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Existence of Nonnegative Solutions for a Class of Systems Involving Fractional (p, q)-Laplacian Operators*

Yongqiang FU[1] Houwang LI[1] Patrizia PUCCI[2]

(*Dedicated to Professor Philippe G. Ciarlet on the occasion of his* 80*th birthday, with high feelings of esteem for his notable contributions in mathematics and great affection*)

Abstract The authors study the following Dirichlet problem of a system involving fractional (p, q)-Laplacian operators:

$$\begin{cases} (-\Delta)_p^s u = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-2}|v|^{\beta}u + \dfrac{\mu(x)}{\alpha\delta}|u|^{\gamma-2}|v|^{\delta}u & \text{in } \Omega, \\ (-\Delta)_q^s v = \lambda c(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta-2}v + \dfrac{\mu(x)}{\beta\gamma}|u|^{\gamma}|v|^{\delta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^N\backslash\Omega, \end{cases}$$

where $\lambda > 0$ is a real parameter, Ω is a bounded domain in $\mathbb{R}^N$, with boundary $\partial\Omega$ Lipschitz continuous, $s \in (0, 1)$, $1 < p \leq q < \infty$, $sq < N$, while $(-\Delta)_p^s u$ is the fractional p-Laplacian operator of u and, similarly, $(-\Delta)_q^s v$ is the fractional q-Laplacian operator of v. Since possibly $p \neq q$, the classical definitions of the Nehari manifold for systems and of the Fibering mapping are not suitable. In this paper, the authors modify these definitions to solve the Dirichlet problem above. Then, by virtue of the properties of the first eigenvalue λ_1 for a related system, they prove that there exists a positive solution for the problem when $\lambda < \lambda_1$ by the modified definitions. Moreover, the authors obtain the bifurcation property when $\lambda \to \lambda_1^-$. Finally, thanks to the Picone identity, a nonexistence result is also obtained when $\lambda \geq \lambda_1$.

Keywords The Nehari manifold, Fractional p-Laplacian, Variational methods
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1 Introduction

Recently, fractional p-Laplacian equations have been greatly studied, since they model several problems in Physics, Biology, Economics and so on. Thus, the research in this field has

Manuscript received August 7, 2017. Revised November 1, 2017.
[1]Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China.
E-mail: fuyongqiang@hit.edu.cn lhwmath@163.com
[2]Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, via Vanvitelli 1, 06123 Perugia, Italy. E-mail: patrizia.pucci@unipg.it
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received a wide attention. Franzina and Palatucci [1] considered an eigenvalue problem associated with the fractional p-Laplacian operator $(-\Delta)_p^s$. In particular, they obtained some useful properties of the first eigenvalue. Later, Iannizzotto and Squassina [2] established the Weyl type estimates for the asymptotic behavior of variational eigenvalues.

A classical method to solve the existence and multiplicity of solutions for the elliptic equations is the Nehari manifold technique, which has received great attention. In [3] Brown and Zhang considered the following Dirichlet problem:

$$\begin{cases} -\Delta u = \lambda a(x)u + b(x)|u|^{\nu-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the Nehari manifold and the Fibering mapping, defined by

$$\Lambda_\lambda = \{u \,:\, \langle I'_\lambda(u), u\rangle = 0\}, \quad \Phi_{\lambda,u}(t) = I_\lambda(tu), \tag{1.1}$$

they obtained the existence and the bifurcation via the Nehari manifold method in the scalar case.

Recently, Chen and Deng [4] yielded with the following fractional p-Laplacian system:

$$\begin{cases} (-\Delta)_p^s u = \lambda|u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta & \text{in } \Omega, \\ (-\Delta)_p^s v = \mu|u|^{q-2}u + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

where $(-\Delta)_p^s u$ is the fractional p-Laplacian operator of u. They defined the Nehari manifold and the Fibering mapping by

$$\Lambda_\lambda = \{(u,v) \,:\, \langle I'_\lambda(u,v), (u,v)\rangle = 0\}, \quad \Phi_{\lambda,u,v}(t) = I_\lambda(tu,tv) \tag{1.2}$$

and obtained the multiplicity of the solutions.

In [5], Zhang, Liu and Liu applied the same method to the (p,q)-Laplacian system:

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-2}|v|^\beta u + \dfrac{\mu(x)}{\alpha\delta}|u|^{\gamma-2}|v|^\delta u & \text{in } \Omega, \\ -\Delta_q v = \lambda c(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta-2}v + \dfrac{\mu(x)}{\beta\gamma}|u|^{\gamma}|v|^{\delta-2} v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

using the same definitions in (1.2). However, since in general $p \neq q$, the definitions in (1.2) are not suitable for problem (1.3) and there are some bugs in [5]. In this paper, we fix these bugs by modifying the definitions of the Nehari manifold and the Fibering mapping in (1.2), and furthermore we generalize the results to the fractional setting.

More specifically, we consider the problem

$$\begin{cases} (-\Delta)_p^s u = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-2}|v|^\beta u + \dfrac{\mu(x)}{\alpha\delta}|u|^{\gamma-2}|v|^\delta u & \text{in } \Omega, \\ (-\Delta)_q^s v = \lambda c(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta-2}v + \dfrac{\mu(x)}{\beta\gamma}|u|^{\gamma}|v|^{\delta-2} v & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.4}$$

where $\lambda > 0$ is a real parameter, Ω is a bounded domain in $\mathbb{R}^N$, with boundary $\partial\Omega$ Lipschitz continuous, $1 < p \leq q < \infty$, $sq < N$, $(-\Delta)_p^s u$ is the fractional p-Laplacian of u, that is,

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \mathrm{d}y,$$

where $B_\varepsilon(x)$ is the ball of $\mathbb{R}^N$ centered at x and of radius $\varepsilon > 0$. Of course, the fractional q-Laplacian $(-\Delta)_q^s v$ of v is defined in a similar way, simply replacing p by q. Initially, we study the associated eigenvalue problem:

$$\begin{cases} (-\Delta)_p^s u = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-2}|v|^{\beta}u & \text{in } \Omega, \\ (-\Delta)_q^s v = \lambda c(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

and obtain some properties of the first eigenvalue λ_1 of problem (1.5).

In this paper, we solve the question on the correct definitions in (1.2), since they are not suitable for the case $p \neq q$ in (1.4). We define the Nehari manifold and the Fibering mapping as

$$\Lambda_\lambda = \Big\{(u,v) : \Big\langle I'_\lambda(u,v), \Big(\frac{u}{p}, \frac{v}{q}\Big)\Big\rangle = 0\Big\}, \quad \Phi_{\lambda,u,v}(t) = I_\lambda(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v). \tag{1.5}$$

Then, we use Λ_λ in (1.5) related to (1.4), and prove that there exists a nonnegative nontrivial solution of (1.4) for all λ, with $0 < \lambda < \lambda_1$, under the natural conditions.

(H_1) The positive numbers α, β, γ and δ satisfy $\frac{\alpha}{p} + \frac{\beta}{q} = 1$, $1 < p < \gamma$, $1 < q < \delta$, $\frac{\gamma}{p_s^*} + \frac{\delta}{q_s^*} < 1$, $\frac{1}{\alpha\delta} + \frac{1}{\beta\gamma} < 1$, where p_s^*, q_s^* are the fractional Sobolev critical exponents $p_s^* = \frac{Np}{N-sp}$, $q_s^* = \frac{Nq}{N-sq}$.

(H_2) The functions a, b and c are nonnegative, smooth and of class $L^\infty(\Omega)$; moreover the sets $\Omega_1^+ = \{x \in \Omega : a(x) > 0\}$, $\Omega_2^+ = \{x \in \Omega : c(x) > 0\}$ have positive Lebesgue measure, that is $|\Omega_1^+| > 0$ and $|\Omega_2^+| > 0$, where $|\cdot|$ denotes the Lebesgue measure of a measurable set of $\mathbb{R}^N$;

(H_3) The measurable function μ may change sign in Ω and $\mu \in L^\infty(\Omega)$.

The natural solution space of (1.4) and (1.5) is $X = W^p \times W^q$, where W^p and W^q are the closed subspaces of $W^{s,p}(\mathbb{R}^N)$ and $W^{s,q}(\mathbb{R}^N)$, respectively, consisting of all functions in $W^{s,p}(\mathbb{R}^N)$ and $W^{s,q}(\mathbb{R}^N)$, respectively, which vanish in $\mathbb{R}^N \setminus \Omega$. More details on X are given in Section 2.

In the second part of the paper, we assume the further condition

$$\int_{\mathbb{R}^N} \mu(x) u_1^\gamma v_1^\delta \mathrm{d}x > 0, \tag{1.6}$$

where μ is the function given in (H_3), while $(u_1, v_1) \in X$ is a normalized positive eigenfunction of (1.5) associated to the first eigenvalue λ_1 of (1.5). Then, we obtain the bifurcation property for (1.4) as $\lambda \to \lambda_1^-$. Thanks to the Picone identity, we finally get a nonexistence result for (1.4) when $\lambda \geq \lambda_1$, provided that μ is nonnegative in Ω.

In addition to the papers already cited, the fractional (p,q)-Laplacian systems, mostly with the same $p = q$, have been widely studied. We refer to [5–7] and the references therein.

In the scalar case, for Dirichlet problems involving general integro-differential operator, with the structure of the fractional p-Laplacian, we cite [8], in which existence of unique weak solutions is proved by the direct method of the calculus of variations. Operators of the type treated in [8] can be used also in our context. We do not perform this extension here, and refer the interested reader to the general systems as explained in all details in Section 5 of [7].

The paper is so organized. In Section 2, we state some notations and preliminary results. In Section 3, we determine useful properties for the first eigenvalue λ_1 of (1.5). In Section 4, we prove the existence of solutions of (1.4) for all λ, with $0<\lambda<\lambda_1$ and the bifurcation property for (1.4) as $\lambda\to\lambda_1^-$. In Section 5, we establish a nonexistence result for (1.4) when $\lambda\geq\lambda_1$.

2 Notations and Preliminaries

Let us introduce for clarity some classical notations, referring to [4, 9] for further details. Let

$$W^{s,p}(\mathbb{R}^N)=\Big\{u\in L^p(\mathbb{R}^N)\,:\,\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\mathrm{d}y<\infty\Big\},$$
$$W^{s,q}(\mathbb{R}^N)=\Big\{u\in L^q(\mathbb{R}^N)\,:\,\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\,\mathrm{d}x\mathrm{d}y<\infty\Big\}$$

denote the standard fractional Sobolev spaces, endowed with the norms

$$\|u\|_{W^{s,p}(\mathbb{R}^N)}=\Big(\|u\|^p_{L^p(\mathbb{R}^N)}+\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{p}},$$
$$\|u\|_{W^{s,q}(\mathbb{R}^N)}=\Big(\|u\|^q_{L^q(\mathbb{R}^N)}+\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\,\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{q}}.$$

The subspaces $W^p=\{u\in W^{s,p}(\mathbb{R}^N):u|_{\mathbb{R}^N\setminus\Omega}\equiv 0\}$, $W^q=\{u\in W^{s,q}(\mathbb{R}^N):u|_{\mathbb{R}^N\setminus\Omega}\equiv 0\}$ of $W^{s,p}(\mathbb{R}^N)$ and $W^{s,q}(\mathbb{R}^N)$, respectively, are clearly closed and the norms

$$\|u\|_p=\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{p}},\quad \|u\|_q=\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\,\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{p}}$$

on W^p and W^q, respectively, are equivalent to $\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}$ and $\|\cdot\|_{W^{s,q}(\mathbb{R}^N)}$, since $\partial\Omega$ is Lipschitz continuous (see [10, Theorem 1.4.2.2]). The solution space $X=W^p\times W^q$, given in the Introduction, is equipped with the norm $\|(u,v)\|_X=\|u\|_p+\|v\|_q$. For the proof of the next lemma we refer to [4, 9, 11].

Lemma 2.1 (1) *The embeddings $W^q\hookrightarrow W^p\hookrightarrow L^\nu(\mathbb{R}^N)$ are continuous for any $\nu\in[1,p_s^*]$ and the latter is compact, whenever $\nu\in[1,p_s^*)$, since $1<p\leq q$ and $sq<N$.*

(2) *$X=(X,\|\cdot\|_X)$ is a real reflexive Banach space, while $W^p=(W^p,\|\cdot\|_p)$ and $W^q=(W^q,\|\cdot\|_q)$ are real uniformly convex Banach spaces.*

For convenience, for all $(u,v)\in X$ we introduce the linear functionals $B_p(u,\cdot):W^p\to\mathbb{R}$ and $B_q(v,\cdot):W^q\to\mathbb{R}$, defined by

$$B_p(u,\phi)=\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(\phi(x)-\phi(y))\,\mathrm{d}x\mathrm{d}y,$$
$$B_q(v,\psi)=\iint_{\mathbb{R}^{2N}}\frac{|v(x)-v(y)|^{q-2}(v(x)-v(y))}{|x-y|^{N+sq}}(\psi(x)-\psi(y))\,\mathrm{d}x\mathrm{d}y$$

for all $\phi\in W^p$ and all $\psi\in W^q$, respectively.

Thanks to the main assumptions (H_1)–(H_3), given in the Introduction, the next two definitions make sense.

Definition 2.1 *We say that $(u,v)\in X$ is a (weak) solution of problem (1.4), if*

$$\frac{\alpha}{p}B_p(u,\phi)+\frac{\beta}{q}B_q(v,\psi)=\frac{\alpha}{p}\lambda\Big(\int_{\mathbb{R}^N}a|u|^{p-2}u\phi\,\mathrm{d}x+\int_{\mathbb{R}^N}b|u|^{\alpha-2}|v|^{\beta}u\phi\,\mathrm{d}x\Big)$$

$$+\frac{\beta}{q}\lambda\Big(\int_{\mathbb{R}^N} c|v|^{q-2}v\psi\,\mathrm{d}x+\int_{\mathbb{R}^N} b|u|^{\alpha}|v|^{\beta-2}v\psi\,\mathrm{d}x\Big)$$
$$+\frac{1}{p\delta}\int_{\mathbb{R}^N}\mu|u|^{\gamma-2}|v|^{\delta}u\phi\,\mathrm{d}x+\frac{1}{q\gamma}\int_{\mathbb{R}^N}\mu|u|^{\gamma}|v|^{\delta-2}v\psi\,\mathrm{d}x$$

for any $(\phi,\psi)\in X$.

Definition 2.2 *We say that $(u,v)\in X$ is an* eigenfunction associated to λ *for problem* (1.5), *if*

$$\begin{aligned}\frac{\alpha}{p}B_p(u,\phi)+\frac{\beta}{q}B_q(v,\psi)=&\frac{\alpha}{p}\lambda\Big(\int_{\mathbb{R}^N} a|u|^{p-2}u\phi\,\mathrm{d}x+\int_{\mathbb{R}^N} b|u|^{\alpha-2}|v|^{\beta}u\phi\,\mathrm{d}x\Big)\\&+\frac{\beta}{q}\lambda\Big(\int_{\mathbb{R}^N} c|v|^{q-2}v\psi\,\mathrm{d}x+\int_{\mathbb{R}^N} b|u|^{\alpha}|v|^{\beta-2}v\psi\,\mathrm{d}x\Big)\end{aligned}$$

for any $(\phi,\psi)\in X$.

It is well known that problem (1.4) has a variational structure, i.e., (weak) solutions of problem (1.4) are exactly the critical points of the associated functional

$$I_\lambda(u,v)=J(u,v)-\lambda K(u,v)-M(u,v)$$

from X into $\mathbb{R}$, where

$$\begin{aligned}J(u,v)&=\frac{\alpha}{p}\|u\|_p^p+\frac{\beta}{q}\|v\|_q^q,\\K(u,v)&=\frac{\alpha}{p}\int_{\mathbb{R}^N}a(x)|u|^p\,\mathrm{d}x+\frac{\beta}{q}\int_{\mathbb{R}^N}c(x)|v|^q\,\mathrm{d}x+\int_{\mathbb{R}^N}b(x)|u|^{\alpha}|v|^{\beta}\,\mathrm{d}x,\\M(u,v)&=\frac{1}{\gamma\delta}\int_{\mathbb{R}^N}\mu(x)|u|^{\gamma}|v|^{\delta}\,\mathrm{d}x.\end{aligned}$$

Lemma 2.2 *The functional $J:X\to\mathbb{R}$ is weakly lower semicontinuous in X, and K, and M are compact in X.*

Proof The weak lower semicontinuity of J in X is obtained from the weak lower semicontinuity in X of the norms $\|\cdot\|_p$ and $\|\cdot\|_q$. Indeed, if $(u_n,v_n)\rightharpoonup(u,v)$ in X as $n\to\infty$, then $u_n\rightharpoonup u$ in W^p and $v_n\rightharpoonup v$ in W^q as $n\to\infty$, so that

$$\begin{aligned}J(u,v)&\le\frac{\alpha}{p}\liminf_{n\to\infty}\|u_n\|_p^p+\frac{\beta}{q}\liminf_{n\to\infty}\|v_n\|_q^q\\&\le\liminf_{n\to\infty}\Big(\frac{\alpha}{p}\|u_n\|_p^p+\frac{\beta}{q}\|v_n\|_q^q\Big)=\liminf_{n\to\infty}J(u_n,v_n).\end{aligned}$$

We refer to [12, Lemma 2.1] for a proof of the fact that K and M are compact in X, since the changes are obvious.

3 Some Properties of the First Eigenvalue

Assume that $u\ge 0$ and $v>0$. Put

$$R(u,v)=|\nabla_y^x u|^p-\nabla_y^x\Big(\frac{u^p}{v^{p-1}}\Big)|\nabla_y^x v|^{p-2}\nabla_y^x v,$$

where

$$\nabla_y^x u=u(x)-u(y).$$

Before getting the main result of this section, we give a lemma.

Lemma 3.1 *Assume that $\lambda > 0$ and that $(u_1, v_1) \in X$ is a positive vector function in Ω, such that for any $(\phi, \psi) \in X$,*

$$\begin{aligned}
&B_p(u_1, \phi) \le \lambda\Big(\int_{\mathbb{R}^N} a|u_1|^{p-2}u_1\phi\,\mathrm{d}x + \int_{\mathbb{R}^N} b|u_1|^{\alpha-2}|v_1|^{\beta}u_1\phi\,\mathrm{d}x\Big),\\
&B_q(v_1, \psi) \le \lambda\Big(\int_{\mathbb{R}^N} c|v_1|^{q-2}v_1\psi\,\mathrm{d}x + \int_{\mathbb{R}^N} b|u_1|^{\alpha}|v_1|^{\beta-2}v_1\psi\,\mathrm{d}x\Big),\\
&u_1 = v_1 = 0 \quad \text{in } \mathbb{R}^N\backslash\Omega.
\end{aligned}$$

Then for any $(u_2, v_2) \in X$, which is positive in Ω and satisfies the inequalities

$$\begin{aligned}
&B_p(u_2, \phi) \ge \lambda\Big(\int_{\mathbb{R}^N} a|u_2|^{p-2}u_2\phi\,\mathrm{d}x + \int_{\mathbb{R}^N} b|u_2|^{\alpha-2}|v_2|^{\beta}u_2\phi\,\mathrm{d}x\Big),\\
&B_q(v_2, \psi) \ge \lambda\Big(\int_{\mathbb{R}^N} c|v_2|^{q-2}v_2\psi\,\mathrm{d}x + \int_{\mathbb{R}^N} b|u_2|^{\alpha}|v_2|^{\beta-2}v_2\psi\,\mathrm{d}x\Big),\\
&u_2 = v_2 = 0 \quad \text{in } \mathbb{R}^N\backslash\Omega
\end{aligned}$$

for any $(\phi, \psi) \in X$, there exists a constant $C > 0$ such that $(u_2, v_2) = (C^{\frac{1}{p}}u_1, C^{\frac{1}{q}}v_1)$.

Proof By the Picone identity (see [13, Theorem 6.2]) we get

$$\begin{aligned}
0 &\le \iint_{\mathbb{R}^{2N}} \frac{R(u_1, u_2)}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y = \|u_1\|_p^p - B_p\Big(u_2, \frac{u_1^p}{u_2^{p-1}}\Big)\\
&\le \|u_1\|_p^p - \lambda\int_{\mathbb{R}^N} \frac{u_1^p}{u_2^{p-1}}(au_2^{p-1} + bu_2^{\alpha-1}v_2^{\beta})\,\mathrm{d}x\\
&\le \lambda\int_{\mathbb{R}^N} b(u_1^{\alpha}v_1^{\beta} - u_1^p u_2^{\alpha-p}v_2^{\beta})\,\mathrm{d}x. \qquad (3.1)
\end{aligned}$$

Using the same method, we have

$$0 \le \lambda\int_{\mathbb{R}^N} b(u_1^{\alpha}v_1^{\beta} - v_1^q u_2^{\alpha}v_2^{\beta-q})\,\mathrm{d}x. \qquad (3.2)$$

Therefore, multiplying (3.1) by $\frac{\alpha}{p}$ and (3.2) by $\frac{\beta}{q}$ and adding them, we get

$$0 \le \int_{\mathbb{R}^N} b\Big(u_1^{\alpha}v_1^{\beta} - \frac{\alpha}{p}u_1^p u_2^{\alpha-p}v_2^{\beta} - \frac{\beta}{q}v_1^q u_2^{\alpha}v_2^{\beta-q}\Big)\,\mathrm{d}x, \qquad (3.3)$$

since $\frac{\alpha}{p} + \frac{\beta}{q} = 1$ by (H_1) and $\lambda > 0$. Now, put $\theta_1 = \frac{\alpha\beta}{q}$ and $\theta_2 = \frac{\alpha\beta}{p}$, then

$$\begin{aligned}
u_1^{\alpha}v_1^{\beta} &= u_1^{\alpha}u_2^{-\theta_1}v_2^{\theta_2} \times v_1^{\beta}u_2^{\theta_1}v_2^{-\theta_2}\\
&\le \frac{\alpha}{p}u_1^p u_2^{-\frac{p\theta_1}{\alpha}}v_2^{\frac{p\theta_2}{\alpha}} + \frac{\beta}{q}v_1^q u_2^{\frac{q\theta_1}{\beta}}v_2^{-\frac{q\theta_2}{\beta}}\\
&= \frac{\alpha}{p}u_1^p u_2^{\alpha-p}v_2^{\beta} + \frac{\beta}{q}v_1^q u_2^{\alpha}v_2^{\beta-q}
\end{aligned}$$

by the Young inequality, since $\frac{\alpha}{p} + \frac{\beta}{q} = 1$ by (H_1). Hence the integral in (3.3) is zero and so, in particular by (3.1)–(3.2),

$$0 \le \frac{\alpha}{p}\int_{\mathbb{R}^N} b(u_1^{\alpha}v_1^{\beta} - u_1^p u_2^{\alpha-p}v_2^{\beta})\,\mathrm{d}x = \frac{\beta}{q}\int_{\mathbb{R}^N} b(v_1^q u_2^{\alpha}v_2^{\beta-q} - u_1^{\alpha}v_1^{\beta})\,\mathrm{d}x \le 0.$$

In other words, $u_1^p u_2^{\alpha-p} v_2^\beta = v_1^q u_2^\alpha v_2^{\beta-q} = u_1^\alpha v_1^\beta$. In conclusion,

$$\Big(\frac{u_1}{u_2}\Big)^p = \Big(\frac{v_1}{v_2}\Big)^q = \Big(\frac{u_1}{u_2}\Big)^\alpha \cdot \Big(\frac{v_1}{v_2}\Big)^\beta = C > 0$$

and in turn $(u_2, v_2) = (C^{\frac{1}{p}} u_1, C^{\frac{1}{q}} v_1)$.

Theorem 3.1 *Let* $\lambda_1 = \inf\limits_{K(u,v)=1} J(u,v)$. *Then*

(1) $\lambda_1 > 0$;

(2) *there exists* $(u_1, v_1) \in X$ *which is the eigenfunction associated to* λ_1 *of the problem* (1.5) *and* (u_1, v_1) *is positive in* Ω;

(3) *the eigenspace associated to* λ_1 *is simple, that is, the first eigenspace*

$$E_1 = \{(u,v) \in X \,:\, \text{there exists } C > 0 \text{ such that } (u,v) = (C^{\frac{1}{p}} u_1, C^{\frac{1}{q}} v_1)\}$$

has dimension 1;

(4) λ_1 *is the only eigenvalue of problem* (1.5) *whose eigenfunctions are positive in* Ω.

Proof (1) It is easy to see that

$$\begin{aligned} \lambda_1 &= \inf_{\substack{(u,v)\in X\\(u,v)\neq(0,0)}} \frac{\frac{\alpha}{p}\|u\|_p^p + \frac{\beta}{q}\|v\|_q^q}{\frac{\alpha}{p}\int_{\mathbb{R}^N} a|u|^p \,\mathrm{d}x + \frac{\beta}{q}\int_{\mathbb{R}^N} c|v|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} b|u|^\alpha |v|^\beta \,\mathrm{d}x} \\ &= \inf_{\substack{(u,v)\in X\\(u,v)\neq(0,0)}} \frac{J(u,v)}{K(u,v)} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} K(u,v) &\le \frac{\alpha}{p}\|a\|_{L^\infty}\|u\|_{L^p}^p + \frac{\beta}{q}\|c\|_{L^\infty}\|v\|_{L^q}^q + \|b\|_{L^\infty}\|u\|_{L^p}^\alpha \|v\|_{L^q}^\beta \\ &\le \frac{\alpha}{p}(\|a\|_{L^\infty} + \|b\|_{L^\infty})\|u\|_{L^p}^p + \frac{\beta}{q}(\|c\|_{L^\infty} + \|b\|_{L^\infty})\|v\|_{L^q}^q \\ &\le \max\{(\|a\|_{L^\infty} + \|b\|_{L^\infty})S_p, (\|b\|_{L^\infty} + \|c\|_{L^\infty})S_q\} J(u,v). \end{aligned}$$

Thus, we have

$$\lambda_1 \ge \frac{1}{\max\{(\|a\|_{L^\infty} + \|b\|_{L^\infty})S_p, (\|b\|_{L^\infty} + \|c\|_{L^\infty})S_q\}} > 0,$$

where S_p, S_q are the best constants of the Sobolev embeddings from W^p into $L^p(\mathbb{R}^N)$ and from W^q into $L^q(\mathbb{R}^N)$, respectively.

(2) Assume that $\{(U_n, V_n)\}_n$ is a sequence in X such that $K(U_n, V_n) = 1$ for all n in $\mathbb{N}$ and $J(U_n, V_n) \to \lambda_1$ as $n \to \infty$. Then $\{(U_n, V_n)\}_n$ is bounded in X, so that, without loss of generality, we may assume that $(U_n, V_n) \rightharpoonup (u_1, v_1)$ in X for some (u_1, v_1) of X. In particular, $K(u_1, v_1) = \lim\limits_{n\to\infty} K(U_n, V_n) = 1$, and $J(u_1, v_1) \le \liminf\limits_{n\to\infty} J(U_n, V_n) = \lambda_1$ by virtue of Lemma 2.2. Hence, $J(u_1, v_1) = \lambda_1$.

On the other hand, (u_1, v_1) is a conditional extremum of J, so that by the Lagrange multiplier method we have

$$L'_{(u,v)}(u_1, v_1, \lambda) = \langle J'(u_1, v_1) - \lambda K'(u_1, v_1), (\phi, \psi)\rangle = 0$$

for all $(\phi, \psi) \in X$, where

$$L(u, v, \lambda) = J(u, v) - \lambda[K(u, v) - 1].$$

Let $(\phi, \psi) = \left(\frac{u_1}{p}, \frac{v_1}{q}\right)$, then $\lambda = \lambda_1$. Furthermore, $(u_1, v_1) \in X$ is the eigenfunction associated to λ_1 of the eigenvalue problem (1.5). Since $J(|u|, |v|) = J(u, v)$, $K(|u|, |v|) = K(u, v)$ for all $(u, v) \in X$, (u_1, v_1) must be nonnegative. Since $(-\Delta)_p^s u_1 \geq 0$ and $(-\Delta)_q^s v_1 \geq 0$ in Ω, with $u_1 = v_1 = 0$ in $\mathbb{R}^N \backslash \Omega$, thanks to the strong maximum principle, as given in [14, Lemma 2.3] and in [15, Theorem 1.2], we get that either $u_1 > 0$ or $u_1 \equiv 0$ in Ω and either $v_1 > 0$ or $v_1 \equiv 0$ in Ω. Thus, the only possible case is that (u_1, v_1) is positive in X by symmetry.

(3) Applying Lemma 3.1 to (u_1, v_1) of part (2), with $\lambda = \lambda_1$, we obtain that for any eigenfunction $(u_2, v_2) \in X$ associated to λ_1 which is positive in Ω, there exists a constant $C > 0$ such that $(u_2, v_2) = (C^{\frac{1}{p}} u_1, C^{\frac{1}{q}} v_1)$.

(4) Assume that (u_2, v_2) is an eigenfunction associated to $\widetilde{\lambda}$ which is positive in X. If $\widetilde{\lambda} \geq \lambda_1$, then, according to Lemma 3.1, there exists a constant $C > 0$ such that $(u_2, v_2) = (C^{\frac{1}{p}} u_1, C^{\frac{1}{q}} v_1)$, that is, $\widetilde{\lambda} = \lambda_1$ by part (3). Similarly, for the case $\widetilde{\lambda} \leq \lambda_1$, we get $\widetilde{\lambda} = \lambda_1$.

4 The Case $0 < \lambda < \lambda_1$

In this section, we use the Nehari manifold method to prove the existence and bifurcation of the solutions for problem (1.4). First, put

$$\Lambda_\lambda = \Big\{(u, v) \in X \ : \ \Big\langle I'_\lambda(u, v), \Big(\frac{u}{p}, \frac{v}{q}\Big)\Big\rangle = 0\Big\}.$$

Clearly, Λ_λ is a closed subset of X, and all critical points of I_λ are in Λ_λ. We continue to call Λ_λ a Nehari manifold even if Λ_λ may not be a manifold. It is easy to see that $(u, v) \in \Lambda_\lambda$ if and only if

$$J(u, v) - \lambda K(u, v) - \Big(\frac{\gamma}{p} + \frac{\delta}{q}\Big) M(u, v) = 0. \tag{4.1}$$

Hence, for $(u, v) \in \Lambda_\lambda$,

$$I_\lambda(u, v) = \Big(\frac{\gamma}{p} + \frac{\delta}{q} - 1\Big) M(u, v).$$

The Nehari manifold Λ_λ can be described by the Fibering mapping, defined for all $t > 0$ by

$$\begin{aligned} \Phi_{\lambda,u,v}(t) &= I_\lambda(t^{\frac{1}{p}} u, t^{\frac{1}{q}} v) \\ &= t[J(u, v) - \lambda K(u, v)] - t^{\frac{\gamma}{p} + \frac{\delta}{q}} M(u, v). \end{aligned} \tag{4.2}$$

Therefore, for all $t > 0$,

$$\Phi'_{\lambda,u,v}(t) = J(u, v) - \lambda K(u, v) - \Big(\frac{\gamma}{p} + \frac{\delta}{q} - 1\Big) t^{\frac{\gamma}{p} + \frac{\delta}{q} - 1} M(u, v).$$

With this starting we are now in a position to prove the following lemma.

Lemma 4.1 *If $(u, v) \in X \backslash \{(0, 0)\}$ and $t > 0$, then $(t^{\frac{1}{p}} u, t^{\frac{1}{q}} v) \in \Lambda_\lambda$ if and only if $\Phi'_{\lambda,u,v}(t) = 0$.*

Proof The result is an immediate consequence of the fact that

$$\Phi'_{\lambda,u,v}(t) = \Big\langle I'_\lambda(t^{\frac{1}{p}} u, t^{\frac{1}{q}} v), \Big(t^{\frac{1}{p}} \frac{u}{p}, t^{\frac{1}{q}} \frac{v}{q}\Big)\Big\rangle,$$

since $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Remark 4.1 From $\Phi'_{\lambda,u,v}(t) = 0$, we obtain at once

$$t_{\lambda,u,v} = t = \Big(\frac{J(u,v) - \lambda K(u,v)}{(\frac{\gamma}{p} + \frac{\delta}{q})M(u,v)}\Big)^{\frac{1}{\frac{\gamma}{p}+\frac{\delta}{q}-1}}.$$

Hence, if $(u,v) \in X\backslash\{(0,0)\}$ and $t_{\lambda,u,v} > 0$, then $(t^{\frac{1}{p}}_{\lambda,u,v}u, t^{\frac{1}{q}}_{\lambda,u,v}v) \in \Lambda_\lambda$ by Lemma 4.1.

The elements of Λ_λ correspond to the stationary points of the maps $\Phi_{\lambda,u,v}(1)$ by Lemma 4.1, i.e.,

$$(u,v) \in \Lambda_\lambda \quad \text{if and only if} \quad \Phi'_{\lambda,u,v}(1) = 0.$$

Hence it is natural to divide Λ_λ into three subsets Λ^+_λ, Λ^-_λ and Λ^0_λ corresponding to the local minima, the local maxima and the saddle points of the fibering mapping. In other words,

$$\begin{aligned}
\Lambda^+_\lambda &= \{(u,v) \in \Lambda_\lambda \,:\, \Phi''_{\lambda,u,v}(1) > 0\},\\
\Lambda^-_\lambda &= \{(u,v) \in \Lambda_\lambda \,:\, \Phi''_{\lambda,u,v}(1) < 0\},\\
\Lambda^0_\lambda &= \{(u,v) \in \Lambda_\lambda \,:\, \Phi''_{\lambda,u,v}(1) = 0\}.
\end{aligned}$$

Now

$$\Phi''_{\lambda,u,v}(1) = \Big(\frac{\gamma}{p} + \frac{\delta}{q}\Big)\Big(1 - \frac{\gamma}{p} - \frac{\delta}{q}\Big)M(u,v).$$

Consequently, since $\frac{\gamma}{p} + \frac{\delta}{q} > 1$ by (H_1),

$$\begin{aligned}
\Lambda^+_\lambda &= \{(u,v) \in \Lambda_\lambda \,:\, M(u,v) < 0\},\\
\Lambda^-_\lambda &= \{(u,v) \in \Lambda_\lambda \,:\, M(u,v) > 0\},\\
\Lambda^0_\lambda &= \{(u,v) \in \Lambda_\lambda \,:\, M(u,v) = 0\}.
\end{aligned}$$

We shall prove the existence of solutions of problem (1.4) by investigating the existence of minimizers of the functional I_λ on Λ_λ. Although Λ_λ is a small subset of X, we shall see that local minimizers on the Nehari manifold Λ_λ are the usual critical points of I_λ in X. Indeed, we have the following result.

Lemma 4.2 *If (u_0, v_0) is a local minimizer of I_λ on Λ_λ and $(u_0, v_0) \notin \Lambda^0_\lambda$, then (u_0, v_0) is a critical point of I_λ in X.*

Proof It is enough to use the same method used in the proof of [4, Lemma 2.2] to obtain the desired conclusion, with obvious changes.

To achieve a detailed characterization of the sets Λ^+_λ, Λ^-_λ and Λ^0_λ, we start by proving a lemma.

Lemma 4.3 *For any λ, with $0 < \lambda < \lambda_1$, there exists a constant $\mu_\lambda > 0$ such that*

$$J(u,v) - \lambda K(u,v) \ge \mu_\lambda \int_{\mathbb{R}^N} (|u|^p + |v|^q)\,\mathrm{d}x \tag{4.3}$$

for all $(u,v) \in X\backslash\{(0,0)\}$.

Proof Otherwise, for any $n \in \mathbb{N}$ there exists $(u_n, v_n) \in X\backslash\{(0,0)\}$ such that

$$J(u_n, v_n) - \lambda K(u_n, v_n) < \frac{1}{n} \int_{\mathbb{R}^N} (|u_n|^p + |v_n|^q)\,\mathrm{d}x.$$

Let $C_n = \|u_n\|_p^p + \|v_n\|_q^q$, $\widetilde{u}_n = C_n^{-\frac{1}{p}} u_n$, $\widetilde{v}_n = C_n^{-\frac{1}{q}} v_n$, which is possible, since $C_n > 0$. Thus, $\|\widetilde{u}_n\|_p^p + \|\widetilde{v}_n\|_q^q = 1$ for all $n \in \mathbb{N}$. Then, up to a subsequence, still denoted by $\{(\widetilde{u}_n, \widetilde{v}_n)\}_n$, we get that $(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (\widetilde{u}_0, \widetilde{v}_0)$ in X for some $(\widetilde{u}_0, \widetilde{v}_0)$ of X. Therefore, $\widetilde{u}_n \to \widetilde{u}_0$ in $L^p(\mathbb{R}^N)$, $\widetilde{v}_n \to \widetilde{v}_0$ in $L^q(\mathbb{R}^N)$ by Lemma 2.1(1).

On the other hand, putting $S = \max\{S_p, S_q\}$, where S_p, S_q are the best constants of the Sobolev embeddings from W^p into $L^p(\mathbb{R}^N)$ and from W^q into $L^q(\mathbb{R}^N)$, respectively, we have

$$\begin{aligned}
\frac{S}{n} &\geq \frac{1}{n}\int_{\mathbb{R}}^{N} (|\widetilde{u}_n|^p + |\widetilde{v}_n|^q)\,\mathrm{d}x \\
&> J(\widetilde{u}_n, \widetilde{v}_n) - \lambda K(\widetilde{u}_n, \widetilde{v}_n) \\
&= \frac{\lambda}{\lambda_1}\Big[\frac{\lambda_1}{\lambda} J(\widetilde{u}_n, \widetilde{v}_n) - \lambda_1 K(\widetilde{u}_n, \widetilde{v}_n)\Big] \\
&= \frac{\lambda}{\lambda_1}\Big[\Big(\frac{\lambda_1}{\lambda} - 1\Big) J(\widetilde{u}_n, \widetilde{v}_n) + J(\widetilde{u}_n, \widetilde{v}_n) - \lambda_1 K(\widetilde{u}_n, \widetilde{v}_n)\Big] \\
&\geq \frac{\lambda}{\lambda_1}\Big(\frac{\lambda_1}{\lambda} - 1\Big)\min\Big\{\frac{\alpha}{p}, \frac{\beta}{q}\Big\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$0 \geq \frac{\lambda}{\lambda_1}\Big(\frac{\lambda_1}{\lambda} - 1\Big)\min\Big\{\frac{\alpha}{p}, \frac{\beta}{q}\Big\} > 0,$$

which is the required contradiction.

Now, if $\lambda \in (0, \lambda_1)$ Lemma 4.3 ensures that $\Lambda_\lambda^+ = \emptyset$ and $\Lambda_\lambda^0 = \{(0,0)\}$ by (4.1) and (4.3). Therefore, $\Lambda_\lambda = \Lambda_\lambda^- \cup \{(0,0)\}$, and $I_\lambda(u,v) > 0$ for all $(u,v) \in \Lambda_\lambda^-$. Hence, $\inf\limits_{\Lambda_\lambda^-} I_\lambda(u,v) \geq 0$.

Lemma 4.4 *If $0 < \lambda < \lambda_1$, then I_λ is coercive on Λ_λ.*

Proof Fix $\lambda \in (0, \lambda_1)$. For all $(u,v) \in \Lambda_\lambda$, using (4.1) and (4.3), we have

$$\Big(\frac{\gamma}{p} + \frac{\delta}{q}\Big) M(u,v) \geq \mu_\lambda \int_{\mathbb{R}^N} (|u|^p + |v|^q)\,\mathrm{d}x,$$

so that, since $\frac{\gamma}{p} + \frac{\delta}{q} > 1$ by (H_1), we get

$$\begin{aligned}
I_\lambda(u,v) &= \Big(\frac{\gamma}{p} + \frac{\delta}{q} - 1\Big) M(u,v) = \Big(\frac{\gamma}{p} + \frac{\delta}{q}\Big)\Big(1 - \frac{1}{\frac{\gamma}{p} + \frac{\delta}{q}}\Big) M(u,v) \\
&\geq \Big(1 - \frac{1}{\frac{\gamma}{p} + \frac{\delta}{q}}\Big)\mu_\lambda \int_{\mathbb{R}^N} (|u|^p + |v|^q)\,\mathrm{d}x.
\end{aligned}$$

The Hölder inequality gives

$$\begin{aligned}
\frac{\alpha}{p}\|u\|_p^p + \frac{\beta}{q}\|v\|_q^q &= \Big(\frac{\alpha}{p}\lambda \int_{\mathbb{R}^N} a|u|^p\,\mathrm{d}x + \frac{\beta}{q}\lambda \int_{\mathbb{R}^N} c|v|^q\,\mathrm{d}x + \int_{\mathbb{R}^N} b|u|^\alpha |v|^\beta\,\mathrm{d}x\Big) \\
&\quad + \Big(\frac{\gamma}{p} + \frac{\delta}{q}\Big) M(u,v) \\
&\leq \lambda \max\Big\{\frac{\alpha}{p}(\|a\|_{L^\infty} + \|b\|_{L^\infty}), \frac{\beta}{q}(\|b\|_{L^\infty} + \|c\|_{L^\infty})\Big\} \int_{\mathbb{R}^N} (|u|^p + |v|^q)\,\mathrm{d}x \\
&\quad + \frac{\frac{\gamma}{p} + \frac{\delta}{q}}{\frac{\gamma}{p} + \frac{\delta}{q} - 1} I_\lambda(u,v)
\end{aligned}$$

$$\le C_{p,q}\, I_\lambda(u,v),$$

where

$$C_{p,q} = \frac{\frac{\gamma}{p}+\frac{\delta}{q}}{\frac{\gamma}{p}+\frac{\delta}{q}-1}\Big(\frac{\lambda}{\mu_\lambda}\max\Big\{\frac{\alpha}{p}(\|a\|_{L^\infty}+\|b\|_{L^\infty}),\frac{\beta}{q}(\|b\|_{L^\infty}+\|c\|_{L^\infty})\Big\}+1\Big)>0.$$

Hence I_λ is coercive on Λ_λ.

Theorem 4.1 *Assume that* (H_1)–(H_3) *hold and let* $0<\lambda<\lambda_1$*. Then there exists a positive solution of problem* (1.4).

Proof Let $\{(u_n,v_n)\}_n \subset \Lambda_\lambda^-$ be a minimizing sequence of I_λ in Λ_λ^-, i.e.,

$$\lim_{n\to\infty} I_\lambda(u_n,v_n) = \inf_{\Lambda_\lambda^-} I_\lambda(u,v).$$

Since I_λ is coercive in $\Lambda_\lambda = \Lambda_\lambda^- \cup \{(0,0)\}$ by Lemma 4.4, $\{(u_n,v_n)\}_n$ is bounded in X. Thus, passing if necessary to a subsequence still denoted by $\{(u_n,v_n)\}_n$, we get $(u_n,v_n)\rightharpoonup(u_0,v_0)$ in X for some $(u_0,v_0)\in X$. Again $u_n\to u_0$ in $L^p(\mathbb{R}^N)$ and $v_n\to v_0$ in $L^q(\mathbb{R}^N)$ by Lemma 2.1(1). Passing if necessary to another subsequence, we assume that

$$\|(u_0,v_0)\|_X \le \lim_{n\to\infty}\|(u_n,v_n)\|_X.$$

Hence, in particular,

$$\|u_0\|_p \le \lim_{n\to\infty}\|u_n\|_p, \quad \|v_0\|_q \le \lim_{n\to\infty}\|v_n\|_q. \tag{4.4}$$

First, we assert that $\inf_{\Lambda_\lambda^-} I_\lambda(u,v)>0$. Indeed, suppose $\inf_{\Lambda_\lambda^-} I_\lambda(u,v)=0$, i.e.,

$$I_\lambda(u_n,v_n)\to 0, \quad M(u_n,v_n)=\int_{\mathbb{R}^N}\mu(x)|u_n|^\gamma|v_n|^\delta\,\mathrm{d}x\to 0$$

as $n\to\infty$. By (4.1) and Lemma 2.2, we have

$$0\le J(u_0,v_0)-\lambda K(u_0,v_0)\le \liminf_{n\to\infty} J(u_n,v_n)-\lambda\lim_{n\to\infty}K(u_n,v_n)=0.$$

Hence $(u_0,v_0)=(0,0)$. Consequently, using also (4.4), we have

$$0=J(0,0)=J(u_0,v_0)=\frac{\alpha}{p}\lim_{n\to\infty}\|u_n\|_p^p+\frac{\beta}{q}\lim_{n\to\infty}\|v_n\|_q^q.$$

Therefore, $(u_n,v_n)\to(0,0)$ in X by Lemma 2.1(2).

Let $C_n=\|u_n\|_p^p+\|v_n\|_q^q$, $\widetilde{u}_n=C_n^{-\frac{1}{p}}u_n$, $\widetilde{v}_n=C_n^{-\frac{1}{q}}v_n$, so that $\|\widetilde{u}_n\|_p^p+\|\widetilde{v}_n\|_q^q=1$ for all $n\in\mathbb{N}$. We may assume that $(\widetilde{u}_n,\widetilde{v}_n)\rightharpoonup(\widetilde{u}_0,\widetilde{v}_0)$ in X as $n\to\infty$ for some $(\widetilde{u}_0,\widetilde{v}_0)\in X$. Put $\theta=\frac{\gamma}{p_s^*}+\frac{\delta}{q_s^*}\in(0,1)$ by (H_1). Hence, $\theta p_s^*>p$ and $\theta q_s^*>q$, so that

$$\begin{aligned}
C_n^{-1}M(u_n,v_n) &\le \frac{\|\mu\|_{L^\infty}\|u_n\|_{\theta p_s^*}^\gamma\|v_n\|_{\theta q_s^*}^\delta}{\|u_n\|_p^p+\|v_n\|_q^q}\\
&\le \|\mu\|_{L^\infty}\big(S_{\theta p_s^*}^{\frac{\gamma}{p}}\|u_n\|_p^{\gamma-p}+S_{\theta q_s^*}^{\frac{\delta}{q}}\|v_n\|_q^{\delta-q}\big)\\
&\to 0
\end{aligned}$$

as $n \to \infty$, since $1 < p < \gamma$ and $1 < q < \delta$ by (H_1). By (4.1), we get

$$J(\widetilde{u}_n, \widetilde{v}_n) - \lambda K(\widetilde{u}_n, \widetilde{v}_n) = \Big(\frac{\gamma}{p} + \frac{\delta}{q}\Big)\frac{1}{C_n} M(u_n, v_n) \to 0 \tag{4.5}$$

as $n \to \infty$. Therefore,

$$0 \le J(\widetilde{u}_0, \widetilde{v}_0) - \lambda K(\widetilde{u}_0, \widetilde{v}_0) \le \lim_{n\to\infty} (J(\widetilde{u}_n, \widetilde{v}_n) - \lambda K(\widetilde{u}_n, \widetilde{v}_n)) = 0,$$

and so $(\widetilde{u}_0, \widetilde{v}_0) = (0, 0)$. Combining with the compactness of K, given in Lemma 2.2, we obtain that $J(\widetilde{u}_n, \widetilde{v}_n) \to 0$ as $n \to \infty$. Therefore, $(\widetilde{u}_n, \widetilde{v}_n) \to (\widetilde{u}_0, \widetilde{v}_0) = (0, 0)$ in X. This contradicts the fact that $\|\widetilde{u}_n\|_p^p + \|\widetilde{v}_n\|_q^q = 1$ for all $n \in \mathbb{N}$ and shows the assertion $\inf\limits_{\Lambda_\lambda^-} I_\lambda(u, v) > 0$.

Now we prove that $(u_n, v_n) \to (u_0, v_0)$ in X as $n \to \infty$. We claim that the limit $(u_0, v_0) \neq (0, 0)$. Otherwise,

$$J(u_n, v_n) = \lambda K(u_n, v_n) + \Big(\frac{\gamma}{p} + \frac{\delta}{q}\Big)M(u_n, v_n) \to 0$$

as $n \to \infty$. This contradicts that $\inf\limits_{\Lambda_\lambda^-} I(u, v) > 0$. By Remark 4.1, there exists $t_0 > 0$ such that $(t_0^{\frac{1}{p}} u_0, t_0^{\frac{1}{q}} v_0) \in \Lambda_\lambda^-$.

If $(u_n, v_n) \nrightarrow (u_0, v_0)$ as $n \to \infty$, then in (4.4) at least one inequality should hold with the strict sign. Otherwise, if both limits in (4.4) are valid with the equality sign, then by Lemma 2.1(2) this would imply that $(u_n, v_n) \to (u_0, v_0)$ in X as $n \to \infty$. Hence

$$\begin{aligned}
\liminf_{n\to\infty} \Phi'_{\lambda,u_n,v_n}(t_0) &= \liminf_{n\to\infty} \Big[J(u_n, v_n) - \lambda K(u_n, v_n) - \Big(\frac{\gamma}{p} + \frac{\delta}{q} - 1\Big)t_0^{\frac{\gamma}{p}+\frac{\delta}{q}-1} M(u_n, v_n)\Big] \\
&> J(u_0, v_0) - \lambda K(u_0, v_0) - \Big(\frac{\gamma}{p} + \frac{\delta}{q} - 1\Big)t_0^{\frac{\gamma}{p}+\frac{\delta}{q}-1} M(u_0, v_0) \\
&= \Phi'_{\lambda,u_0,v_0}(t_0) = 0
\end{aligned}$$

by Lemma 2.2. That is, $\Phi'_{\lambda,u_n,v_n}(t_0) > 0$ for n large enough. Since $(u_n, v_n) \in \Lambda_\lambda^-$ for all n, one has $\Phi''_{\lambda,u_n,v_n}(t) < 0$ for all $t > 0$, and $\Phi'_{\lambda,u_n,v_n}(1) = 0$ for all n. Thus we must have $0 < t_0 < 1$. On the other hand, since $\Phi'_{\lambda,u_n,v_n}(t) > 0$ for all $t \in (0, 1)$, we have $\Phi_{\lambda,u_n,v_n}(t_0) < \Phi_{\lambda,u_n,v_n}(1)$, and so

$$\Phi_{\lambda,u_0,v_0}(t_0) < \liminf_{n\to\infty} \Phi_{\lambda,u_n,v_n}(t_0) \le \liminf_{n\to\infty} \Phi_{\lambda,u_n,v_n}(1),$$

in other words,

$$I_\lambda(t_0^{\frac{1}{p}} u_0, t_0^{\frac{1}{q}} v_0) < \liminf_{n\to\infty} I_\lambda(u_n, v_n) = \inf_{\Lambda_\lambda^-} I_\lambda(u, v),$$

which is a contradiction. Consequently, $(u_n, v_n) \to (u_0, v_0)$ in X as $n \to \infty$.

Lemma 4.3 implies that (u_0, v_0) is a critical point of $I_\lambda(u, v)$ in X. Since $I_\lambda(|u|, |v|) = I_\lambda(u, v)$ for all $(u, v) \in X$, the solution (u_0, v_0) must be nonnegative in Ω. Again, as explained in the proof of Theorem 3.1, the strong maximum principle, given in [14, Lemma 2.3] and in [15, Theorem 1.2], ensures that actually (u_0, v_0) is a positive solution of (1.4) in Ω by symmetry.

Theorem 4.2 *Assume that* (H_1)–(H_3) *and condition* (1.6) *hold. If* $\lambda_n \to \lambda_1^-$ *as* $n \to \infty$, *then*

(1) $\lim\limits_{n\to\infty} \inf\limits_{\Lambda_{\lambda_n}^-} I_{\lambda_n}(u, v) = 0$;

(2) *if* $(u_n, v_n) \in X$ *is a minimizer for* I_λ *on* Λ_{λ_n}, *then* $(u_n, v_n) \to (0, 0)$ *as* $n \to \infty$.

Proof (1) For $(u_1,v_1)\in X$, by Remark 4.1, there exists t_n, with

$$t_n=t_{\lambda_n,u_1,v_1}=\Big(\frac{J(u_1,v_1)-\lambda_n K(u_1,v_1)}{\frac{\gamma}{p}+\frac{\delta}{q}}M(u_1,v_1)\Big)^{\frac{1}{\frac{\gamma}{p}+\frac{\delta}{q}-1}}>0$$

such that $(t_n^{\frac{1}{p}}u_1,t_n^{\frac{1}{q}}v_1)\in\Lambda_\lambda^-$. Hence, by (4.2),

$$\begin{aligned}\inf_{\Lambda_{\lambda_n}^-} I_{\lambda_n}(u,v)&\le I_{\lambda_n}(t_n^{\frac{1}{p}}u_1,t_n^{\frac{1}{q}}v_1)\\ &=t_n[J(u_1,v_1)-\lambda_nK(u_1,v_1)]-t_n^{\frac{\gamma}{p}+\frac{\delta}{q}}M(u_1,v_1)\\ &=t_n\Big(1-\frac{1}{\frac{\gamma}{p}+\frac{\delta}{q}}\Big)[J(u_1,v_1)-\lambda_nK(u_1,v_1)]\\ &=(\lambda_1-\lambda_n)t_n\Big(1-\frac{1}{\frac{\gamma}{p}+\frac{\delta}{q}}\Big)K(u_1,v_1).\end{aligned}$$

Then $\lim\limits_{n\to\infty}\inf\limits_{\Lambda_{\lambda_n}^-} I_{\lambda_n}(u,v)=0$, as desired.

(2) Let us first show that $\{(u_n,v_n)\}_n$ is a bounded sequence in X. Otherwise, we may assume either $\|u_n\|_p\to\infty$ or $\|v_n\|_q\to\infty$ as $n\to\infty$. Put $C_n=\|u_n\|_p^p+\|v_n\|_q^q$, $\widetilde{u}_n=C_n^{-\frac{1}{p}}u_n$, $\widetilde{v}_n=C_n^{-\frac{1}{q}}v_n$, so that $\|\widetilde{u}_n\|_p^p+\|\widetilde{v}_n\|_q^q=1$ for all $n\in\mathbb{N}$ and $C_n\to\infty$ as $n\to\infty$. Hence, up to a subsequence, $(\widetilde{u}_n,\widetilde{v}_n)\rightharpoonup(\widetilde{u}_0,\widetilde{v}_0)$ in X as $n\to\infty$ for some $(\widetilde{u}_0,\widetilde{v}_0)\in X$. Then, since $I_{\lambda_n}(u_n,v_n)\to 0$ by part (1), we obtain

$$M(u_n,v_n)=\int_{\mathbb{R}^N}\mu(x)|u_n|^\gamma|v_n|^\delta\,\mathrm{d}x\to 0$$

as $n\to\infty$. Hence

$$0\le J(u_n,v_n)-\lambda_nK(u_n,v_n)=\Big(\frac{\gamma}{p}+\frac{\delta}{q}\Big)M(u_n,v_n)\to 0$$

as $n\to\infty$. Multiplying both sides by C_n^{-1}, we get

$$J(\widetilde{u}_n,\widetilde{v}_n)-\lambda_nK(\widetilde{u}_n,\widetilde{v}_n)\to 0,\quad C_n^{\frac{\gamma}{p}+\frac{\delta}{q}-1}\int_{\mathbb{R}^N}\mu(x)|\widetilde{u}_n|^\gamma|\widetilde{v}_n|^\delta\,\mathrm{d}x\to 0$$

as $n\to\infty$. Since $C_n\to\infty$ as $n\to\infty$ and $\frac{\gamma}{p}+\frac{\delta}{q}>1$ by (H_1),

$$\int_{\mathbb{R}^N}\mu(x)|\widetilde{u}_0|^\gamma|\widetilde{v}_0|^\delta\,\mathrm{d}x=\lim_{n\to\infty}\int_{\mathbb{R}^N}\mu(x)|\widetilde{u}_n|^\gamma|\widetilde{v}_n|^\delta\,\mathrm{d}x=0$$

as $n\to\infty$.

Now we claim that $(\widetilde{u}_n,\widetilde{v}_n)\to(\widetilde{u}_0,\widetilde{v}_0)$ in X as $n\to\infty$. Otherwise, as in the proof of Theorem 4.1, passing if necessary to a subsequence, we have

$$\|(\widetilde{u}_0,\widetilde{v}_0)\|_X\le\lim_{n\to\infty}\|(\widetilde{u}_n,\widetilde{v}_n)\|_X.$$

Hence, in particular,

$$\|\widetilde{u}_0\|_p\le\lim_{n\to\infty}\|\widetilde{u}_n\|_p,\quad\|\widetilde{v}_0\|_q\le\lim_{n\to\infty}\|\widetilde{v}_n\|_q.\tag{4.6}$$

Therefore, in (4.6) at least one inequality should hold with the strict sign by Lemma 2.1(2). Thus,

$$J(\widetilde{u}_0,\widetilde{v}_0)-\lambda_1 K(\widetilde{u}_0,\widetilde{v}_0)<\liminf_{n\to\infty} J(\widetilde{u}_n,\widetilde{v}_n)-\lim_{n\to\infty}\lambda_n K(\widetilde{u}_n,\widetilde{v}_n)=0$$

by Lemma 2.2. On the other hand,

$$J(\widetilde{u}_0,\widetilde{v}_0)-\lambda_1 K(\widetilde{u}_0,\widetilde{v}_0)=J(\widetilde{u}_0,\widetilde{v}_0)-\lim_{n\to\infty}\lambda_n K(\widetilde{u}_0,\widetilde{v}_0)\geq 0.$$

This contradiction implies the claim and so $(\widetilde{u}_n,\widetilde{v}_n)\to(\widetilde{u}_0,\widetilde{v}_0)$ in X as $n\to\infty$. Therefore, we immediately obtain

$$J(\widetilde{u}_0,\widetilde{v}_0)-\lambda_1 K(\widetilde{u}_0,\widetilde{v}_0)=\lim_{n\to\infty}[J(\widetilde{u}_n,\widetilde{v}_n)-\lambda_n K(\widetilde{u}_n,\widetilde{v}_n)]=0.$$

Hence $(\widetilde{u}_0,\widetilde{v}_0)$ is a eigenfunction associated with λ_1 for problem (1.5), and there exists a constant $C>0$ such that $\widetilde{u}_0=C^{\frac{1}{p}}u_1$, $\widetilde{v}_0=C^{\frac{1}{q}}v_1$ by Theorem 3.1(3). As

$$\int_{\mathbb{R}^N}\mu(x)|\widetilde{u}_0|^{\gamma}|\widetilde{v}_0|^{\delta}\,\mathrm{d}x=0,$$

it follows that $C=0$. But $\|\widetilde{u}_n\|_p^p+\|\widetilde{v}_n\|_q^q=1$ for all $n\in\mathbb{N}$. This is clearly impossible. Consequently, the sequence $\{(u_n,v_n)\}_n$ is bounded in X, as claimed.

Therefore, eventually up to a subsequence, $(u_n,v_n)\rightharpoonup(u_0,v_0)$ in X as $n\to\infty$. Hence, applying to $\{(u_n,v_n)\}_n$ the same argument used for $\{(\widetilde{u}_n,\widetilde{v}_n)\}_n$, we show that $(u_n,v_n)\to(0,0)$ as $n\to\infty$. This completes the proof.

5 The Case $\lambda\geq\lambda_1$

In this section, we prove a nonexistence result for problem (1.4) via the Picone identity. We shall assume

$(\mathrm{H}_3)'$ $\mu\in L^\infty(\Omega)$ is a nonnegative function in Ω.

Theorem 5.1 *Assume that* (H_1)–(H_2) *and* $(\mathrm{H}_3)'$ *hold. Then problem* (1.4) *has no nonnegative nontrivial solutions for every* $\lambda\geq\lambda_1$.

Proof Suppose on the contrary that the assertion is not true and let $(u,v)\in X\backslash\{(0,0)\}$ be a nonnegative nontrivial solution of (1.4) corresponding to some $\lambda>\lambda_1$.

Let u_n, $v_n\in C_0^\infty(\Omega)$, with u_n, $v_n>0$ in Ω, such that the sequence $\{(u_n,v_n)\}_n$ converges to some (u,v) in X. Fix $\varepsilon>0$. Applying the Picone identity to the functions u_n, $u+\varepsilon$ and v_n, $v+\varepsilon$, we obtain

$$\|u_n\|_p^p-B_p\Big(u,\frac{u_n^p}{(u+\varepsilon)^{p-1}}\Big)\geq 0,\quad \|v_n\|_q^q-B_q\Big(u,\frac{v_n^q}{(v+\varepsilon)^{q-1}}\Big)\geq 0.$$

Using Definition 2.2 and the fact that $\frac{\alpha}{p}+\frac{\beta}{q}=1$ by (H_1), we have

$$\begin{aligned}
&\frac{\alpha}{p}\|u_n\|_p^p+\frac{\beta}{q}\|v_n\|_q^q-\frac{\alpha}{p}\lambda\int_{\mathbb{R}^N}au^{p-1}\frac{u_n^p}{(u+\varepsilon)^{p-1}}\,\mathrm{d}x-\frac{\beta}{q}\lambda\int_{\mathbb{R}^N}cv^{q-1}\frac{v_n^q}{(v+\varepsilon)^{q-1}}\,\mathrm{d}x\\
\geq{}&\frac{\alpha}{p}\lambda\int_{\mathbb{R}^N}cu^{\alpha-1}v^{\beta}\frac{u_n^p}{(u+\varepsilon)^{p-1}}\,\mathrm{d}x+\frac{\beta}{q}\lambda\int_{\mathbb{R}^N}bu^{\alpha}v^{\beta-1}\frac{v_n^q}{(v+\varepsilon)^{q-1}}\,\mathrm{d}x\\
&+\frac{1}{p\delta}\int_{\mathbb{R}^N}\mu u^{\gamma-1}v^{\delta}\frac{u_n^p}{(u+\varepsilon)^{p-1}}\,\mathrm{d}x+\frac{1}{q\gamma}\int_{\mathbb{R}^N}\mu u^{\gamma}v^{\delta-1}\frac{v_n^q}{(v+\varepsilon)^{q-1}}\,\mathrm{d}x.
\end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain

$$\begin{aligned}&\frac{\alpha}{p}\|u_n\|_p^p + \frac{\beta}{q}\|v_n\|_q^q - \frac{\alpha}{p}\lambda\int_{\mathbb{R}^N} a u_n^p\,\mathrm{d}x - \frac{\beta}{q}\lambda\int_{\mathbb{R}^N} c v_n^q\,\mathrm{d}x\\ &\geq \frac{\alpha}{p}\lambda\int_{\mathbb{R}^N} b u^{\alpha-p}v^\beta u_n^p\,\mathrm{d}x + \frac{\beta}{q}\lambda\int_{\mathbb{R}^N} b u^\alpha v^{\beta-q} v_n^q\,\mathrm{d}x\\ &\quad+\frac{1}{p\delta}\int_{\mathbb{R}^N}\mu u^{\gamma-p}v^\delta u_n^p\,\mathrm{d}x + \frac{1}{q\gamma}\int_{\mathbb{R}^N}\mu u^\gamma v^{\delta-q}v_n^q\,\mathrm{d}x. \end{aligned}\tag{5.1}$$

Now put $\theta_1 = \frac{\alpha\beta}{q}$ and $\theta_2 = \frac{\alpha\beta}{p}$. Since $\frac{\alpha}{p} + \frac{\beta}{q} = 1$ by (H_1), the Young inequality yields

$$\begin{aligned} u_n^\alpha v_n^\beta &= u_n^\alpha u^{-\theta_1}v^{\theta_2} \times v_n^\beta u^{\theta_1}v^{-\theta_2}\\ &\leq \frac{\alpha}{p}\,u_n^p u^{-\frac{p\theta_1}{\alpha}} v^{\frac{p\theta_2}{\alpha}} + \frac{\beta}{q}\,v_n^q u^{\frac{q\theta_1}{\beta}} v^{-\frac{q\theta_2}{\beta}}\\ &= \frac{\alpha}{p}\,u_n^p u^{\alpha-p}v^\beta + \frac{\beta}{q}\,v_n^q u^\alpha v^{\beta-q}. \end{aligned}$$

Moreover, $\lambda > 0$ and $b \geq 0$ in Ω by (H_2), so that

$$\lambda\int_{\mathbb{R}^N} b u_n^\alpha v_n^\beta\,\mathrm{d}x \leq \frac{\alpha}{p}\lambda\int_{\mathbb{R}^N} b u^{\alpha-p}v^\beta u_n^p\,\mathrm{d}x + \frac{\beta}{q}\lambda\int_{\mathbb{R}^N} b u^\alpha v^{\beta-q}v_n^q\,\mathrm{d}x. \tag{5.2}$$

Similarly, since $\frac{\gamma}{p} + \frac{\delta}{q} > 1$ by (H_1),

$$u_n^\gamma v_n^\delta < \frac{\gamma}{p}\,u_n^p u^{\gamma-p}v^\delta + \frac{\delta}{q}\,v_n^q u^\gamma v^{\delta-q}.$$

Now, $\mu \geq 0$ in Ω by $(\mathrm{H}_3)'$ implies

$$\begin{aligned}&\frac{1}{p\delta}\int_{\mathbb{R}^N}\mu u^{\gamma-p}v^\delta u_n^p\,\mathrm{d}x + \frac{1}{q\gamma}\int_{\mathbb{R}^N}\mu u^\gamma v^{\delta-q}v_n^q\,\mathrm{d}x\\ &= \frac{1}{\delta\gamma}\Big(\frac{\gamma}{p}\int_{\mathbb{R}^N}\mu u^{\gamma-p}v^\delta u_n^p\,\mathrm{d}x + \frac{\delta}{q}\int_{\mathbb{R}^N}\mu u^\gamma v^{\delta-q}v_n^q\,\mathrm{d}x\Big)\\ &\geq \frac{1}{\delta\gamma}\int_{\mathbb{R}^N}\mu u_n^\gamma v_n^\delta\,\mathrm{d}x. \end{aligned}\tag{5.3}$$

Combining (5.1)–(5.3), we get

$$\begin{aligned}&\frac{\alpha}{p}\,\|u_n\|_p^p + \frac{\beta}{q}\|v_n\|_q^q - \frac{\alpha}{p}\lambda\int_{\mathbb{R}^N} a u_n^p\,\mathrm{d}x - \frac{\beta}{q}\lambda\int_{\mathbb{R}^N} c v_n^q\,\mathrm{d}x\\ &\geq \lambda\int_{\mathbb{R}^N} b u_n^\alpha v_n^\beta\,\mathrm{d}x + \frac{1}{\delta\gamma}\int_{\mathbb{R}^N}\mu u_n^\gamma v_n^\delta\,\mathrm{d}x, \end{aligned}$$

that is,

$$J(u_n, v_n) - \lambda K(u_n, v_n) \geq M(u_n, v_n).$$

Since $\{(u_n, v_n)\}_n$ converges to (u, v) in X,

$$J(u_n, v_n) - \lambda K(u_n, v_n) \to J(u, v) - \lambda K(u, v), \quad M(u_n, v_n) \to M(u, v)$$

as $n \to \infty$. Hence

$$J(u, v) - \lambda K(u, v) \geq M(u, v) > 0,$$

that is $\lambda < \lambda_1$. This is impossible since $\lambda \geq \lambda_1$ by assumption. The proof is now complete.

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