Chin. Ann. Math. Ser. B
39(2), 2018, 297–314
DOI: 10.1007/s11401-018-1065-5

Chinese Annals of Mathematics, Series B

Poincaré's Lemma on Some Non-Euclidean Structures

Alexandru KRISTÁLY[1]

(*Dedicated to Philippe G. Ciarlet on the occasion of his* 80*th birthday*)

Abstract The author proves the Poincaré lemma on some $(n+1)$-dimensional corank 1 sub-Riemannian structures, formulating the $\frac{(n-1)n(n^2+3n-2)}{8}$ necessarily and sufficiently "curl-vanishing" compatibility conditions. In particular, this result solves partially an open problem formulated by Calin and Chang. The proof in this paper is based on a Poincaré lemma stated on Riemannian manifolds and a suitable Cesàro-Volterra path integral formula established in local coordinates. As a byproduct, a Saint-Venant lemma is also provided on generic Riemannian manifolds. Some examples are presented on the hyperbolic space and Carnot/Heisenberg groups.

Keywords Poincaré lemma, Cesàro-Volterra path integral, Sub-Riemannian manifolds
2000 MR Subject Classification 53C17

1 Introduction and Main Result

Let $\Omega \subseteq \mathbb{R}^n$ be an open, simply connected set, and $\boldsymbol{a} = (a_i) \in \boldsymbol{C}^1(\Omega; \mathbb{R}^n)$, $n \geq 2$. The classical Poincaré lemma says that there exists $u \in C^2(\Omega)$ with

$$\nabla u = \boldsymbol{a} \quad \text{in } \Omega,$$

if and only if curl $\boldsymbol{a} = \boldsymbol{0}$ in $\boldsymbol{C}(\Omega; \mathbb{R}^n)$, i.e.,

$$\partial_{x_i} a_j = \partial_{x_j} a_i \quad \text{in } C(\Omega) \text{ for every } i, j = 1, \cdots, n.$$

Here, as usual, $\nabla u = (\partial_{x_i} u) \in \boldsymbol{C}^1(\Omega; \mathbb{R}^n)$. For a weak version of the Poincaré lemma (e.g. in $L^2(\Omega)$) and its equivalent formulation in terms of fundamental results in the theory of PDEs, we refer the reader to Amrouche, Ciarlet and Mardare [3–4] and to the comprehensive monograph by Ciarlet [12, Chapter 6].

Very recently, Poincaré's lemma has been extended to some specific low-dimensional sub-Riemannian structures with rank 2 distributions; e.g., the first Heisenberg group $\mathbb{H}^1$, Engel-type manifolds, Grushin and Martinet type distributions, and the sub-Riemannian 3-dimensional sphere $\mathbb{S}^3$ (see Calin, Chang and Eastwood [6–7] and Calin, Chang and Hu [8–10]). In the sub-Riemannian setting, the number of equations in the system which is going to be solved is strictly

Manuscript received September 22, 2017. Revised October 10, 2017.
[1]Department of Economics, Babeş-Bolyai University, 400591 Cluj-Napoca; Romania & Institute of Applied Mathematics, Óbuda University, 1034 Budapest, Hungary.
E-mail: alexandrukristaly@yahoo.com kristaly.alexandru@nik.uni-obuda.hu

less than the space dimension. Accordingly, the solvability of such gradient-type systems deeply depends on the Lie bracket generating properties of the sub-Riemannian distributions, and it turns out that the "curl-vanishing" characterization of the solvability of the sub-Riemannian system becomes a system of PDEs containing higher-order derivatives. In order to visualize this phenomenon, we consider the first Heisenberg group $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$ endowed with its usual group operation and left-invariant vector fields $X_1 = \partial_{x_1} - 2x_2\partial_{x_3}$ and $X_2 = \partial_{x_2} + 2x_1\partial_{x_3}$. The sub-Riemannian system

$$X_1 u = a_1, \quad X_2 u = a_2 \tag{1.1}$$

is solvable in $\mathcal{F}(\mathbb{H}^1)$(= the space of smooth functions on $\mathbb{H}^1$) for $\boldsymbol{a} = (a_1, a_2) \in \boldsymbol{C}^1(\mathbb{H}^1; \mathbb{R}^2)$ if and only if

$$X_1^2 a_2 = (X_1X_2 + [X_1, X_2])a_1, \quad X_2^2 a_1 = (X_2X_1 + [X_2, X_1])a_2 \tag{1.2}$$

(see, e.g. Calin and Chang [5, Theorem 2.9.8]). In addition, the solution u of (1.1) can be given the work done by the force vector field $X = a_1X_1 + a_2X_2$ along any horizontal curve starting from $\mathbf{0} \in \mathbb{H}^1$, called also as the Cesàro-Volterra horizontal path integral.

The purpose of our paper is to prove Poincaré lemmas on some sub-Riemannian structures of arbitrary dimension with corank 1 distribution, including for instance step-two Carnot groups with not necessarily trivial kernel. In the sequel, we present our main result (see Section 3 for the notions used below).

Let $(M, \mathcal{D}, g)$ be an $(n+1)$-dimensional sub-Riemannian manifold ($n \geq 2$), and consider the distribution $\mathcal{D}$ in a given local coordinate system $(x_i)_{i=1,\cdots,n+1}$ containing vector fields of the form

$$X_i = \partial_{x_i} + A_i\partial_{x_{n+1}}, \quad i = 1, \cdots, n, \tag{1.3}$$

where $A_i : M \to \mathbb{R}$ are smooth functions depending only on the first n variables, i.e., $A_i = A_i(x_1, \cdots, x_n)$. We assume that

$$\partial_{x_i}A_j - \partial_{x_j}A_i = c_{ij} \in \mathbb{R} \quad \text{for every } i, j = 1, \cdots, n \tag{1.4}$$

and

$$I_0 = \{(i, j) : c_{ij} \neq 0\} \neq \emptyset.$$

Due to the latter assumptions, the rank n distribution $\mathcal{D}$ is nonholonomic on M, since

$$[X_i, X_j] = c_{ij}\partial_{x_{n+1}} \quad \text{for every } i, j = 1, \cdots, n. \tag{1.5}$$

Given $\boldsymbol{a} \in \Gamma(\mathcal{D})$(= the set of horizontal vector fields on M), we are going to study the solvability of the system

$$\nabla_H u = \boldsymbol{a} \quad \text{in } M, \tag{1.6}$$

where $u \in \mathcal{F}(M)$ and ∇_H denotes the horizontal gradient. Our main result, the Poincaré lemma on sub-Riemannian manifolds, reads as follows.

Theorem 1.1 *Let $(M, \mathcal{D}, g)$ be an $(n+1)$-dimensional simply connected sub-Riemannian manifold ($n \geq 2$), where the distribution $\mathcal{D}$ is given by the vector fields in* (1.3) *with functions A_i depending only on the first n variables, verifying* (1.4) *and $I_0 \neq \emptyset$.*

Given $\boldsymbol{a} \in \Gamma(\mathcal{D})$, the sub-Riemannian system (1.6) *has a solution $u \in \mathcal{F}(M)$ if and only if*

$$\begin{cases} c_{kl}(X_i\widetilde{a}_j - X_j\widetilde{a}_i) = c_{ij}(X_k\widetilde{a}_l - X_l\widetilde{a}_k) & \text{for every } \; i,j,k,l = 1,\cdots,n, & (1.7) \\ X_kX_i\widetilde{a}_j - X_kX_j\widetilde{a}_i = [X_i, X_j]\widetilde{a}_k & \text{for every } \; i,j,k = 1,\cdots,n, & (1.8) \end{cases}$$

where $\boldsymbol{a} = a_iX_i$ and $\widetilde{a}_j = g_{ij}a_i$ (the summations being from 1 *to* n*), and (g_{ij}) are the components of g with respect to the distribution $\mathcal{D}$. Moreover, if $x_0 \in M$, the solution $u : M \to \mathbb{R}$ for the system* (1.6) *can be obtained by*

$$u(x) = c_0 + \int_0^1 g(\boldsymbol{a}(\gamma(t)), \dot{\gamma}(t))\mathrm{d}t, \quad x \in M, \tag{1.9}$$

where $c_0 = u(x_0) \in \mathbb{R}$ and $\gamma : [0,1] \to M$ is any horizontal curve joining x_0 with x.

Some remarks are in order.

Remark 1.1 (a) Although (1.7) and (1.8) contain n^4 and n^3 conditions, a simple combinatorial reasoning shows that it is enough to verify at most

$$s_n = \frac{(n-2)(n-1)n(n+1)}{8}$$

and

$$s'_n = \frac{(n-1)n^2}{2}$$

conditions, respectively. Thus, the number of compatibility conditions is

$$s_n + s'_n = \frac{(n-1)n(n^2+3n-2)}{8}.$$

(b) Theorem 1.1 provides an answer to the open question of Calin and Chang [5, p. 55] whenever the sub-Riemannian manifold with arbitrarily dimension has corank 1 distribution. We note that the existing results in the literature solve the system (1.6) only for two components, i.e., the distributions contain two vector fields. In particular, if $M = \mathbb{H}^1$ is the first Heisenberg group, the solvability of the system (1.1) can be recovered by Theorem 1.1; indeed, in this particular case, $n = 2$, $\mathcal{D} = \{X_1, X_2\}$ and $g_{ij} = \delta_{ij}$. Moreover, $A_1 = -2x_2$, $A_2 = 2x_1$; thus $c_{12} = -c_{21} = 4$ and $c_{11} = c_{22} = 0$ in (1.4). Notice that the first-ordered relations in (1.7) are trivially satisfied (supported also by the fact that $s_2 = 0$, thus nothing should be checked), while the second-ordered ones (1.8) reduce precisely to (1.2), containing $s'_2 = 2$ conditions. In higher-dimensional Heisenberg groups $\mathbb{H}^d$, $d \geq 2$, the first-ordered assumptions are indispensable as well.

(c) There are more involved, non-Heisenberg-type vector fields which verify also the assumptions of Theorem 1.1. Indeed, let $(\mathbb{R}^5, \mathcal{D}, g)$ be the sub-Riemannian manifold with the vector fields X_i, $i = 1, \cdots, 4$ from (1.3) with $A_1 = -2x_2 + x_1x_4^2$, $A_2 = 2x_1$, $A_3 = -x_4$, $A_4 = x_3 + x_1^2x_4$.

In this case we have that the elements from (1.4) are $c_{12} = 4 = -c_{21}$, $c_{34} = 2 = -c_{43}$, while the rest of c_{ij}'s are zero.

(d) Note that Theorem 1.1 can be formulated on any simply connected open domain instead of the whole M.

Organization of the paper In Section 2 we prove the Poincaré lemma on generic Riemannian manifolds. As a direct byproduct, we also state a Saint-Venant lemma on Riemannian manifolds whose proof is presented in the Appendix (Section 6). The Poincaré lemma on generic Riemannian manifolds turns to be indispensable in the proof of our main theorem, which will be provided in Section 3. Here, we shall explore basic properties of the Riemannian manifolds as the metric compatibility and torsion-freeness (or symmetry) of the Levi-Civita connection with respect to the Riemannian metric. In fact, we shall reduce our original sub-Riemannian system (defined on the distribution) to a differential system on a Riemannian manifold where we can apply the Riemannian Poincaré lemma and Cesàro-Volterra integral formula. An elegant computation connects the force vector fields in these two settings, proving in this way relation (1.9). In Section 4 we give some examples, the first on the hyperbolic spaces, the second one on Carnot/Heisenberg groups. In Section 5 we formulate some problems for further investigations.

2 Poincaré Lemma on Riemannian Manifolds: A Local Version

Let (M, g) be an m-dimensional Riemannian manifold; here (g_{ij}) are the components of the Riemannian metric g in a given local coordinate system $(x_i)_{i=1,\cdots,m}$.

Let $u : M \to \mathbb{R}$ be a C^1-functional on M; the differential of u at x, denoted by $\mathrm{d}u(x)$, belongs to the cotangent space T_x^*M and is defined by

$$\mathrm{d}u(x)(v) = \langle \nabla_g u(x), v\rangle_g \quad \text{for all } v \in T_xM; \tag{2.1}$$

in the sequel, we prefer to use $\langle \cdot, \cdot \rangle_g$ instead of g. If the local components of $\mathrm{d}u$ are denoted by $u_k = \partial_{x_k} u$, then the local components of $\nabla_g u$ are $u^i = g^{ik}u_k$; here, g^{ij} are the local components of $g^{-1} = (g_{ij})^{-1}$.

Let $\Omega \subseteq M$ be an open set and $\boldsymbol{V} \in T\Omega = \bigcup_{x\in\Omega} T_xM$ be an arbitrary vector field in Ω which is represented in local coordinates as

$$\boldsymbol{V} = V_k \partial_{x_k}.$$

The main result of the present section is the Poincaré lemma on Riemannian manifolds.

Theorem 2.1 *Let (M, g) be an m-dimensional Riemannian manifold and $\Omega \subseteq M$ be a simply connected open set. Given a vector field $\boldsymbol{V} \in \boldsymbol{C}^1(\Omega, T\Omega)$, the system*

$$\nabla_g u = \boldsymbol{V} \quad in\ \Omega \tag{2.2}$$

is solvable in $C^2(\Omega)$ if and only if we have

$$\partial_{x_i} \widetilde{V}_j = \partial_{x_j} \widetilde{V}_i \quad in\ \Omega\ for\ every\ i, j = 1, \cdots, m, \tag{2.3}$$

where $\widetilde{V}_j = g_{jk}V_k$.

Moreover, if $x_0 \in \Omega$ is fixed and (2.3) *holds, the solution $u : \Omega \to \mathbb{R}$ for* (2.2) *can be obtained by*

$$u(x) = c_0 + \int_0^1 \langle \boldsymbol{V}(\gamma(t)), \dot{\gamma}(t)\rangle_g \mathrm{d}t, \quad x \in \Omega, \tag{2.4}$$

where $c_0 = u(x_0) \in \mathbb{R}$ and $\gamma : [0,1] \to \Omega$ is any curve joining x_0 with x.

Proof (2.2)⇒(2.3). First of all, (2.2) is equivalent to

$$g^{ik}\partial_{x_k} u = V_i, \quad i = 1, \cdots, m.$$

Multiplying both sides by g_{ji}, we have

$$\partial_{x_j} u = g_{ji} V_i = \widetilde{V}_j, \quad j = 1, \cdots, m.$$

Deriving these relations, (2.3) yields at once by the symmetry of second-order derivatives.

(2.3)⇒(2.2). We closely follow the proof from Ciarlet [12, Theorem 6.17-2]. Let $x_0 \in \Omega$ be given and fix $x \in \Omega$. Since Ω is simply connected, there exists a path $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x_0$ and $\gamma(1) = x$. If there exists $u \in C^2(\Omega)$ which satisfies (2.2), then the function $P : [0,1] \to \mathbb{R}$ defined by $P(t) = u(\gamma(t))$ verifies

$$\frac{\mathrm{d}P}{\mathrm{d}t}(t) = \mathrm{d}u(\gamma(t))(\dot{\gamma}(t)) = \langle \nabla_g u(\gamma(t)), \dot{\gamma}(t)\rangle_g, \quad t \in [0,1].$$

The latter equation together with the Cauchy data $P(0) = P_0 \in \mathbb{R}$ provides a unique solution $P : [0,1] \to \mathbb{R}$ which depends on the path γ.

We are going to show that the value $P(1)$ does not depend on the choice of the path γ whenever (2.3) holds. To see this, let $\gamma_0, \gamma_1 : [0,1] \to \Omega$ be two smooth paths such that $\gamma_i(0) = x_0$ and $\gamma_i(1) = x$, $i \in \{0,1\}$. Since Ω is simply connected, we can find a smooth homotopy $H : [0,1] \times [0,1] \to \Omega$ between γ_0 and γ_1, i.e.,

$$\begin{aligned} &H(\cdot, 0) = \gamma_0, \quad H(\cdot, 1) = \gamma_1, \\ &H(0, \lambda) = x_0, \quad H(1, \lambda) = x, \quad \forall \lambda \in [0,1]. \end{aligned}$$

For every $\lambda \in [0,1]$, let $P(\cdot, \lambda) : [0,1] \to \mathbb{R}$ be the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial P}{\partial t}(t,\lambda) = \Big\langle \boldsymbol{V}(H(t,\lambda)), \dfrac{\partial H}{\partial t}(t,\lambda)\Big\rangle_g \quad \text{for } t \in [0,1], \\ P(0,\lambda) = P_0 \in \mathbb{R}. \end{cases} \tag{$\mathcal{C}_\lambda$}$$

We claim that

$$\frac{\partial P}{\partial \lambda}(1,\lambda) = 0 \quad \text{for every } \lambda \in [0,1]. \tag{2.5}$$

To see this, let us consider the function $\sigma : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$\sigma(t,\lambda) = \frac{\partial P}{\partial \lambda}(t,\lambda) - \Big\langle \boldsymbol{V}(H(t,\lambda)), \frac{\partial H}{\partial \lambda}(t,\lambda)\Big\rangle_g.$$

Since the Levi-Civita connection is compatible with the Riemannian metric, it follows from [16, Proposition 3.2] that

$$\frac{\partial\sigma}{\partial t}(t,\lambda)=\frac{\partial}{\partial t}\Big(\frac{\partial P}{\partial\lambda}\Big)(t,\lambda)-\Big\langle\frac{D\boldsymbol{V}}{\partial t}(H(t,\lambda)),\frac{\partial H}{\partial\lambda}(t,\lambda)\Big\rangle_g-\Big\langle\boldsymbol{V}(H(t,\lambda)),\frac{D}{\partial t}\frac{\partial H}{\partial\lambda}(t,\lambda)\Big\rangle_g,$$

where D denotes the covariant derivation on (M,g). Concerning the latter term, we know from the torsion-freeness of the Levi-Civita connection on (M,g) that

$$\frac{D}{\partial t}\frac{\partial H}{\partial\lambda}(t,\lambda)=\frac{D}{\partial\lambda}\frac{\partial H}{\partial t}(t,\lambda), \tag{2.6}$$

(see [16, Lemma 3.4]). The sophisticated part is to show that

$$\Big\langle\frac{D\boldsymbol{V}}{\partial t}(H(t,\lambda)),\frac{\partial H}{\partial\lambda}(t,\lambda)\Big\rangle_g=\Big\langle\frac{D\boldsymbol{V}}{\partial\lambda}(H(t,\lambda)),\frac{\partial H}{\partial t}(t,\lambda)\Big\rangle_g. \tag{2.7}$$

To prove (2.7) we recall the following well-known facts: If $\boldsymbol{W}=(w_1,\cdots,w_m)$ is a vector field along a path (x), its covariant derivative can be expressed by

$$\frac{D\boldsymbol{W}}{\mathrm{d}t}=\Big(\frac{\mathrm{d}w_k}{\mathrm{d}t}+\Gamma_{ij}^k w_j\frac{\mathrm{d}x_i}{\mathrm{d}t}\Big)\partial_{x_k},$$

where Γ_{ij}^k are the Christofel symbols for which we have

$$g_{ks}\Gamma_{ij}^k=\frac{1}{2}(\partial_{x_i}g_{js}+\partial_{x_j}g_{is}-\partial_{x_s}g_{ij}). \tag{2.8}$$

Coming back to (2.7), we have

$$\begin{aligned}LHS:=\Big\langle\frac{D\boldsymbol{V}}{\partial t}(H(t,\lambda)),\frac{\partial H}{\partial\lambda}(t,\lambda)\Big\rangle_g&=g_{kj}\Big(\partial_{x_i}V_k\frac{\partial H_i}{\partial t}+\Gamma_{il}^kV_l\frac{\partial H_i}{\partial t}\Big)\frac{\partial H_j}{\partial\lambda}\\&=g_{kj}(\partial_{x_i}V_k+\Gamma_{il}^kV_l)\frac{\partial H_i}{\partial t}\frac{\partial H_j}{\partial\lambda}.\end{aligned}$$

In a similar way,

$$\begin{aligned}RHS&:=\Big\langle\frac{D\boldsymbol{V}}{\partial\lambda}(H(t,\lambda)),\frac{\partial H}{\partial t}(t,\lambda)\Big\rangle_g\\&=g_{ki}\Big(\partial_{x_j}V_k\frac{\partial H_j}{\partial\lambda}+\Gamma_{jl}^kV_l\frac{\partial H_j}{\partial\lambda}\Big)\frac{\partial H_i}{\partial t}\\&=g_{ki}(\partial_{x_j}V_k+\Gamma_{jl}^kV_l)\frac{\partial H_i}{\partial t}\frac{\partial H_j}{\partial\lambda}.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}(2.7)\text{ holds}&\Leftrightarrow LHS-RHS=0\\&\Leftrightarrow[g_{kj}(\partial_{x_i}V_k+\Gamma_{il}^kV_l)-g_{ki}(\partial_{x_j}V_k+\Gamma_{jl}^kV_l)]\frac{\partial H_i}{\partial t}\frac{\partial H_j}{\partial\lambda}=0\\&\Leftrightarrow[g_{kj}\partial_{x_i}V_k-g_{ki}\partial_{x_j}V_k+(g_{kj}\Gamma_{il}^k-g_{ki}\Gamma_{jl}^k)V_l]\frac{\partial H_i}{\partial t}\frac{\partial H_j}{\partial\lambda}=0\\&\overset{(2.8)}{\Leftrightarrow}[g_{kj}\partial_{x_i}V_k-g_{ki}\partial_{x_j}V_k+(\partial_{x_i}g_{lj}-\partial_{x_j}g_{li})V_l]\frac{\partial H_i}{\partial t}\frac{\partial H_j}{\partial\lambda}=0\\&\Leftrightarrow[g_{kj}\partial_{x_i}V_k-g_{ki}\partial_{x_j}V_k+(\partial_{x_i}g_{kj}-\partial_{x_ij}g_{ki})V_k]\frac{\partial H_i}{\partial t}\frac{\partial H_j}{\partial\lambda}=0\\&\Leftrightarrow[\partial_{x_i}(g_{jk}V_k)-\partial_{x_j}(g_{ik}V_k)]\frac{\partial H_i}{\partial t}\frac{\partial H_j}{\partial\lambda}=0,\end{aligned}$$

where the latter relation holds true due to (2.3). Consequently, by relations (2.6)–(2.7) and the Cauchy problem $(\mathcal{C}_\lambda)$ we have

$$\begin{aligned}\frac{\partial\sigma}{\partial t}(t,\lambda) &= \frac{\partial}{\partial\lambda}\Big(\frac{\partial P}{\partial t}\Big)(t,\lambda) - \Big\langle \frac{D\boldsymbol{V}}{\partial\lambda}(H(t,\lambda)), \frac{\partial H}{\partial t}(t,\lambda)\Big\rangle_g - \Big\langle \boldsymbol{V}(H(t,\lambda)), \frac{D}{\partial\lambda}\frac{\partial H}{\partial t}(t,\lambda)\Big\rangle_g \\ &= \frac{\partial}{\partial\lambda}\Big(\frac{\partial P}{\partial t}(t,\lambda) - \Big\langle \boldsymbol{V}(H(t,\lambda)), \frac{\partial H}{\partial t}(t,\lambda)\Big\rangle_g\Big), \\ &= 0,\end{aligned}$$

i.e., $t \mapsto \sigma(t,\lambda)$ is constant. Since $P(0,\lambda) = P_0 \in \mathbb{R}$ and $H(0,\lambda) = x_0$, it turns out that

$$\sigma(0,\lambda) = \frac{\partial P}{\partial\lambda}(0,\lambda) - \Big\langle \boldsymbol{V}(H(0,\lambda)), \frac{\partial H}{\partial\lambda}(0,\lambda)\Big\rangle_g = 0 \quad \text{for every } \lambda \in [0,1].$$

In particular,

$$0 = \sigma(1,\lambda) = \frac{\partial P}{\partial\lambda}(1,\lambda) - \Big\langle \boldsymbol{V}(H(1,\lambda)), \frac{\partial H}{\partial\lambda}(1,\lambda)\Big\rangle_g.$$

Since $H(1,\lambda) = x_0$ for every $\lambda \in [0,1]$, it follows the claim (2.5), showing that the value $P(1)$ is not depending on the particular choice of the path.

For every $x \in \Omega$, let $u : \Omega \to \mathbb{R}$ be defined by

$$u(x) = P(1),$$

where P is the unique solution to the Cauchy problem $(\mathcal{C}_\lambda)$ having the initial data $P(0) = P_0$ and using any path joining x_0 and x; thus, the function u is well-defined.

To conclude the proof, we show the validity of (2.2). Let $x \in \Omega$ and $v \in T_xM$ be arbitrarily fixed elements. Let $\gamma : [0,1] \to \Omega$ be a path such that $\gamma(0) = x_0$, $\gamma(1) = x$ and $\dot\gamma(1) = v \in T_xM$, and let P be the solution of the Cauchy problem associated to this path, thus, $P(t) = u(\gamma(t))$. Therefore, the latter relation yields that

$$\frac{\mathrm{d}P}{\mathrm{d}t}(t) = \langle \nabla_g u(\gamma(t)), \dot\gamma(t)\rangle_g, \quad t \in [0,1].$$

On the other hand, by the Cauchy problem we have

$$\frac{\mathrm{d}P}{\mathrm{d}t}(t) = \langle \boldsymbol{V}(\gamma(t)), \dot\gamma(t)\rangle_g, \quad t \in [0,1].$$

Accordingly, for the moment $t = 1$, it follows that

$$\langle \nabla_g u(x), v\rangle_g = \langle \boldsymbol{V}(x), v\rangle_g$$

and the arbitrariness of $v \in T_xM$ concludes the proof of (2.2).

If $\gamma : [0,1] \to \Omega$ is any path joining the points x_0 and x, the Cesàro-Volterra path integral formula easily follows as

$$u(x) - u(x_0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t))\mathrm{d}t = \int_0^1 \langle \nabla_g u(\gamma(t)), \dot\gamma(t)\rangle_g \mathrm{d}t = \int_0^1 \langle \boldsymbol{V}(\gamma(t)), \dot\gamma(t)\rangle_g \mathrm{d}t,$$

which is precisely (2.4).

Remark 2.1 Poincaré's lemma can be also proved by using 1-forms, see, e.g. Abraham, Marsden and Ratiu [1]. However, we preferred here a direct proof based on local coordinates for two reasons: (a) It highlights the importance of the Riemannian structure, i.e., the metric compatibility and torsion-freeness of the Levi-Civita connection, which is not valid anymore on non-Riemannian Finsler settings (see Section 5 for details); (b) The proof provides directly a Cesàro-Volterra path integral formula.

As a byproduct of the Poincaré lemma (Theorem 2.1), we state a Saint-Venant lemma on generic Riemannian manifolds; its proof is sketched in the Appendix. To present it, fix $\boldsymbol{e}_i \in T\Omega$, $i = 1, \cdots, m$, and assume that they can be represented as

$$\boldsymbol{e}_i = e_{ik}\partial_{x_k}.$$

The m-vector field $\boldsymbol{e} = (\boldsymbol{e}_1, \cdots, \boldsymbol{e}_m) \in \boldsymbol{C}^2(\Omega, T\Omega^m)$ is called symmetric if $e_{ij} = e_{ji} \in C^2(\Omega)$ for every $i, j = 1, \cdots, m$.

Proposition 2.1 *Let (M, g) be an m-dimensional Riemannian manifold and $\Omega \subseteq M$ be a simply connected open set. Given $\boldsymbol{e} = (\boldsymbol{e}_1, \cdots, \boldsymbol{e}_m) \in \boldsymbol{C}^2(\Omega, T\Omega^m)$ a symmetric m-vector field on Ω, the system*

$$\nabla_{s,g} \boldsymbol{V} = \boldsymbol{e} \quad \textit{in } \Omega, \tag{2.9}$$

has a vector field solution $\boldsymbol{V} = (V_1, \cdots, V_m) \in \boldsymbol{C}^3(\Omega, \mathbb{R}^m)$, where the components of the symmetric gradient $\nabla_{s,g} \boldsymbol{V}$ are given by

$$\frac{1}{2}(\partial_{x_i}(g_{jk}V_k) + \partial_{x_j}(g_{ik}V_k)), \quad i, j = 1, \cdots, m,$$

if and only if the Saint-Venant compatibility relations hold (in local coordinate system) in Ω, i.e.,

$$\partial^2_{x_l x_j} e_{ik} + \partial^2_{x_k x_i} e_{jl} - \partial^2_{x_l x_i} e_{jk} - \partial^2_{x_j x_k} e_{il} = 0, \quad i, j, k, l = 1, \cdots, m. \tag{2.10}$$

Moreover, if $x_0 \in M$ is fixed and (2.10) holds, then the solution of (2.9) is obtained by

$$V_k = g^{ks} u_s, \quad k = 1, \cdots, m,$$

where

$$u_i(x) = c_0^i + \int_0^1 \langle \boldsymbol{U}_i(\gamma(t)), \dot{\gamma}(t) \rangle_g \mathrm{d}t, \quad x \in \Omega$$

with $\boldsymbol{U}_i = g^{ls}(p_{is} + e_{is})\partial_{x_l}$,

$$p_{ij}(x) = c_0^{ij} + \int_0^1 \langle \boldsymbol{W}_{ij}(\gamma(t)), \dot{\gamma}(t) \rangle_g \mathrm{d}t, \quad x \in \Omega$$

and $\boldsymbol{W}_{ij} = g^{ls}(\partial_{x_j} e_{is} - \partial_{x_i} e_{js})\partial_{x_l}$ for some numbers c_0^s, c_0^{ij}, and the curve $\gamma : [0, 1] \to \Omega$ is arbitrary fixed joining x_0 with $x \in \Omega$.

Remark 2.2 (a) Note that $\nabla_{s,g}\boldsymbol{V}$ is a kind of *symmetric Lie derivative* of the vector field $\boldsymbol{V}$ with respect to the Riemannian metric g; indeed, the latter notion appears in [11, p. 518], where $\nabla_{s,g}\boldsymbol{V}$ is an $\mathcal{L}-$type tensor of the form

$$\nabla_{s,g}\boldsymbol{V} = \frac{1}{2}(g_{jk}\partial_{x_i}V_k + g_{ik}\partial_{x_j}V_k + C_{ijk}V_k)\mathrm{d}x_i \otimes \mathrm{d}x_j.$$

In our setting, the elements C_{ijk} are expressed by means of the Christoffel symbols as

$$C_{ijk} = \partial_{x_i}g_{jk} + \partial_{x_j}g_{ik} = g_{lj}\Gamma^l_{ki} + g_{li}\Gamma^l_{kj} + 2g_{lk}\Gamma^l_{ij}.$$

(b) Proposition 2.1 provides a curved version of the Saint-Venant lemma; further curvilinear versions of the Saint-Venant lemma can be found in the papers by Ciarlet, Gratie, Mardare and Shen [13], Ciarlet and Mardare [14], and Ciarlet, Mardare and Shen [15].

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we first recall some basic notions from the theory of sub-Riemannian manifolds; for further details, see Agrachev, Barilari and Boscain [2], Calin and Chang [5] and Figalli and Rifford [17].

Let M be a smooth connected $(n+1)$-dimensional manifold ($n \geq 2$), $\mathcal{D}$ be a smooth nonholonomic distribution of rank $m \leq n$ on M (i.e., a rank m subbundle of the tangent bundle TM) and g be a Riemannian metric on $\mathcal{D}$. Without loss of generality, we may assume that g is defined on the whole tangent bundle TM (not necessarily in a unique way); we shall keep the same notation of g on TM. The triplet $(M, \mathcal{D}, g)$ is a sub-Riemannian manifold. As usual, the distribution $\mathcal{D}$ is said to be nonholonomic if for every $x \in M$ there exists an m-tuple $X^x_1, \cdots, X^x_m$ of smooth vector fields on a neighborhood N_x of x such that all the Lie brackets generated by these vectors at y generate T_yM for every $y \in N_x$. A curve $\gamma : [0, 1] \to M$ is horizontal with respect to $\mathcal{D}$ if it belongs to $W^{1,2}([0, 1]; M)$ and $\dot\gamma(t) \in \mathcal{D}(\gamma(t))$ for a.e. $t \in [0, 1]$. If $\mathcal{D}$ is nonholonomic on M, by the Chow-Rashewsky theorem, every two points of M can be joined by a horizontal path. Let $\Gamma(\mathcal{D})$ be the set of horizontal vector fields on M, and $\mathcal{F}(M)$ be the set of smooth functions on M. If $u \in \mathcal{F}(M)$, the horizontal gradient $\nabla_H u \in \Gamma(\mathcal{D})$ of u is defined by $g(\nabla_H u, X) = X(u)$ for every $X \in \Gamma(\mathcal{D})$.

Now, let us put ourselves into the context of Theorem 1.1. Accordingly, let $(M, \mathcal{D}, g)$ be an $(n+1)$-dimensional sub-Riemannian manifold ($n \geq 2$), and the rank n distribution $\mathcal{D}$ in a local coordinate system $(x_i)_{i=1,\cdots,n+1}$ formed by the vector fields given in (1.3) and verifying (1.4). Since

$$\begin{aligned} X_iX_j &= (\partial_{x_i} + A_i\partial_{x_{n+1}})(\partial_{x_j} + A_j\partial_{x_{n+1}}) \\ &= \partial^2_{x_ix_j} + \partial_{x_i}A_j\partial_{x_{n+1}} + A_j\partial^2_{x_ix_{n+1}} + A_i\partial^2_{x_jx_{n+1}} + A_iA_j\partial^2_{x_{n+1}}, \end{aligned}$$

by (1.4) we obtain (1.5), i.e.,

$$[X_i, X_j] = X_iX_j - X_jX_i = (\partial_{x_i}A_j - \partial_{x_i}A_j)\partial_{x_{n+1}} = c_{ij}\partial_{x_{n+1}} \quad \text{for every } i, j = 1, \cdots, n.$$

Therefore, since $I_0 = \{(i, j) : c_{ij} \neq 0\} \neq \emptyset$, the distribution $\mathcal{D}$ is nonholonomic on M.

Let $\boldsymbol{a} \in \Gamma(\mathcal{D})$ be fixed. The system (1.6), i.e.,

$$\nabla_H u = \boldsymbol{a},$$

in local coordinates reads as

$$X_j(u) = g_{ij} a_i =: \widetilde{a}_j, \quad j = 1, \cdots, n, \tag{3.1}$$

where $g_{ij} = g(X_i, X_j)$ and $\boldsymbol{a} = a_i X_i$. With this preparatory part in our mind, we now present the proof of our main result.

Proof of Theorem 1.1 (1.6)⇒(1.7)–(1.8). Assume that the sub-Riemannian system (1.6) has a solution $u \in \mathcal{F}(M)$. First, by (1.5) applied to u, we have

$$[X_i, X_j]u = c_{ij} \partial_{x_{n+1}} u, \quad i, j = 1, \cdots, n.$$

This relation and (3.1) give that

$$X_i \widetilde{a}_j - X_j \widetilde{a}_i = c_{ij} \partial_{x_{n+1}} u, \quad i, j = 1, \cdots, n. \tag{3.2}$$

If $\partial_{x_{n+1}} u(x) = 0$ for some $x \in M$, then $X_i \widetilde{a}_j(x) - X_j \widetilde{a}_i(x) = 0$ for every $i, j = 1, \cdots, n$, thus (1.7) clearly holds. If $\partial_{x_{n+1}} u(x) \neq 0$ for some $x \in M$, then by writing the relation (3.2) for (k, l) instead of (i, j), and eliminating $\partial_{x_{n+1}} u(x) \neq 0$, we obtain (1.7).

Deriving (3.2) with respect to the vector field X_k, $k = 1, \cdots, n$, and taking into account that $[X_k, \partial_{x_{n+1}}] = X_k \partial_{x_{n+1}} - \partial_{x_{n+1}} X_k = 0$, it turns out by (3.1) and (1.5) that

$$X_k X_i \widetilde{a}_j - X_k X_j \widetilde{a}_i = c_{ij} X_k \partial_{x_{n+1}} u = c_{ij} \partial_{x_{n+1}} X_k u = [X_i, X_j] \widetilde{a}_k,$$

which is precisely relation (1.8).

(1.7)–(1.8)⇒(1.6). Since $I_0 \neq \emptyset$, let $(i_0, j_0) \in I_0$ and introduce the function

$$\widetilde{a} = \frac{X_{i_0} \widetilde{a}_{j_0} - X_{j_0} \widetilde{a}_{i_0}}{c_{i_0 j_0}},$$

where $\widetilde{a}_j = g_{ij} a_i$. With these notations, we consider the system

$$\begin{cases} \partial_{x_j} u = \widetilde{a}_j - A_j \widetilde{a} \quad \text{for } j = 1, \cdots, n, \\ \partial_{x_{n+1}} u = \widetilde{a}. \end{cases} \tag{3.3}$$

Let

$$\widetilde{V}_j = \widetilde{a}_j - A_j \widetilde{a}, \quad j = 1, \cdots, n \quad \text{and} \quad \widetilde{V}_{n+1} = \widetilde{a}.$$

We are going to prove that

$$\partial_{x_i} \widetilde{V}_j = \partial_{x_j} \widetilde{V}_i, \quad i, j = 1, \cdots, n+1. \tag{3.4}$$

To do this, we distinguish three cases:

Case 1 $i = j = n + 1$. (3.4) trivially holds.

Case 2 $i \in \{1, \cdots, n\}$ and $j = n+1$. On one hand, (3.4) is equivalent to $\partial_{x_i}\widetilde{a} = \partial_{x_{n+1}}(\widetilde{a}_i - A_i\widetilde{a})$, which can be written as $X_i\widetilde{a} = \partial_{x_{n+1}}\widetilde{a}_i$. On the other hand, by the definition of $\widetilde{a}$, (1.8) and (1.5) we have that

$$X_i\widetilde{a} = \frac{X_iX_{i_0}\widetilde{a}_{j_0} - X_iX_{j_0}\widetilde{a}_{i_0}}{c_{i_0j_0}} = \frac{[X_{i_0}, X_{j_0}]\widetilde{a}_i}{c_{i_0j_0}} = \partial_{x_{n+1}}\widetilde{a}_i,$$

which is the required relation.

Case 3 $i, j \in \{1, \cdots, n\}$. We have the following chain of equivalences:

$$\begin{aligned}
(3.4) \text{ holds} &\Leftrightarrow \partial_{x_i}\widetilde{a}_j - \widetilde{a}\partial_{x_i}A_j - A_j\partial_{x_i}\widetilde{a} = \partial_{x_j}\widetilde{a}_i - \widetilde{a}\partial_{x_j}A_i - A_i\partial_{x_j}\widetilde{a} \\
&\Leftrightarrow \partial_{x_i}\widetilde{a}_j - \widetilde{a}\partial_{x_i}A_j - A_jX_i\widetilde{a} = \partial_{x_j}\widetilde{a}_i - \widetilde{a}\partial_{x_j}A_i - A_iX_j\widetilde{a} \\
&\Leftrightarrow \partial_{x_i}\widetilde{a}_j - A_jX_i\widetilde{a} = \partial_{x_j}\widetilde{a}_i + c_{ij}\widetilde{a} - A_iX_j\widetilde{a} \\
&\Leftrightarrow \partial_{x_i}\widetilde{a}_j - A_j\frac{[X_{i_0}, X_{j_0}]\widetilde{a}_i}{c_{i_0j_0}} = \partial_{x_j}\widetilde{a}_i + c_{ij}\widetilde{a} - A_i\frac{[X_{i_0}, X_{j_0}]\widetilde{a}_j}{c_{i_0j_0}} \qquad \text{(by (1.8))} \\
&\Leftrightarrow \partial_{x_i}\widetilde{a}_j - A_j\partial_{x_{n+1}}\widetilde{a}_i = \partial_{x_j}\widetilde{a}_i + c_{ij}\widetilde{a} - A_i\partial_{x_{n+1}}\widetilde{a}_j \qquad \text{(by (1.5))} \\
&\Leftrightarrow X_i\widetilde{a}_j - X_j\widetilde{a}_i = c_{ij}\widetilde{a}.
\end{aligned}$$

By the definition of $\widetilde{a}$, let us observe that the latter relation is nothing but (1.7) with the choice $(k, l) = (i_0, j_0)$, which concludes the proof of (3.4).

According to Theorem 2.1 (applied for $(M, \widetilde{g})$ with $\widetilde{g}_{ij} = g(\partial_{x_i}, \partial_{x_j})$, $i, j = 1, \cdots, n+1$) and relation (3.4), it turns out that the system (3.3) has a solution in $C^2(M)$, which can be obtained by

$$u(x) = c_0 + \int_0^1 \langle \boldsymbol{V}(\gamma(t)), \dot{\gamma}(t)\rangle_{\widetilde{g}}\mathrm{d}t, \quad x \in M, \tag{3.5}$$

where $\boldsymbol{V} = \sum\limits_{i=1}^{n+1} V_i\partial_{x_i}$ with $V_i = \sum\limits_{j=1}^{n+1} \widetilde{g}^{ij}\widetilde{V}_j$ and $\widetilde{g}^{ij} = (\widetilde{g}_{ij})^{-1}$; here, $\gamma : [0, 1] \to M$ is any curve joining an $x_0 \in M$ with $x \in M$ with $c_0 = u(x_0)$.

By (3.3) we clearly have for every $j = 1, \cdots, n$ that

$$X_j(u) = \partial_{x_j}u + A_j\partial_{x_{n+1}}u = (\widetilde{a}_j - A_j\widetilde{a}) + A_j\widetilde{a} = \widetilde{a}_j,$$

which is equivalent to $\nabla_H u = \boldsymbol{a}$, see (3.1), i.e., $u \in C^2(M)$ is a solution to (1.6).

It remains to prove the sub-Riemannian Cesàro-Volterra path integral formula (1.9). To do this, let us fix an arbitrary horizontal path $\gamma : [0, 1] \to M$, joining x_0 with $x \in M$. If γ has the local representation $\gamma = (\gamma_1, \cdots, \gamma_{n+1})$, its horizontality means that

$$\dot{\gamma}_{n+1} = \sum_{k=1}^{n} A_k\dot{\gamma}_k.$$

Considering every term at the moment $t \in [0,1]$ in the following computations, we have

$$\begin{aligned}
\langle \boldsymbol{V}(\gamma(t)), \dot\gamma(t)\rangle_{\widetilde g} &= \sum_{i,k=1}^{n+1} \widetilde g_{ik} V_i \dot\gamma_k = \sum_{i,k,j=1}^{n+1} \widetilde g_{ik} \widetilde g^{ij} \widetilde V_j \dot\gamma_k = \sum_{k=1}^{n+1}\Big(\sum_{j=1}^{n+1}\Big(\sum_{i=1}^{n+1} \widetilde g_{ik}\widetilde g^{ij}\Big)\widetilde V_j\Big)\dot\gamma_k \\
&= \sum_{k=1}^{n+1}\Big(\sum_{j=1}^{n+1}\delta_{kj}\widetilde V_j\Big)\dot\gamma_k = \sum_{k=1}^{n+1}\widetilde V_k\dot\gamma_k = \sum_{k=1}^{n}\widetilde V_k\dot\gamma_k + \widetilde V_{n+1}\dot\gamma_{n+1} \\
&= \sum_{k=1}^{n}(\widetilde V_k + A_k\widetilde V_{n+1})\dot\gamma_k \\
&= \sum_{k=1}^{n}(\widetilde a_k - A_k\widetilde a + A_k\widetilde a)\dot\gamma_k = \sum_{k=1}^{n}\widetilde a_k\dot\gamma_k = \sum_{k=1}^{n} g_{ik}a_i\dot\gamma_k \\
&= g(\boldsymbol{a}(\gamma(t)), \dot\gamma(t)).
\end{aligned}$$

Thus, by (3.5) and the latter computation we obtain (1.9), which concludes our proof.

4 Examples

In this section we provide some computational examples as applications to Theorems 1.1 and 2.1 and Proposition 2.1, respectively.

4.1 Hyperbolic space

Let $\mathbb{B}^m = \{x \in \mathbb{R}^m : |x| < 1\}$ be the set endowed with the Riemannian metric

$$g_{\rm hyp}(x) = (g_{ij}(x))_{i,j=1,\cdots,m} = p(x)^2\delta_{ij},$$

where

$$p(x) = \frac{2}{1-|x|^2}.$$

The pair $(\mathbb{B}^m, g_{\rm hyp})$ is a model of the m-dimensional hyperbolic space with constant sectional curvature -1.

Example 4.1 We solve the problem

$$\nabla_{g_{\rm hyp}} u = \frac{x}{p} \quad \text{in } \mathbb{B}^m, \tag{4.1}$$

where $\nabla_{g_{\rm hyp}}$ denotes the hyperbolic gradient.

A direct computation shows that $\partial_{x_i}(px_j) = \partial_{x_j}(px_i)$ for every $i,j = 1,\cdots,m$, thus we may apply Theorem 2.1 on $(\mathbb{B}^m, g_{\rm hyp})$, which implies the solvability of (4.1). Moreover, if $\gamma : [0,1] \to \mathbb{B}^m$ is $\gamma(t) = tx$ with an arbitrarily fixed $x \in \mathbb{B}^m$, the solution u can be obtained as

$$\begin{aligned}
u(x) &= c_0 + \int_0^1 \Big\langle \frac{\gamma(t)}{p(\gamma(t))}, \dot\gamma(t)\Big\rangle_{g_{\rm hyp}} {\rm d}t = c_0 + \int_0^1 p(\gamma(t))\langle\gamma(t),\dot\gamma(t)\rangle {\rm d}t \\
&= c_0 + 2\int_0^1 \frac{|x|^2 t}{1-|x|^2t^2}{\rm d}t = c_0 - \ln(1-|x|^2) \\
&= c_0 + \ln\Big(\frac{p(x)}{2}\Big)
\end{aligned}$$

for any $c_0 \in \mathbb{R}$.

For simplicity, in the next example we consider only the hyperbolic plane $(\mathbb{B}^2, g_{\text{hyp}})$.

Example 4.2 We solve the problem

$$\nabla_{s,g_{\text{hyp}}} \boldsymbol{V} = \boldsymbol{e} \quad \text{on } \mathbb{B}^2, \tag{4.2}$$

where $\nabla_{s,g_{\text{hyp}}}$ denotes the symmetric hyperbolic gradient and $\boldsymbol{e} = (\boldsymbol{e}_1, \boldsymbol{e}_2) \in \boldsymbol{C}^\infty(\mathbb{B}^2, (T\mathbb{B}^2)^2)$ has the components $\boldsymbol{e}_1 = -\frac{x_1}{p}\partial_{x_2}$ and $\boldsymbol{e}_2 = -\frac{1}{p}(x_1\partial_{x_1} + 2x_2\partial_{x_2})$.

First, we have $e_{11} = 0$, $e_{12} = e_{21} = -\frac{x_1}{p}$ and $e_{22} = -\frac{2x_2}{p}$. It is easily seen that the Saint-Venant relations (2.10) are verified; for instance, if $i = k = 1$ and $j = l = 2$ then the components in (2.10) are $\partial^2_{x_2x_2} e_{11} = 0$, $\partial^2_{x_1x_1} e_{22} = 2x_2$ and $\partial^2_{x_1x_2} e_{12} = x_2$. Therefore, we may apply Proposition 2.1, guaranteeing the solvability of (4.2). By keeping the same notations as in Proposition 2.1, since $g^{-1}_{\text{hyp}} = p(x)^{-2}\delta_{ij}$, after some computation it turns out that

$$\boldsymbol{W}_{11} = \boldsymbol{W}_{22} = 0, \quad \boldsymbol{W}_{12} = -\boldsymbol{W}_{21} = \frac{1}{2p^2}(1 - |x^2| - 2x_1^2)\partial_{x_1} - \frac{x_1x_2}{p^2}\partial_{x_2}.$$

Accordingly, for every $x \in \mathbb{B}^2$ one has that $p_{11}(x) = c_0^{11}$, $p_{22}(x) = c_0^{22}$ for some $c_0^{11}, c_0^{22} \in \mathbb{R}$ and if we fix $\gamma : [0,1] \to \mathbb{B}^2$ with $\gamma(t) = tx = (tx_1, tx_2)$, then

$$p_{12}(x) = -p_{21}(x) = c_0^{12} + \int_0^1 \langle \boldsymbol{W}_{12}(\gamma(t)), \dot{\gamma}(t)\rangle_{g_{\text{hyp}}} \mathrm{d}t = c_0^{12} + \frac{1}{2}(x_1 - x_1^3 - x_1x_2^2)$$

for some $c_0^{12} \in \mathbb{R}$. Thus

$$\boldsymbol{U}_1 = \frac{1}{p^2}(c_0^{11}\partial_{x_1} + c_0^{12}\partial_{x_2})$$

and

$$\boldsymbol{U}_2 = \frac{1}{p^2}((-c_0^{12} - x_1 + x_1^3 + x_1x_2^2)\partial_{x_1} + (c_0^{22} - x_2 + x_1^2x_2 + x_2^3)\partial_{x_2}).$$

Therefore, for every $x \in \mathbb{B}^2$, if $\gamma : [0,1] \to \mathbb{B}^2$ is again the curve given by $\gamma(t) = tx = (tx_1, tx_2)$, then the latter vector fields provide the functions

$$u_1(x) = c_0^1 + \int_0^1 \langle \boldsymbol{U}_1(\gamma(t)), \dot{\gamma}(t)\rangle_{g_{\text{hyp}}} \mathrm{d}t = c_0^1 + c_0^{11}x_1 + c_0^{12}x_2$$

and

$$u_2(x) = c_0^2 + \int_0^1 \langle \boldsymbol{U}_2(\gamma(t)), \dot{\gamma}(t)\rangle_{g_{\text{hyp}}} \mathrm{d}t = c_0^2 - \frac{1}{4} - c_0^{12}x_1 + c_0^{22}x_2 + \frac{1}{p^2(x)}.$$

Consequently, $\boldsymbol{V} = (V_1, V_2)$ is a solution of (4.2), where $V_i = \frac{u_i}{p^2}$, $i = 1, 2$, with $c_0^{11} = c_0^{22} = 0$ and c_0^1, c_0^2 and c_0^{12} arbitrarily fixed.

4.2 Carnot and Heisenberg groups

Let $\mathbb{G}$ be an $(n+1)$-dimensional corank 1 Carnot group with the Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\dim\mathfrak{g}_1 = n$ and $\dim\mathfrak{g}_2 = 1$. Usually, the operation on $\mathfrak{g}$ (in exponential coordinates on $\mathbb{R}^n \times \mathbb{R}$) is given by

$$x \circ y = \Big(x_1 + y_1, \cdots, x_n + y_n, x_{n+1} + y_{n+1} - \frac{1}{2}\sum_{i,j=1}^n \mathcal{A}_{ij}x_jy_i\Big),$$

where $x = (x_1, \cdots, x_{n+1})$, $y = (y_1, \cdots, y_{n+1})$, and without loss of generality, $\mathcal{A}$ is represented by

$$\mathcal{A} = \begin{bmatrix} 0_{n-2d} & & & 0 \\ & \alpha_1 J & & \\ 0 & & \ddots & \\ & & & \alpha_d J \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4.3}$$

(see, e.g. [19]). Here $0 < \alpha_1 \leq \cdots \leq \alpha_d$, and 0_{n-2d} is the $(n-2d) \times (n-2d)$ square null-matrix. The layers $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are generated by the left-invariant vector fields

$$X_i = \partial_{x_i} - \frac{1}{2} \sum_{j=1}^{k} \mathcal{A}_{ij} x_j \partial_{x_{n+1}}, \quad i = 1, \cdots, n. \tag{4.4}$$

Note that $[X_i, X_j] = \mathcal{A}_{ij} \partial_{x_{n+1}}$, $i, j = 1, \cdots, n$.

If $n = 2d$ (thus the kernel of $\mathcal{A}$ is trivial) and $\alpha_1 = \cdots = \alpha_d = 4$, the Carnot group $\mathbb{G}$ reduces to the usual Heisenberg group $\mathbb{H}^d = \mathbb{R}^{2d} \times \mathbb{R}$.

For our example, we shall consider a 6-dimensional corank 1 Carnot group with the left-invariant vector fields given by (4.4), by choosing $d = 2$, $n = 5$, $\alpha_1 = 4$ and $\alpha_2 = 2$. To be more explicit, the distribution $\mathcal{D}$ on $(\mathbb{G}, \circ)$ is formed by the vector fields given by

$$\begin{cases} X_1 = \partial_{x_1}, \\ X_2 = \partial_{x_2} - 2x_3 \partial_{x_6}, \\ X_3 = \partial_{x_3} + 2x_2 \partial_{x_6}, \\ X_4 = \partial_{x_4} - x_5 \partial_{x_6}, \\ X_5 = \partial_{x_5} + x_4 \partial_{x_6}. \end{cases} \tag{4.5}$$

Let $\boldsymbol{a} = (a_1, a_2, a_3, a_4, a_5) \in \Gamma(\mathcal{D})$ be given by the functions

$$\begin{cases} a_1 = x_3^2 x_5, \\ a_2 = 2x_2 x_4 x_6 (x_6 - 2x_2 x_3), \\ a_3 = 3x_1 x_3 x_5 + 4x_2^3 x_4 x_6, \\ a_4 = x_2^2 x_6 (x_6 - 2x_4 x_5), \\ a_5 = x_1 x_3^2 + 2x_2^2 x_4^2 x_6. \end{cases} \tag{4.6}$$

Example 4.3 We solve the problem

$$X_i u = a_i \quad \text{in } \mathbb{G}, \ i = 1, \cdots, 5. \tag{4.7}$$

To do this, we are going to fully explore Theorem 1.1; by using the same notations, we identify $A_1 = 0$, $A_2 = -2x_3$, $A_3 = 2x_2$, $A_4 = -x_5$, $A_5 = x_4$. Moreover, $c_{23} = 4 = -c_{32}$, $c_{45} = 2 = -c_{54}$, and the rest of the elements of the matrix $C = (c_{ij})$ are zero, $i, j = 1, \cdots, 5$. In order to solve (4.7), we have to check relations (1.7) and (1.8), respectively. It is easy to observe that (1.7) is relevant only for $(i, j) = (2, 3)$ and $(k, l) = (4, 5)$ (the other choices giving always zero), where simple computations give that $X_2 a_3 - X_3 a_2 = 8x_2^2 x_4 x_6$ and $X_4 a_5 - X_5 a_4 = 4x_2^2 x_4 x_6$; thus, (1.7) holds. Another simple reasoning shows that relation (1.8) is also verified; for instance, $X_3 X_2 a_3 - X_3 X_3 a_2 = 16x_2^3 x_4 = [X_2, X_3] a_3$, the other relations following in the same way.

Thus, Theorem 1.1 implies that the system (4.7) is solvable in $\mathcal{F}(\mathbb{G})$; let $x_0 = \mathbf{0} \in \mathbb{G}$ and any horizontal curve $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) : [0, 1] \to \mathbb{G}$ with $\gamma(0) = \mathbf{0}$ and $\gamma(1) = x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{G}$. Note that the horizontality of γ means that

$$\dot{\gamma}_6 = -2\gamma_3\dot{\gamma}_2 + 2\gamma_2\dot{\gamma}_3 - \gamma_5\dot{\gamma}_4 + \gamma_4\dot{\gamma}_5.$$

Due to the latter relation and (1.9), some suitable rearrangements and $\gamma(0) = \mathbf{0}$ give that

$$\begin{aligned} u(x) - c_0 &= \int_0^1 \sum_{i=1}^5 a_i(\gamma(t))\dot{\gamma}_i(t)\mathrm{d}t \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_1(t)\gamma_3^2(t)\gamma_5(t))\mathrm{d}t + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_2(t)\gamma_4(t)\gamma_6^2(t))\mathrm{d}t \\ &= \gamma_1(1)\gamma_3^2(1)\gamma_5(1) + \gamma_2(1)\gamma_4(1)\gamma_6^2(1) \\ &= x_1x_3^2x_5 + x_2^2x_4x_6^2 \end{aligned}$$

for some $c_0 \in \mathbb{R}$, which provides the solution of system (4.7).

5 Final Remarks

We conclude the paper with two remarks which can be considered as starting points of further investigations.

(I) **Poincaré lemma on Finsler manifolds** Let (M, F) be an m-dimensional, not necessarily reversible Finsler manifold and $\Omega \subseteq M$ be a simply connected domain. Given a vector field $\boldsymbol{V} \in \boldsymbol{C}^1(\Omega, T\Omega)$, we are asking about the solvability of the equation

$$\nabla_F u = \boldsymbol{V} \quad \text{in } \Omega, \tag{5.1}$$

where ∇_F denotes the Finslerian gradient. Here, as usual $\nabla_F u(x) = J^*(x, Du(x))$, where $J^* : T^*M \to TM$ is the Legendre transform associating to each element $\alpha \in T_x^*M$ the unique maximizer on T_xM of the map $y \mapsto \alpha(y) - \frac{1}{2}F^2(x, y)$ and $Du(x) \in T_x^*M$ is the derivative of u at $x \in M$ (see [18]). Note that in general, $u \mapsto \nabla_F u$ is not linear. In order to solve (5.1), a necessarily curl-vanishing condition can be formulated by using the inverse Legendre transform $J = (J^*)^{-1}$ and fundamental form of the Finsler metric F. However, we cannot adapt the proof of Theorem 2.1 into the Finsler setting. Indeed, we recall that in the proof of Theorem 2.1 we explored the metric compatibility and torsion-freeness of the Levi-Civita connection with respect to the given Riemannian metric; as we know, such properties are not simultaneously valid on a generic Finsler manifold unless it is Riemannian.

(II) **Saint-Venant lemma on sub-Riemannian structures** For simplicity, we shall consider only the usual Heisenberg group $(\mathbb{H}^d, \mathcal{D}, g)$, where $\mathcal{D} = \{X_1, \cdots, X_{2d}\}$ with

$$X_{2i-1} = \partial_{x_{2i-1}} - 2x_{2i}\partial_{x_{2d+1}}, \quad X_{2i} = \partial_{x_{2i}} + 2x_{2i-1}\partial_{x_{2d+1}}, \quad i = 1, \cdots, d,$$

and g is the natural Riemannian metric on $\mathcal{D}$ (see (4.4)). Given a symmetric vector field $\boldsymbol{e} = (\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{2d}) \in \Gamma(\mathcal{D})^{2d}$ on $\Omega \subseteq \mathbb{H}^d$, i.e., $e_{ij} = e_{ji}$ for every $i, j = 1, \cdots, 2d$ where $\boldsymbol{e}_i = \sum\limits_{j=1}^{2d} e_{ij}X_j$, the question concerns the solvability of the sub-Riemannian system

$$\nabla_{s,H} \boldsymbol{V} = \boldsymbol{e} \quad \text{in } \Omega \tag{5.2}$$

for the unknown vector field $\boldsymbol{V} = (V_1, \cdots, V_{2d}) \in \boldsymbol{C}^\infty(\Omega, \mathbb{R}^{2d})$, where the components of the symmertric horizontal gradient $\nabla_{s,H}$ are given by

$$\frac{1}{2}(X_i V_k + X_k V_i), \quad i, k = 1, \cdots, 2d.$$

The first challenging problem is to establish the necessary Saint-Venant compatibility relations associated to problem (5.2) and then to apply Proposition 2.1; note that Schwartz type properties are not valid in this setting since usually $X_i X_j \neq X_j X_i$ for $i \neq j$. Moreover, weaker versions of the Saint-Venant lemma on $\mathbb{H}^d$ would provide a sub-Riemannian Korn-type inequality as well. Clearly, more general sub-Riemannian structures can also be considered instead of Heisenberg groups verifying the assumptions of Theorem 1.1.

6 Appendix: Proof of the Saint-Venant Lemma (Proposition 2.1)

A direct computation shows that if (2.9) has a solution, then the Saint-Venant compatibility relations (2.10) trivially hold.

Conversely, the Saint-Venant compatibility relations (2.10) can be written into the form

$$\partial_{x_l}(\partial_{x_j} e_{ik} - \partial_{x_i} e_{jk}) = \partial_{x_k}(\partial_{x_j} e_{il} - \partial_{x_i} e_{jl}),$$

which is equivalent to

$$\partial_{x_l}(g_{kt} g^{ts}(\partial_{x_j} e_{is} - \partial_{x_i} e_{js})) = \partial_{x_k}(g_{lt} g^{ts}(\partial_{x_j} e_{is} - \partial_{x_i} e_{js})). \tag{6.1}$$

If $\boldsymbol{W}_{ij}$ is a vector field on Ω with the representation

$$\boldsymbol{W}_{ij} = W_{ijt}\partial_{x_t} = g^{ts}(\partial_{x_j} e_{is} - \partial_{x_i} e_{js})\partial_{x_t},$$

relation (6.1) can be written equivalently into the form

$$\partial_{x_l}(g_{kt} W_{ijt}) = \partial_{x_k}(g_{lt} W_{ijt}).$$

Thus, we may apply Theorem 2.1, i.e., there exists $p_{ij} \in C^2(\Omega)$ such that

$$\nabla_g p_{ij} = \boldsymbol{W}_{ij} \quad \text{on } \Omega,\ \forall i, j = 1, \cdots, m.$$

By components, the latter relation means that

$$g^{ts}\partial_{x_s} p_{ij} = W_{ijt} = g^{ts}(\partial_{x_j} e_{is} - \partial_{x_i} e_{js}).$$

Multiplying from left by g_{tl} and adding them, we have

$$\partial_{x_l} p_{ij} = \partial_{x_j} e_{il} - \partial_{x_i} e_{jl}, \quad \forall i, j, l = 1, \cdots, n. \tag{6.2}$$

Since $\partial_{x_l} p_{ij} + \partial_{x_l} p_{ji} = 0$, we can assume without loss of generality that $p_{ij} + p_{ji} = 0$.

If $q_{ij} = p_{ij} + e_{ij}$, then by (6.2) we have

$$\partial_{x_k} q_{ij} = \partial_{x_k} p_{ij} + \partial_{x_k} e_{ij} = \partial_{x_j} e_{ik} - \partial_{x_i} e_{jk} + \partial_{x_k} e_{ij} = \partial_{x_j} e_{ik} + \partial_{x_j} p_{ik} = \partial_{x_j} q_{ik}.$$

Again, the latter relation can be transformed into

$$\partial_{x_k}(g_{tj} g^{ts} q_{is}) = \partial_{x_j}(g_{tk} g^{ts} q_{is}).$$

Therefore, if

$$\boldsymbol{U}_i = U_{il} \partial_{x_l} = g^{ls} q_{is} \partial_{x_l},$$

Theorem 2.1 implies the existence of $u_i \in C^2(\Omega)$ such that

$$\nabla_g u_i = \boldsymbol{U}_i, \quad \forall i = 1, \cdots, m.$$

If we write the components of the latter relation, it yields that

$$\partial_{x_l} u_i = q_{il},, \quad \forall i, l = 1, \cdots, m. \tag{6.3}$$

Let $\boldsymbol{V} = (V_1, \cdots, V_m)$ with $V_k = g^{ks} u_s, \quad k = 1, \cdots, m$. Consequently, by (6.3), we have

$$\begin{aligned}
\frac{1}{2}(\partial_{x_i}(g_{jk} V_k) + \partial_{x_j}(g_{ik} V_k)) &= \frac{1}{2}(\partial_{x_i}(g_{jk} g^{ks} u_s) + \partial_{x_j}(g_{ik} g^{ks} u_s)) \\
&= \frac{1}{2}(\partial_{x_i} u_j + \partial_{x_i} u_i) = \frac{1}{2}(q_{ij} + q_{ji}) \\
&= e_{ij},
\end{aligned}$$

which is nothing but $\nabla_{s,g} \boldsymbol{V} = \boldsymbol{e}$, i.e., relation (2.9). The Cesàro-Volterra integral formula follows at once by combining the above steps.

Acknowledgements The author thanks Professor Philippe G. Ciarlet for his invitation to the City University of Hong Kong where the present work has been initiated. He is also grateful to Professors Ovidiu Calin and Der-Chen Chang for their suggestions and remarks.

References

[1] Abraham, R., Marsden, J. E. and Ratiu, T., Manifolds, Tensor Analysis, and Applications, 2nd edition, Applied Mathematical Sciences, **75**, Springer-Verlag, New York, 1988.

[2] Agrachev, A., Barilari, D. and Boscain, U., Introduction to Geodesics in Sub-Riemannian Geometry, Geometry Analysis and Dynamics on Sub-Kiemannian Manifolds, **II**, EMS Ser. Lect. Math., Eur. Math. Soc., Zürich, 2016.

[3] Amrouche, Ch., Ciarlet, P. G. and Mardare, C., Remarks on a lemma by Jacques-Louis Lions, *C. R. Math. Acad. Sci. Paris*, **352**(9), 2014, 691–695.

[4] Amrouche, Ch., Ciarlet, P. G. and Mardare, C., On a lemma of Jacques-Louis Lions and its relation to other fundamental results, *J. Math. Pures Appl.* (9), **104**(2), 2015, 207–226.

[5] Calin, O. and Chang, D.-C., Sub-Riemannian Geometry, General Theory and Examples, Encyclopedia Math. Appl., **126**, Cambridge University Press, Cambridge, 2009.

[6] Calin, O., Chang, D.-C. and Eastwood, M., Integrability conditions for the Grushin and Martinet distributions, *Bull. Inst. Math. Acad. Sin.* (*N.S.*), **8**(2), 2013, 159–168.

[7] Calin, O., Chang, D.-C. and Eastwood, M., Integrability conditions for Heisenberg and Grushin-type distributions, *Anal. Math. Phys.*, **4**(1–2), 2014, 99–114.

[8] Calin, O., Chang, D.-C. and Hu, J., Poincaré's lemma on the Heisenberg group, *Adv. in Appl. Math.*, **60**, 2014, 90–102.

[9] Calin, O., Chang, D.-C. and Hu, J., Integrability conditions on Engel-type manifolds, *Anal. Math. Phys.*, **5**(3), 2015, 217–231.

[10] Calin, O., Chang, D.-C. and Hu, J., Integrability conditions on a sub-Riemannian structure on $\mathbb{S}^3$, *Anal. Math. Phys.*, **7**(1), 2017, 9–18.

[11] Chen, W. and Jost, J., A Riemannian version of Korn's inequality, *Calc. Var. Partial Differential Equations*, **14**, 2012, 517–530.

[12] Ciarlet, P. G., Linear and nonlinear functional analysis with applications, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[13] Ciarlet, P. G., Gratie, L., Mardare, C. and Shen, M., Saint Venant compatibility equations on a surface application to intrinsic shell theory, *Math. Models Methods Appl. Sci.*, **18**(2), 2008, 165–194.

[14] Ciarlet, P. G. and Mardare, S., Nonlinear Saint-Venant compatibility conditions and the intrinsic approach for nonlinearly elastic plates, *Math. Models Methods Appl. Sci.*, **23**(12), 2013, 2293–2321.

[15] Ciarlet, P. G., Mardare, S. and Shen, M., Saint Venant compatibility equations in curvilinear coordinates, *Anal. Appl.* (*Singap.*), **5**(3), 2007, 231–251.

[16] do Carmo, M. P., Riemannian Geometry, Birkhäuser, Boston, 1992.

[17] Figalli, A. and Rifford, L., Mass transportation on sub-Riemannian manifolds, *Geom. Funct. Anal.*, **20**(1), 2010, 124–159.

[18] Ohta, S. and Sturm, K.-T., Heat flow on Finsler manifolds, *Comm. Pure Appl. Math.*, **62**(10), 2009, 1386–1433.

[19] Rizzi, L., Measure contraction properties of Carnot groups, *Calc. Var. Partial Differential Equations*, **55**(3), 2016, 20 pages.