may assume that $\lambda > 0$ (since every eigenvalue of the Laplace operator with Dirichlet boundary conditions is positive). We multiply the first line of (3.5) by d_{22} and the second line of (3.5) by d_{11} , and we subtract the result. We obtain that

$$d_{11} \int_{\Omega} a_{12}(\xi, x)\varphi^1(\xi)d\xi - d_{11} \int_{\Omega} a_{22}(\xi, x)\varphi^1(t, \xi)d\xi = \lambda(d_{22} - d_{11})\varphi^1. \quad (3.6)$$

We apply the Laplace operator to this equation, we use the symmetry of the coefficients a_{12} , a_{22} and we perform some integrations by parts. We obtain that

$$d_{11} \int_{\Omega} a_{12}(\xi, x)\Delta\varphi^1(\xi)d\xi - d_{11} \int_{\Omega} a_{22}(\xi, x)\varphi^1(t, \xi)d\xi = \lambda(d_{22} - d_{11})\Delta\varphi^1.$$

Now, we replace $\Delta\varphi^1$ thanks to the first line of (3.5) and we obtain

$$-\lambda d_{11} \int_{\Omega} a_{12}(\xi, x)\varphi^1(\xi)d\xi + \lambda d_{11} \int_{\Omega} a_{22}(\xi, x)\varphi^1(t, \xi)d\xi = \lambda^2(d_{22} - d_{11})\varphi^1. \quad (3.7)$$

Multiplying (3.6) by λ and using (3.7) lead to $\varphi^1 = 0$ since $\lambda \neq 0$ and $d_{11} \neq d_{22}$, so that we also have $\varphi^2 = 0$ by the third line of (3.4).

References

- [1] Ammar-Khodja, F., Benabdallah, A., González-Burgos, M. and de Teresa, L., Recent results on the controllability of linear coupled parabolic problems: A survey, *Mathematical Control and Related Fields*, **1**(3), 2011, 267–306.
- [2] Ammar-Khodja, F., Benabdallah, A., González-Burgos, M. and de Teresa, L., The Kalman condition for the boundary controllability of coupled parabolic systems, bounds on biorthogonal families to complex matrix exponentials, *J. Math. Pures Appl.* (9), **96**(6), 2011, 555–590.
- [3] Ammar-Khodja, F., Benabdallah, A., González-Burgos, M. and de Teresa, L., Minimal time for the null controllability of parabolic systems: The effect of the condensation index of complex sequences, *J. Funct. Anal.*, **267**(7), 2014, 2077–2151.
- [4] Ammar Khodja, F., Benabdallah, A., González-Burgos, M. and de Teresa, L., New phenomena for the null controllability of parabolic systems: Minimal time and geometrical dependence, *J. Math. Anal. Appl.*, **444**(2), 2016, 1071–1113.
- [5] Benabdallah, A., Boyer, F., González-Burgos, M. and Olive, G., Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N -dimensional boundary null controllability in cylindrical domains, *SIAM J. Control Optim.*, **52**(5), 2014, 2970–3001.
- [6] Boyer, F. and Olive, G., Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients, *Math. Control Relat. Fields*, **4**(3), 2014, 263–287.
- [7] Ciarlet, P. G., *Linear and Nonlinear Functional Analysis with Applications*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.
- [8] Coron, J.-M., Guerrero, S. and Rosier, L., Null controllability of a parabolic system with a cubic coupling term, *SIAM Journal on Control and Optimization*, **48**(8), 2010, 5629–5653.
- [9] Coron, J.-M. and Guilleron, J.-P., Control of three heat equations coupled with two cubic nonlinearities, *SIAM J. Control Optim.*, **55**(2), 2016, 989–1019.
- [10] Duprez, M. and Lissy, P., Indirect controllability of some linear parabolic systems of m equations with $m - 1$ controls involving coupling terms of zero or first order, *J. Math. Pures Appl.* (9), **106**(5), 2016, 905–934.
- [11] Duprez, M. and Lissy, P., Positive and negative results on the internal controllability of parabolic equations coupled by zero and first order terms, *J. Evol. Equ.*, 2016, 1–22, DOI: 10.1007/s00028-017-0415-1.
- [12] Ervedoza, S. and Zuazua, E., Sharp observability estimates for heat equations, *Archive for Rational Mechanics and Analysis*, **202**, 2011, 975–1017.
- [13] Fattorini, H. O., Some remarks on complete controllability, *SIAM J. Control*, **4**(4), 1966, 686–694.
- [14] Fernández-Cara, E., González-Burgos, M. and de Teresa, L., Controllability of linear and semilinear non-diagonalizable parabolic systems, *ESAIM Control Optim. Calc. Var.*, **21**(4), 2015, 1178–1204.
- [15] Fernández-Cara, E., Lü, Q. and Zuazua, E., Null controllability of linear heat and wave equations with nonlocal spatial terms, *SIAM J. Control Optim.*, **54**(4), 2016, 2009–2019.
- [16] Fernández-Cara, E. and Zuazua, E., The cost of approximate controllability for heat equations: The linear case, *Adv. Differential Equations*, **5**(4–6), 2000, 465–514.
- [17] Ladyzenskaja, O. A., Solonnikov, V. A. and Ural’ceva, N. N., *Linear and quasilinear equations of parabolic type*, **23**, American Mathematical Society, Providence, R I., 1968.
- [18] Léautaud, M., Spectral inequalities for non-selfadjoint elliptic operators and application to the null-controllability of parabolic systems, *J. Funct. Anal.*, **258**(8), 2010, 2739–2778.
- [19] Lions, J.-L., Exact controllability, stabilization and perturbations for distributed systems, *SIAM Rev.*, **30**(1), 1988, 1–68.
- [20] Lorenzi, A., Two severely ill-posed linear parabolic problems, Alexandru Myller Mathematical Seminar, AIP Conf. Proc., **1329**, Amer. Inst. Phys., Melville, NY, 2011, 150–169.
- [21] Lissy, P. and Zuazua, E., Internal observability for coupled systems of linear partial differential equations, HAL, 2017, <https://hal.archives-ouvertes.fr/hal-01480301/document>.
- [22] Micu, S. and Takahashi, T., Local controllability to stationary trajectories of a one-dimensional simplified model arising in turbulence, HAL, 2017, <https://hal.archives-ouvertes.fr/hal-01572317>.
- [23] Miller, L., A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups, *Discrete Contin. Dyn. Syst. Ser. B*, **14**(4), 2010, 1465–1485.
- [24] Okubo, A. and Levin, S. A., *Diffusion and Ecological Problems: Modern Perspectives*, Interdisciplinary Applied Mathematics, **14**, Springer-Verlag, New York, 2001.

- [25] Russell, D. L., Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions, *SIAM Rev.*, **20**(4), 1978, 639–739.
- [26] Zuazua, E., Stable observation of additive superpositions of partial differential equations, *Systems Control Lett.*, **93**, 2016, 21–29.