Chin. Ann. Math. Ser. B
39(2), 2018, 335–344
DOI: 10.1007/s11401-018-1067-3

Chinese Annals of Mathematics, Series B

Some Remarks on Korn Inequalities

Alain DAMLAMIAN[1]

(*Dedicated to Philippe G. Ciarlet on the occasion of his* 80*th birthday*)

Abstract A recent joint paper with Doina Cioranescu and Julia Orlik was concerned with the homogenization of a linearized elasticity problem with inclusions and cracks (see [Cioranescu, D., Damlamian, A. and Orlik, J., Homogenization via unfolding in periodic elasticity with contact on closed and open cracks, *Asymptotic Analysis*, **82**, 2013, 201–232]). It required uniform estimates with respect to the homogenization parameter. A Korn inequality was used which involves unilateral terms on the boundaries where a no-penetration condition is imposed. In this paper, the author presents a general method to obtain many diverse Korn inequalities including the unilateral inequalities used in [Cioranescu, D., Damlamian, A. and Orlik, J., Homogenization via unfolding in periodic elasticity with contact on closed and open cracks, *Asymptotic Analysis*, **82**, 2013, 201–232]. A preliminary version was presented in [Damlamian, A., Some unilateral Korn inequalities with application to a contact problem with inclusions, *C. R. Acad. Sci. Paris, Ser. I*, **350**, 2012, 861–865].

Keywords Korn domains, Korn-Wirtinger domains, Unilateral Korn inequality
2010 MR Subject Classification 35Q74, 74B05, 35J86

1 Introduction

In the joint paper with Doina Cioranescu and Julia Orlik [2], the homogenization of static linearized elasticity problems in the presence of cracks and inclusions was studied. This can be reduced to minimizing a convex functional but its coerciveness is not straightforward, mainly because of the presence of (infinitesimal) rigid motions. In the homogenization process, the first difficulty is to obtain uniform estimates. These are intimately connected with the uniform coerciveness of the convex functionals involved.

There are multiple "classical" results on unilateral contact. Many works of Gaetano Fichera [5–8] are the first concerning the existence of solutions for such problems. More recently, the following books consider this problem: Hlaváček, I., Haslinger, J., Nečas, J. and Lovíček, J. [11] (1982), Kikuchi, N. and Oden, J. T. [12] (1988), Eck, C., Jarušek, J. and Krbec, M. [4] (2005).

All these results give conditions under which a solution exists, but none give explicit estimates which, in the case of homogenization, would be uniform with respect to the homogenization parameter. Furthermore, none seem to treat the case of Tresca friction for inclusions or cracks. In [2], the problem was solved by first proving a unilateral Korn inequality adapted to inclusions.

Manuscript received August 9, 2017. Revised October 24, 2017.
[1]Laboratoire d'Analyse et de Mathématiques Appliquées CNRS UMR 8050, Université Paris-Est Créteil Val-de-Marne, 94010 Créeil Cedex, France. E-mail: damla@u-pec.fr

Section 2 of this paper presents some definitions related to the classical Korn inequalities. In Section 3, we give a procedure to construct Korn inequalities starting from "semi-norm-like functions" on the space of rigid motions. Section 4 gives examples. For simplicity, they are set in the natural space $\mathbb{R}^3$ but extensions to higher dimensions are straightforward.

Notations Let v be a vector field on a domain of $\mathbb{R}^3$ and S an orientable surface therein[1];

- a choice of unit normal vector to the surface S is denoted ν (the other would be $-\nu$), the scalar normal component of v on S is $v \cdot \nu$, denoted v_ν, its tangential component $v - v_\nu \nu$ is denoted v_τ;
- ∇v is the gradient of v, defined as the matrix field:

$$(\nabla v)_{ij} \doteq \frac{\partial v_i}{\partial x_j}, \quad \forall (i,j) \in \{1,2,3\}^2;$$

- $e(v)$ is the strain tensor (symmetric gradient) of v, defined as the symmetric matrix field:

$$e(v)_{ij} \doteq \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big), \quad \forall (i,j) \in \{1,2,3\}^2;$$

- $\mathcal{R}$ is the kernel of e in connected domains, i.e., the space of infinitesimal rigid motions:

$$\mathcal{R} = \{x \mapsto a + Bx,\ a \in \mathbb{R}^3,\ B \text{ a skew-symmetric matrix}\};$$

In the case of $\mathbb{R}^3$ (or an open connected subdomain in $\mathbb{R}^3$), this is also

$$\mathcal{R} \doteq \{x \mapsto v_{a,b}(x) = a + b \wedge x;\ a \text{ and } b \in \mathbb{R}^3\}, \tag{1.1}$$

where $\wedge$ indicates the vector product (i.e., cross product or wedge product) in $\mathbb{R}^3$;

- in estimates, C is a generic constant (function of the domain only);
- for a given domain O of $\mathbb{R}^3$, the spaces of scalar fields, vector fields and matrix fields $L^2(O;\mathbb{R})$, $L^2(O;\mathbb{R}^3)$ and $L^2(O;\mathbb{R}^9)$, will all be referred to as $L^2(O)$ (there will be no ambiguity from the context). Similarly, the spaces of scalar fields and vector fields $H^1(O;\mathbb{R})$ and $H^1(O;\mathbb{R}^3)$ will often be denoted simply $H^1(O)$. The latter is endowed with the norm

$$\|u\|^2_{H^1(O)} \doteq \|u\|^2_{L^2(O)} + \sum_{i,j=1}^{3} \Big\|\frac{\partial u_i}{\partial x_j}\Big\|^2_{L^2(O)}.$$

2 Korn Domains and Korn-Wirtinger Domains

Korn inequalities are inequalities bounding the L^2-norm of the gradient of a vector field in a domain (or its H^1-norm) by that of its symmetric gradient together with some extra terms, if necessary.

The first Korn inequality is classical for the space H^1_0 of any domain.

Proposition 2.1 *Let $\mathcal{O}$ be an open set in $\mathbb{R}^3$. Then for every u in $H^1_0(\mathcal{O})$,*

$$2\|e(u)\|\|^2_{L^2(\mathcal{O})} = \|\nabla u\|^2_{L^2(\mathcal{O})} + \|\operatorname{div} u\|^2_{L^2(\mathcal{O})}.$$

[1] All surfaces in this paper are assumed to be orientable.

Consequently

$$\|\nabla u\|_{L^2(\mathcal{O})} \le \sqrt{2}\|e(u)\|_{L^2(\mathcal{O})}, \tag{2.1}$$

and if the Poincaré inequality holds in $\mathcal{O}$ with the constant $C_P(\mathcal{O})$ (e.g., $\mathcal{O}$ is bounded in one direction)

$$\|u\|_{H_0^1(\mathcal{O})} \le \sqrt{2}\, C_P(\mathcal{O})\|e(u)\|_{L^2(\mathcal{O})}. \tag{2.2}$$

Proof Using the summation convention and the Stokes formula (for the cross terms)

$$\begin{aligned}
4\int_{\mathcal{O}} |e(u)|^2 \,\mathrm{d}x &= \int_{\mathcal{O}} \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)^2 \mathrm{d}x \\
&= \int_{\mathcal{O}} \Big(\frac{\partial u_i}{\partial x_j}^2 + \frac{\partial u_j}{\partial x_i}^2 + 2\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\Big)\mathrm{d}x \\
&= \int_{\mathcal{O}} \Big(\frac{\partial u_i}{\partial x_j}^2 + \frac{\partial u_j}{\partial x_i}^2 + 2\frac{\partial u_i}{\partial x_i}\frac{\partial u_j}{\partial x_j}\Big)\mathrm{d}x \\
&= \int_{\mathcal{O}} (2|\nabla u|^2 + 2|\mathrm{div}\, u|^2)\,\mathrm{d}x.
\end{aligned}$$

Definition 2.1 *A domain O is a Korn domain if the second Korn inequality holds for $H^1(O)$, i.e., there exists a constant $C_{\mathrm{K}}(= C_{\mathrm{K}}(O))$ such that*

$$\forall v \in H^1(O), \quad \|v\|_{H^1(O)} \le C_{\mathrm{K}}(\|v\|_{L^2(O)} + \|e(v)\|_{L^2(O)}). \tag{2.3}$$

In 1962, Gobert [9] gave the first proof that a bounded domain with Lipschitz boundary is a Korn domain. More recent proofs (none of them straightforward) can be found in the book [13] of Oleinik, Shamaev and Yosifian and the paper [1] of P. Ciarlet and P. G. Ciarlet.

It is obvious that the union of a finite number of Korn domains is a Korn domain. This is the case of domains with a finite number of Lipschitz cracks provided the cracks which touch the boundary are not tangent to it.

The following is the analogue for vector fields of the Poincar?-Wirtinger inequality for scalar functions.

Definition 2.2 *A bounded connected open domain O is a Korn-Wirtinger domain if there exists a constant $C_{\mathrm{KW}}(= C_{\mathrm{KW}}(O))$ such that for every v in $H^1(O)$ there is an $r(v)$ in $\mathcal{R}$ with*

$$\|v - r(v)\|_{H^1(O)} \le C_{\mathrm{KW}}\|e(v)\|_{L^2(O)}. \tag{2.4}$$

Obviously, $r(v)$ can be chosen as the orthonormal projection of v on $\mathcal{R}$ in the Hilbert space $H^1(O)$ so that r is linear and $v - r(v)$ is orthogonal to $\mathcal{R}$.

Remark 2.1 It is straightforward to check that the Poincar?-Wirtinger for scalar functions holds in $H^1(O)$ when O is a Korn-Wirtinger domain (it suffices to consider vector fields with only one non-zero component). The converse seems open.[2]

The following proposition gives examples of Korn-Wirtinger domains.

[2]The converse for certain weighted norms is a consequence of Theorem 2.3 of [10].

Proposition 2.2 *Suppose that O is a connected and bounded Korn domain. If the natural injection from $H^1(O)$ to $L^2(O)$ is compact, then, O is a Korn-Wirtinger domain. This is true in particular as soon as the boundary of O is Lipschitz.*

The proof is classical but we give it for the sake of completeness.

Proof By contradiction, if (2.4) holds for no constant $C_{\rm KW}$, then one can construct a sequence $\{u_n\}_{n\in\mathbb{N}}$ in the orthogonal complement of $\mathcal{R}$ in $H^1(O)$ such that

$$n\|e(u_n)\|_{L^2(O)} \le \|u_n\|_{H^1(O)} \le C_{\rm K}(\|u_n\|_{L^2(O)} + \|e(u_n)\|_{L^2(O)}).$$

By scaling, one can assume that $\|u_n\|_{L^2(O)} \equiv 1$.

It then follows that $\|e(u_n)\|_{L^2(O)} \le \frac{C_{\rm K}}{n-C_{\rm K}} \to 0$ so that $\|u_n\|_{H^1(O)}$ is bounded. Thus, $\{u_n\}_n$ admits weak limit points in $H^1(O)$. Let v be one such weak limit point. By weak lower semi-continuity, $e(v) = 0$ so that v belongs to $\mathcal{R}$. On the other hand, v is orthogonal to $\mathcal{R}$ since all the u_n's are. Therefore $v = 0$. But, by the compact injection of $H^1(O)$ to $L^2(O)$, $\|v\|_{L^2(O)} = 1$, a contradiction.

Remark 2.2 As is expected in a proof by contradiction, the constant C is not explicit. The same is true for every statement below.

Here is a way to obtain more Korn-Wirtinger domains.

Proposition 2.3 *The union of two Korn-Wirtinger domains whose intersection is not empty is a Korn-Wirtinger domain. The same holds true for two Korn-Wirtinger domains whose boundaries intersect along a subset which contains a portion of a Lipschitz hypersurface with non-zero superficial measure. This can be generalized to a finite union of Korn-Wirtinger domains.*

Proof Let u be in $H^1(O)$ with $O = O_1 \cup O_2$. The hypotheses imply that there exist two rigid motions r_1 and r_2 such that

$$\|u - r_i\|_{H^1(O_i)} \le C\|e(u)\|_{L^2(O)}, \quad i = 1, 2.$$

In particular,

$$\|u - r_i\|_{H^1(O_1\cap O_2)} \le C\|e(u)\|_{L^2(O)}, \quad i = 1, 2,$$

and consequently,

$$\|r_1 - r_2\|_{H^1(O_1\cap O_2)} \le 2C\|e(u)\|_{L^2(O)}.$$

Since all the norms are equivalent on the finite dimensional space $\mathcal{R}$, it is also true that

$$\|r_1 - r_2\|_{H^1(O_2)} \le 2CC'\|e(u)\|_{L^2(O)}.$$

Consequently,

$$\|u - r_1\|_{H^1(O_2)} \le C(1 + 2C')\|e(u)\|_{L^2(O)}$$

and

$$\|u - r_1\|_{H^1(O)} \le 2C(1 + C')\|e(u)\|_{L^2(O)}.$$

In the second case denoting Σ a part of the common boundary which is included in a Lipschitz hypersurface and with finite measure, the proof is the same but makes use of the trace theorem from $H^1(O_i)$, $i = 1, 2$, to $L^2(\Sigma)$ (instead of the restriction from $H^1(O_i)$) to $H^1(O_1 \cap O_2)$).

3 Construction of Korn Inequalities

The point of Korn inequalities is that there is no single one which can be applied in every problem. Each problem requires an adapted Korn inequality. Here is a method to generate Korn inequalities on Korn-Wirtinger domains. The constants exist but are not very easy to track. Sharper estimates of the constant in each case may be of interest but is out of the scope of this paper.

Theorem 3.1 *Let O be a Korn-Wirtinger domain. If $F : H^1(O)) \to \mathbb{R}$ is a Lipschitz map whose restriction to the subspace $\mathcal{R}$ of rigid motions is bounded below by a norm on $\mathcal{R}$. Then there exists a constant C such that*

$$\forall u \in H^1(O), \quad \|u\|_{H^1(O)} \leq C(F(u) + \|e(u)\|_{L^2(O)}). \tag{3.1}$$

Proof Let u be in $H^1(O)$ and $r(u)$ be an element in $\mathcal{R}$ such that (2.4) holds. By the hypothesis on F, it follows that

$$\|r(u)\|_{\mathcal{R}} \leq F(r(u)) \leq F(u) + C\|u - r(u)\|_{H^1(O)} \leq F(u) + C\|e(u)\|_{L^2(O)}.$$

Since all norms are equivalent on $\mathcal{R}$, this implies

$$\|r(u)\|_{H^1(O)} \leq F(u) + C\|e(u)\|_{L^2(O)}$$

and going back to (2.4) completes the proof.

4 Some Examples

4.1 Some standard Korn inequalities

Proposition 4.1 *Let O be a Korn-Wirtinger domain. Let ω be a non-empty open subset of O or an open subset of the boundary of a Lipschitz subdomain of O. Then, there is a constant C such that*

$$\forall u \in H^1(O), \quad \|u\|_{H^1(O)} \leq C(\|u(x)\|_{L^1(\omega)} + \|e(u)\|_{L^2(O)}).$$

In particular,

$$\forall u \in H^1(O) \text{ such that } u_{|\omega} \equiv 0,$$

then

$$\|u\|_{H^1(O)} \leq C\|e(u)\|_{L^2(O)}.$$

Proof Set

$$F(u) \doteq \|u\|_{L^1(\omega)}.$$

Clearly, F is Lipschitz on $H^1(O)$ by injection into $L^2(O)$ (resp. by a trace theorem into $L^2(\omega)$) and $F_{|\mathcal{R}}$ is a semi-norm on $\mathcal{R}$. Furthermore, $F(r) = 0$ implies that r vanishes at every point of ω. Since ω is open (resp. is open in a Lipschitz surface), it contains at least three non-aligned points where r vanishes. One concludes that r vanishes everywhere in $\mathbb{R}^3$ (one easy argument is to use the equi-projectivity of every rigid motion).

This Korn inequality applies for example when the vector field u satisfies a homogeneous or non-homogeneous Dirichlet condition on a non-empty open subset of the boundary ∂O.

Remark 4.1 The previous result can be extended to the case where the norm on ω is taken in $L^2(\omega;\mu)$ where μ is a non zero-measure of finite energy (i.e., in the dual space of $H^1(O)$).

4.2 Some non-standard Korn inequalities

This section starts with a trivial remark concerning rigid motions: Every rigid motion is divergence-free.

Proposition 4.2 *Let ω be a bounded open set with Lipschitz boundary in $\mathbb{R}^3$. Then the map $r \mapsto \|r_\nu^+\|_{L^1(\partial\omega)}$ is a semi-norm on $\mathcal{R}$. Its kernel consists of all the rigid motions which are tangent to $\partial\omega$.*

Proof By the Stokes theorem applied in ω, and since $\operatorname{div} r = 0$, it follows that $\int_{\partial\omega} r_\nu \,\mathrm{d}\sigma = 0$. Consequently,

$$\|(r_\nu)^-\|_{L^1(\partial\omega)} = \|(r_\nu)^+\|_{L^1(\partial\omega)} = \frac{1}{2}\|r_\nu\|_{L^1(\partial\omega)}. \tag{4.1}$$

The conclusions follow.

We discuss now under which conditions this map is a norm on $\mathcal{R}$.

Definition 4.1 (Locked Domains) *Let ω be a bounded open domain with Lipschitz boundary in $\mathbb{R}^3$. It is said to be locked if the map $r \mapsto \|(r_\nu)^+\|_{L^1(\partial\omega)}$ is a norm on $\mathcal{R}$. Because of* (4.1)*, a domain is locked if and only if the only rigid motion tangent to its boundary is* 0*. Making use of the exponential map one can see that a domain is locked if and only if its isometry group is discrete.*

Consequently, the only domains which are not locked in $\mathbb{R}^3$ are euclidean balls, zones between two concentric euclidean spheres (the isometry group is SO_3), and domains of revolution around an axis (the isometry group is isomorphic to S^1). For a domain ω, $\mathcal{R}_\omega$ will denote the set of rigid motions which are tangent to $\partial\omega$. It is reduced to zero for locked domains.

Since the boundary of a domain ω can have several connected components, for ω to be locked, it is enough that one of these components not be of revolution. This component then locks the domain (e.g., the complement of a small ball in a larger cube). It may also be that several components are needed to lock the domain (e.g., the zone between two non intersecting spheres is not locked if the spheres are concentric, but is locked if they are not).

We now introduce notations for the classical moments of a vector field.

Definition 4.2 (Moments) *Let ω be a bounded open subset of $\mathbb{R}^3$ (resp. of a Lipschitz surface in $\mathbb{R}^3$). Let z be a point in $\mathbb{R}^3$ and $\mathbf{d}$ a unit vector in $\mathbb{R}^3$. For a vector field φ in $L^1(\omega)$ its (vector) moment $\mathcal{M}^z_\omega(\varphi)$ at the point z and its (scalar) moment $\mathcal{M}^{z,\mathbf{d}}_\omega(\varphi)$ with respect to the axis with direction $\mathbf{d}$ going through z are*

$$\mathcal{M}^z_\omega(\varphi) = \int_\omega (x-z)\wedge\varphi(x)\,\mathrm{d}\mu, \quad \mathcal{M}^{z,\mathbf{d}}_\omega(\varphi) = \int_\omega \mathbf{d}\cdot[(x-z)\wedge\varphi(x)]\,\mathrm{d}\mu,$$

where μ is the Lebesgue measure on ω (resp. the superficial measure on ω). When z is the origin, it will be omitted in the notation of the moments.

These functions are clearly linear continuous on the space $L^1(\omega)$, hence on the space $H^1(O)$ as soon as O contains ω as a subdomain (resp. when ω is an open subset of the boundary of a Lipschitz subdomain of O).

One can also use a measure different from the Lebesgue measure in ω (resp. the superficial measure on ω) provided it is also of finite energy (i.e., in the dual space of $H^1(\Omega)$) so that the moments are continuous on $H^1(O)$. In particular, one can use a density measure $\rho(x)\mathrm{d}x$ with ρ non-negative and in $L^2(O)$.

Note that if ω is an open subset of a sphere of center z and radius R, $\mathcal{M}^z_\omega(\varphi)$ is bounded in norm by $R\,\|\varphi_\tau\|_{L^1(\omega)}$ (recall that φ_τ is the tangential component of φ), and if it is an open subset of a surface of revolution of radius R around the axis going through z with direction $\mathbf{d}$, the absolute value of $\mathcal{M}^{z,\mathbf{d}}_\omega(\varphi)$ is bounded by $R\,\|\mathbf{d}\wedge\varphi_\tau\|_{L^1(\omega)}$ ($\mathbf{d}\wedge\varphi_\tau$ is the podal component of φ).

Moments are a source of semi-norms on $\mathcal{R}$.

Lemma 4.1 (*Norms and Semi-norms on* $\mathcal{R}$)

Locked domains *Let O be a locked domain and $\Sigma(O)$ a union of connected components of ∂O which locks O. Then,*

$$r \mapsto \|r\|_l \doteq \|(r_\nu)^+\|_{L^1(\Sigma(O))} \text{ is a norm on } \mathcal{R}. \tag{4.2}$$

For non-locked domains this map is only a semi-norm on $\mathcal{R}$.

Domains with spherical symmetry *Let Σ be an euclidean sphere centered at the origin (i.e., a connected component of the boundary of a domain with spherical symmetry with respect to the origin). Then, there is a constant C and a linear map $\mathbf{b}$ from $\mathcal{R}$ to $\mathbb{R}^3$ such that*

$$\|r - \mathbf{b}(r)\wedge \mathrm{Id}\,\|_{\mathcal{R}} \le C\,\|(r_\nu)^+\|_{L^1(\Sigma)}. \tag{4.3}$$

Furthermore, if ω is a bounded open subset of $\mathbb{R}^3$ (resp. of a Lipschitz surface in $\mathbb{R}^3$), then

$$r \mapsto \|r\|_{\mathcal{S}} \doteq \|(r_\nu)^+\|_{L^1(\Sigma)} + |\mathcal{M}_\omega(r1_\omega)| \text{ is a norm on } \mathcal{R}. \tag{4.4}$$

Domains of cylindrical revolution *Let O be an open domain of revolution around an axis going through the origin and with direction $\mathbf{d}$. Let Σ be a Lipschitz connected component of its boundary. Then, there is a constant C and a linear map ℓ from $\mathcal{R}$ to $\mathbb{R}$ such that*

$$\|r - \ell(r)\,\mathbf{d}\wedge \mathrm{Id}\,\|_{\mathcal{R}} \le C\,\|(r_\nu)^+\|_{L^1(\Sigma)}. \tag{4.5}$$

Furthermore, if ω is a bounded open subset of $\mathbb{R}^3$ (resp. of a Lipschitz surface in $\mathbb{R}^3$), then

$$r \mapsto \|r\|_{\mathcal{C}} \doteq \|(r_\nu)^+\|_{L^1(\Sigma)} + |\mathcal{M}^{\mathbf{d}}_\omega(r1_\omega)| \text{ is a norm on } \mathcal{R}. \tag{4.6}$$

Proof Locked domains First remark that every connected component of the boundary of (any domain) O is the boundary of a bounded domain (its "interior"). Therefore Proposition 4.2 applies for each component and $\|\cdot\|_l$ is a semi-norm on $\mathcal{R}$. If $\|r\|_l = 0$, it implies that $r_{|\Sigma(O)}$ is tangent on every component of $\Sigma(O)$. Since the latter is not globally of revolution, this implies that $r = 0$, so that $\|\cdot\|_l$ is a norm.

Domains with spherical symmetry Let Σ be a sphere centered at the origin. Here also, the map $\|\cdot\|_l$ is a semi-norm on $\mathcal{R}$ whose kernel consists of all the rigid motions which are tangent to Σ, namely $\mathcal{R}_\Sigma \doteq \{\mathfrak{b}\wedge \mathrm{Id} \,|\, \mathfrak{b}\in\mathbb{R}^3\}$. Therefore, it is a norm on an orthogonal of $\mathcal{R}_\Sigma$ in $\mathcal{R}$ (for this, any scalar product on the finite dimensional space $\mathcal{R}$ will do), and $\mathbf{b}\wedge\mathrm{Id}$ is simply the orthogonal projection on $\mathcal{R}_\Sigma$.

The map $\|\cdot\|_{\mathcal{S}}$ of formula (4.4) is also a semi norm. But if r is in its kernel, it is both tangent to Σ, hence of the form $\mathfrak{b}\wedge\mathrm{Id}$ and with vanishing moment on ω. This condition reads

$$\int_\omega x\wedge(\mathfrak{b}\wedge x)\mathrm{d}x = 0,$$

hence

$$\int_\omega \mathfrak{b}\cdot(x\wedge(\mathfrak{b}\wedge x))\mathrm{d}x = 0.$$

Now $\mathfrak{b}\cdot(x\wedge(\mathfrak{b}\wedge x)) = |x|^2|\mathfrak{b}|^2-(\mathfrak{b}\cdot x)^2$ which is non-negative and can only vanish for x collinear with $\mathfrak{b}$ if the latter is not zero. But this cannot hold for every x in ω (because it is of dimension at least 2). Therefore, $\mathfrak{b}$ has to be 0 and the kernel is reduced to 0.

Domains with cylindrical symmetry The reasoning is the same here. The elements of $\mathcal{R}_O$ are the rigid motions of the form $\{k\,\mathbf{d}\wedge\mathrm{Id}\,|\,k\in\mathbb{R}\}$. Imposing further that the moment with respect to $\mathbf{d}$ at the origin vanish reads

$$O = \int_\omega \mathbf{d}\cdot[x\wedge(k\,\mathbf{d}\wedge x)]\mathrm{d}x = k\int_\omega[|\mathbf{d}|^2|x|^2-(\mathbf{d}\cdot x)^2]\mathrm{d}x,$$

but, as above, the last integral vanishing implies $\mathbf{d}=0$.

The maps $\|\cdot\|_l, \|\cdot\|_{\mathcal{S}}$ and $\|\cdot\|_{\mathcal{C}}$ extends in the obvious way (and with the same notations) to the space $H^1(O)$ as Lipschitz functions, leading to simple corollaries of Theorem 3.1 which give some unilateral Korn inequalities.

Corollary 4.1 *Let O be a Korn-Wirtinger domain which is locked by a subset Σ of its boundary. Let ν be a choice of unit normal to Σ. Then, there exists a constant C such that the following generalized Korn inequality holds*[3]*:*

$$\forall v\in H^1(O),\quad \|v\|_{H^1(O)}\le C(\|e(v)\|_{L^2(O)}+\|(v_\nu)^+\|_{L^1(\Sigma)}). \tag{4.7}$$

Corollary 4.2 *Let O be a Korn-Wirtinger domain and Σ a spherical connected component of its boundary (for simplicity, centered at the origin). Then, there exists a constant C and a continuous linear map $\mathbf{b}: H^1(O)\to\mathbb{R}^3$ such that*

$$\forall v\in H^1(O),\quad \|v-\mathbf{b}(v)\wedge\mathrm{Id}\,\|_{H^1(O)}\le C(\|e(v)\|_{L^2(O)}+\|(v_\nu)^+\|_{L^1(\Sigma)}). \tag{4.8}$$

Moreover, if ω is a bounded open subset of O (resp. of the boundary of a Lipschitz subdomain of O), then there exists a constant C' such that

$$\forall v\in H^1(O),\quad \|v\|_{H^1(O)}\le C'(\|e(v)\|_{L^2(O)}+\|(v_\nu)^+\|_{L^1(\Sigma)}+\|\mathcal{M}_\omega(v)\|). \tag{4.9}$$

Corollary 4.3 *Let O be a Korn-Wirtinger domain and Σ a connected component of its boundary invariant under the rotations around the axis with unit vector $\mathbf{d}$ (for simplicity,*

[3]Since there are 2^k choices of ν (where k is the number of distinct connected components making up Σ) there are as many distinct inequalities!

through the origin). Then, there exists a constant C *and a continuous linear map* $\boldsymbol{\ell} : H^1(O) \to \mathbb{R}$ *such that*

$$\|v - \boldsymbol{\ell}(v)\mathbf{d} \wedge \mathrm{Id}\,\|_{H^1(O)} \leq C(\|e(v)\|_{L^2(O)} + \|(v_\nu)^+\|_{L^1(\Sigma)}). \tag{4.10}$$

Moreover, if ω *is a bounded open subset of* O *(resp. of a Lipschitz surface in* $\overline{O}$*), then there exists a constant* C' *such that*

$$\forall v \in H^1(O), \quad \|v\|_{H^1(O)} \leq C'(\|e(v)\|_{L^2(O)} + \|(v_\nu)^+\|_{L^1(\Sigma)} + \|\mathcal{M}^{\mathbf{d}}_\omega(v)\|). \tag{4.11}$$

Proof of Corollary 4.1 Theorem 3.1 applies with $F(u) \doteq \|u\|_l$ which is Lipschitz continuous on $H^1(O)$.

Proof of Corollary 4.2 The second part follows in the same way as above from the fact that the map $F(u) \doteq \|u\|_{\mathcal{S}}$ extends the map defined in (4.4) and is Lipschitz continuous on $H^1(O)$.

The proof of the first part goes as follows. Given u in $H^1(O)$ and using the definition of a Korn-Wirtinger domain, there is a $r(u)$ in $\mathcal{R}$ with inequality (2.4). Applying the map $\mathbf{b}$ given in (4.3) to $r(u)$ then gives (recall that all norms are equivalent on $\mathcal{R}$)

$$\|u - \mathbf{b}\big(r(u)\big) \wedge \mathrm{Id}\,\| \leq C_{\mathrm{KW}}\|e(u)\|_{L_2(O)} + C\,\|(r(u)_\nu)^+\|_{L^1(\Sigma)}.$$

From (2.4) again, and using the trace theorem on Σ,

$$\|(r(u)_\nu)^+\|_{L^1(\Sigma)} \leq \|(u_\nu)^+\|_{L^1(\Sigma)} + C_{\mathrm{KW}}\|e(u)\|_{L_2(O)}.$$

Combining these two inequalities gives inequality (4.8).

The proof of Corollary 4.3 is similar.

Acknowledgement The author expresses his thanks to the (anonymous) referee who found many typos and made several suggestions to improve the manuscript.

References

[1] Ciarlet, P. and Ciarlet, P. G., Another approach to linearized elasticity and a new proof of Korn's inequality, *Mathematical Models and Methods in Applied Sciences*, **15**(2), 2005, 259–271.

[2] Cioranescu, D., Damlamian, A. and Orlik, J., Homogenization via unfolding in periodic elasticity with contact on closed and open cracks, *Asymptotic Analysis*, **82**, 2013, 201–232.

[3] Damlamian, A., Some unilateral Korn inequalities with application to a contact problem with inclusions, *C. R. Acad. Sci. Paris, Ser. I*, **350**, 2012, 861–865.

[4] Eck, C., Jarušek, J. and Krbec, M., Unilateral Contact Problems: Variational Methods and Existence Theorems, CRC Press, Boca Raton, 2005.

[5] Fichera, G., Sul problema elastostatico di Signorini con ambigue condizioni al contorno, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **34**(8), 1963, 138–142 (in Italian).

[6] Fichera, G., Problemi elastostatici con vincoli unilaterali: Il problema di Signorini con ambigue condizioni al contorno, *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I,* **7**(8), 1963/1964, 91–140 (in Italian).

[7] Fichera, G., Elastostatics problems with unilateral constraints, *Séminaire Jean Leray*, **3**, 1966–1967, 64–68, http:// www.numdam.org/item?id=SJL_1966-1967___3_64_0.

[8] Fichera, G., Unilateral constraints in elasticity, *Actes, Congrès Intern. Math. Nice*, **3**, 1970, p. 7.

[9] Gobert, J., Une inégalité fondamentale de la théorie de l'élasticité, *Bull. Soc. Roy. Sci. Liège*, **31**, 1962, 182–191.

[10] Griso, G., Decompositions of displacements of thin structures, *J. Math. Pures Appl.,* **89**, 2008, 199–223.

[11] Hlaváček, I., Haslinger, J., Nečas J. and Lovíček, J., Solution of Variational Inequalities in Mechanics, Springer-Verlag, New York, Berlin, Heidelberg, London Paris, Tokyo, 1988.

[12] Kikuchi, N. and Oden, J. T., Contact problems in elasticity, *SIAM Studies in Applied Mathematics*, SIAM, Philadelphia, 1988.

[13] Oleinik, O. A., Shamaev, A. S. and Yosifian, G. A., Mathematical Problems in Elasticity and Homogenization, North-Holland, Amsterdam, 1992.