Chin. Ann. Math. Ser. B
39(3), 2018, 597–606
DOI: 10.1007/s11401-018-0083-7

Chinese Annals of
Mathematics, Series B

On Some Elliptic Problems in Unbounded Domains*

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(*Dedicated to Philippe G. Ciarlet on the occasion of his* 80*th birthday*)

Abstract The author presents a method allowing to obtain existence of a solution for some elliptic problems set in unbounded domains, and shows exponential rate of convergence of the approximate solution toward the solution.

Keywords Elliptic, Exponential convergence, Unbounded domains
2000 MR Subject Classification 35J25, 35J60, 35J65

1 Introduction

The goal of this note is to prove existence and uniqueness of solutions for some classes of elliptic problems set in unbounded domains. More precisely if Ω is an unbounded domain in $\mathbb{R}^n$, let

$$A = A(x) = (a_{ij}(x))$$

be an $n \times n$ matrix with entries $a_{ij} \in L_\infty(\mathbb{R}^n)$ for $i, j = 1, \cdots, n$, which satisfies the usual ellipticity condition

$$A(x)\xi \cdot \xi \geq \lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,\ \text{a.e. } x \in \Omega, \tag{1.1}$$

and

$$|A(x)\xi| \leq \Lambda|\xi|, \quad \forall \xi \in \mathbb{R}^n,\ \text{a.e. } x \in \Omega. \tag{1.2}$$

Here "$\cdot$" denotes the usual Euclidean product in $\mathbb{R}^n$, $|\cdot|$ the Euclidean norm, λ and Λ some positive constants.

If f denotes some distribution on Ω we would like to consider for instance problems of the type

$$\begin{cases} -\nabla \cdot (A(x)\nabla u(x)) + \beta(x, u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega_D, \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega_N, \end{cases} \tag{1.3}$$

where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ is the boundary of Ω which is split into two parts where we impose Dirichlet or Neumann boundary conditions, ν denotes the outward unit normal to $\partial\Omega$ supposed

Manuscript received October 10, 2017. Revised December 12, 2017.
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*This work was supported by the Ministry of Education and Science of the Russian Federation (No. 02.a03.21.0008).

to be possibly smooth. We refer to [11, 13–14] for the classical notation and results on Sobolev spaces.

We make now more precise our assumptions on β — a Carathéodory function — such that for some $a \in L^\infty(\Omega),\ a \geq 0,\ a \not\equiv 0,\ \Lambda > 0$ one has

$$(\beta(x,u) - \beta(x,v))(u-v) \geq a(x)(u-v)^2 \quad \text{a.e. } x \in \Omega,\ \forall u, v \in \mathbb{R}, \tag{1.4}$$

$$|\beta(x,u) - \beta(x,v)| \leq \Lambda |u-v| \quad \text{a.e. } x \in \Omega,\ \forall u, v \in \mathbb{R}, \tag{1.5}$$

$$\beta(x,0) = 0 \quad \text{a.e. } x \in \Omega. \tag{1.6}$$

Note that we made the choice of uniformizing the constant Λ appearing here and in (1.2). This can be done w. l. o. g. at the expense of choosing this constant bigger.

In a bounded domain $f \in H^{-1}(\Omega)$ allows to solve (1.3) relatively easily. Unfortunately when Ω is unbounded many simple functions fail to belong to $H^{-1}(\Omega)$ as it is the case for the constant functions (see for instance [4, 9]). Thus, in this case, some new techniques have to be developed.

These kinds of problems were attacked by the Russian school in the past decades. For instance one will find in [17], (see also [12, 16]), some technique of resolution of (1.3) in the distributional sense in the case, where

$$\beta(x,u) = a(x)|u|^{p-1}u \quad \text{a.e. } x \in \Omega$$

with $p > 1$, $a(x) \geq \lambda > 0$ under the boundary conditions below. Our approach is different and allows for instance the case

$$\beta(x,u) = a(x)u \quad \text{a.e. } x \in \Omega \tag{1.7}$$

with $a \geq 0$. In addition we establish an exponential rate of convergence of the approximate solutions to (1.3) toward u. Note that here we do not assume Ω bounded in one direction and we are not relying on the Poincaré inequality (see [4–8, 10]).

Let us introduce further notation. If ω is a bounded, convex open set of $\mathbb{R}^n$ containing 0 for $\ell > 0$ we denote by Ω_ℓ the set

$$\Omega_\ell = \ell\omega \cap \Omega, \quad \ell\omega = \{\ell x \mid x \in \omega\}. \tag{1.8}$$

Let V_ℓ denote the closed subspace of $H^1(\Omega_\ell)$ defined as

$$V_\ell = \{v \in H^1(\Omega_\ell) \mid v = 0 \text{ on } \partial\Omega_D \cap \partial\Omega_\ell\}. \tag{1.9}$$

V_ℓ could be define for instance as the closure for the $H^1(\Omega_\ell)$-norm of the space of $C^1(\overline{\Omega}_\ell)$-functions vanishing on $\partial\Omega_D \cap \partial\Omega_\ell$. We will suppose V_ℓ equipped with the usual norm of $H^1(\Omega_\ell)$.

For $f \in V_\ell^*$ the dual of V_ℓ and for ℓ large enough there exists a unique u_ℓ solution to

$$\begin{cases} u_\ell \in V_\ell, \\ \displaystyle\int_{\Omega_\ell} A(x)\nabla u_\ell(x) \cdot \nabla v(x) + \beta(x,u_\ell)v\,\mathrm{d}x = \langle f, v\rangle, \quad \forall v \in V_\ell, \end{cases} \tag{1.10}$$

where $\langle\ ,\ \rangle$ denotes the duality bracket between V_ℓ^* and V_ℓ.

To see this, one sets

$$\langle \mathcal{A}u, v\rangle = \int_{\Omega_\ell} A(x)\nabla u(x)\cdot\nabla v(x) + \beta(x,u)v\mathrm{d}x, \quad \forall u, v \in V_\ell. \tag{1.11}$$

First note that for $u \in V_\ell$ one has

$$|\beta(x,u)| = |\beta(x,u) - \beta(x,0)| \le \Lambda|u| \in L^2(\Omega_\ell) \tag{1.12}$$

and thus the right-hand side integral in (1.11) is well defined. It is then easy to see that $\mathcal{A}$ defines a monotone operator from V_ℓ into V_ℓ^* (see below). Moreover this operator is hemicontinuous and coercive. Indeed to see this last point note that for all $u, u_0 \in V_\ell$ one has

$$\begin{aligned}\langle \mathcal{A}u - \mathcal{A}u_0, u-u_0\rangle &= \int_{\Omega_\ell} A(x)\nabla(u-u_0)\cdot\nabla(u-u_0)\\ &\qquad + \{\beta(x,u)-\beta(x,u_0)\}(u-u_0)\mathrm{d}x\\ &\ge \lambda\int_{\Omega_\ell}|\nabla(u-u_0)|^2\ \mathrm{d}x + \int_{\Omega_\ell} a(x)(u-u_0)^2\ \mathrm{d}x\\ &\ge \delta\int_{\Omega_\ell}|\nabla(u-u_0)|^2 + (u-u_0)^2\mathrm{d}x\end{aligned} \tag{1.13}$$

for some δ small enough. Note that for $a \ge 0$, $a \not\equiv 0$ the norms

$$\Big\{\int_{\Omega_\ell}|\nabla u|^2 + a(x)u^2\mathrm{d}x\Big\}^{\frac12},\ \Big\{\int_{\Omega_\ell}|\nabla u|^2 + u^2\mathrm{d}x\Big\}^{\frac12} = |u|_{H^1(\Omega_\ell)} \tag{1.14}$$

are equivalent in $H^1(\Omega_\ell)$ (see [2] or Section 3 below). The existence of u_ℓ follows then from classical results (see [3, 15]).

Set

$$V_\ell^0 = \{v \in V_\ell \mid v = 0 \text{ on } \partial\Omega_\ell\cap\Omega\}. \tag{1.15}$$

Suppose that there exists some constant that w. l. o. g. we can denote by λ such that for ℓ large enough

$$\int_{\Omega_\ell} A(x)\nabla u\cdot\nabla u + a(x)u^2\mathrm{d}x \ge \lambda\int_{\Omega_\ell}|\nabla u|^2 + u^2\mathrm{d}x = \lambda|u|^2_{H^1(\Omega_\ell)}, \quad \forall u \in H^1(\Omega_\ell). \tag{1.16}$$

We will give in Section 3 below some conditions on a for this to hold.

Then we can prove the following theorem.

Theorem 1.1 *Under the above assumptions suppose that*

$$|f|_{V_\ell^*} = O(\mathrm{e}^{\sigma\ell}) \tag{1.17}$$

for some positive constant σ which will be chosen later on. Then there exists a unique u_∞ such that for ℓ large enough

$$\begin{cases} u_\infty \in V_\ell, \quad \forall\ell,\\ \displaystyle\int_{\Omega_\ell} A(x)\nabla u_\infty\cdot\nabla v + \beta(x,u_\infty)v\mathrm{d}x = \langle f, v\rangle, \quad \forall v \in V_\ell^0,\ \forall\ell,\\ |u_\infty|_{V_\ell} = O(\mathrm{e}^{2\sigma\ell}).\end{cases} \tag{1.18}$$

Moreover one has for some positive constants C and β independent of ℓ

$$|u_\ell - u_\infty|_{V_{\frac{\ell}{2}}} \le C\mathrm{e}^{-\beta\ell}. \tag{1.19}$$

Remark 1.1 Note that due to the second equation of (1.18) u_∞ solves the first equation of (1.11) in the distributional sense.

2 Proof of Theorem 1.1

We first will need the following lemma.

Lemma 2.1 *u_ℓ is a Cauchy "sequence".*

Proof We do not precise here in what space u_ℓ is a Cauchy "sequence" —note that $\ell \in \mathbb{R}$— it will be clear later on. We assume that ℓ is large enough in such a way that (1.16) holds. Let $r \in [0,1]$. We are going to estimate first $u_\ell - u_{\ell+r}$.

Let R be such that

$$B(0,R) \subset \omega, \tag{2.1}$$

where $B(0,R)$ denotes the Euclidean ball in $\mathbb{R}^n$ with center 0 and radius R. Set for $\ell_1 \le \ell - 1$,

$$\rho_{\ell_1}(x) = \rho(x) = 0 \vee \Big(1 - \frac{\operatorname{dist}(x, \ell_1\omega)}{R}\Big),$$

where $\vee$ stands for the maximum of two numbers and dist denotes the Euclidean distance. Clearly one has

$$0 \le \rho \le 1, \quad \rho = 1 \quad \text{on } \ell_1\omega, \quad \rho = 0 \quad \text{outside of } (\ell_1+1)\omega, \tag{2.2}$$

$$|\nabla\rho| \le \frac{1}{R}. \tag{2.3}$$

To prove the last claim of (2.2) it is enough to show that if $x \notin (\ell_1+1)\omega$, then $\operatorname{dist}(x, \ell_1\omega) \ge R$. If not, for some $y \in \omega$ one has

$$|x - \ell_1 y| < R.$$

This implies by (2.1) that $x - \ell_1 y = z$ where $z \in \omega$. It follows that

$$x = (\ell_1+1)\Big(\frac{\ell_1}{\ell_1+1}y + \frac{1}{\ell_1+1}z\Big) \in (\ell_1+1)\omega$$

by the convexity of ω. This completes the proof of the last claim of (2.2).

Since clearly $(u_\ell - u_{\ell+r})\rho \in V_\ell \cap V_{\ell+r}$ using the equation of (1.10) for ℓ and $\ell + r$ we get

$$\int_{\Omega_\ell} A(x)\nabla(u_\ell - u_{\ell+r}) \cdot \nabla\{(u_\ell - u_{\ell+r})\rho\} + \{\beta(x,u_\ell) - \beta(x,u_{\ell+r})\}(u_\ell - u_{\ell+r})\rho \mathrm{d}x = 0. \tag{2.4}$$

Using (2.2) we obtain

$$\int_{\Omega_{\ell_1+1}} \{A(x)\nabla(u_\ell - u_{\ell+r}) \cdot \nabla(u_\ell - u_{\ell+r})\}\rho$$

$$+ \{\beta(x,u_\ell) - \beta(x,u_{\ell+r})\}(u_\ell - u_{\ell+r})\rho \mathrm{d}x$$
$$= -\int_{D_{\ell_1}} A(x)\nabla(u_\ell - u_{\ell+r}) \cdot \nabla\rho\ (u_\ell - u_{\ell+r})\mathrm{d}x, \tag{2.5}$$

where $D_{\ell_1} = \Omega_{\ell_1+1}\backslash\Omega_{\ell_1}$. Thus by (1.4),

$$\int_{\Omega_{\ell_1+1}} \{A(x)\nabla(u_\ell - u_{\ell+r}) \cdot \nabla(u_\ell - u_{\ell+r})\}\rho + a(u_\ell - u_{\ell+r})^2\rho \mathrm{d}x$$
$$\leq \int_{D_{\ell_1}} \Lambda|\nabla(u_\ell - u_{\ell+r})||\nabla\rho||(u_\ell - u_{\ell+r})|\ \mathrm{d}x. \tag{2.6}$$

Using again (2.2)–(2.3) and (1.16), we derive, thanks to the Young Inequality,

$$\lambda \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 + (u_\ell - u_{\ell+r})^2 \mathrm{d}x$$
$$\leq \frac{\Lambda}{2R}\int_{D_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 + (u_\ell - u_{\ell+r})^2 \mathrm{d}x.$$

This can be written as

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 + (u_\ell - u_{\ell+r})^2 \mathrm{d}x$$
$$\leq \frac{\Lambda}{2\lambda R}\int_{D_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 + (u_\ell - u_{\ell+r})^2\ \mathrm{d}x$$

and thus for any $\ell_1 \leq \ell - 1$ we get

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 + (u_\ell - u_{\ell+r})^2 \mathrm{d}x \leq c\int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 + (u_\ell - u_{\ell+r})^2 \mathrm{d}x,$$

where

$$c = \frac{\frac{\Lambda}{2\lambda R}}{1 + \frac{\Lambda}{2\lambda R}} < 1.$$

Let us denote by [ ] the integer part of a number. Choosing for ℓ large enough $\ell_1 = \frac{\ell}{2}$ and iterating the inequality above $\left[\frac{\ell}{2}\right]$-times we get

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 + (u_\ell - u_{\ell+r})^2 \mathrm{d}x \leq c^{[\frac{\ell}{2}]}\int_{\Omega_{\frac{\ell}{2}+[\frac{\ell}{2}]}} |\nabla(u_\ell - u_{\ell+r})|^2 + (u_\ell - u_{\ell+r})^2 \mathrm{d}x.$$

Since $\frac{\ell}{2} - 1 < [\frac{\ell}{2}] \leq \frac{\ell}{2}$, this leads to

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 + (u_\ell - u_{\ell+r})^2 \mathrm{d}x$$
$$\leq c^{\frac{\ell}{2}-1}\int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 + (u_\ell - u_{\ell+r})^2 \mathrm{d}x$$
$$\leq \frac{2}{c}\mathrm{e}^{-\frac{\ell}{2}\log\frac{1}{c}}\int_{\Omega_\ell} |\nabla u_\ell|^2 + u_\ell^2 + |\nabla u_{\ell+r}|^2 + u_{\ell+r}^2 \mathrm{d}x. \tag{2.7}$$

We would like now to estimate the right-hand side of (2.7). Taking $v = u_\ell$ in (1.10) leads to

$$\int_{\Omega_\ell} A(x)\nabla u_\ell(x) \cdot \nabla u_\ell(x) + \beta(x,u_\ell)u_\ell \mathrm{d}x = \langle f, u_\ell\rangle \leq |f|_{V_\ell^*}|u_\ell|_{V_\ell}.$$

Thus by (1.16)–(1.17) for some constant C one has

$$\lambda \int_{\Omega_\ell} |\nabla u_\ell|^2 + u_\ell^2 \mathrm{d}x \le C\mathrm{e}^{\sigma\ell} |u_\ell|_{V_\ell}.$$

From this it follows that

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 + u_\ell^2 \mathrm{d}x \le \Big(\frac{C}{\lambda}\Big)^2 \mathrm{e}^{2\sigma\ell}, \tag{2.8}$$

where C is independent of ℓ. Going back to (2.7) we obtain

$$\begin{aligned}
&\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 + (u_\ell - u_{\ell+r})^2 \mathrm{d}x \\
&\le \frac{2}{c}\mathrm{e}^{-\frac{\ell}{2}\log\frac{1}{c}} \Big\{ \Big(\frac{C}{\lambda}\Big)^2 \mathrm{e}^{2\sigma\ell} + \Big(\frac{C}{\lambda}\Big)^2 \mathrm{e}^{2\sigma(\ell+r)} \Big\} \\
&= \frac{2}{c}\Big(\frac{C}{\lambda}\Big)^2 \mathrm{e}^{-\frac{\ell}{2}\log\frac{1}{c}} \mathrm{e}^{2\sigma\ell}\{1 + \mathrm{e}^{2\sigma r}\} \\
&\le \frac{2}{c}\Big(\frac{C}{\lambda}\Big)^2 \mathrm{e}^{-\frac{\ell}{2}\log\frac{1}{c}} \mathrm{e}^{2\sigma\ell}\{1 + \mathrm{e}^{2\sigma}\}.
\end{aligned} \tag{2.9}$$

We choose then σ small enough such that

$$4\sigma < \frac{1}{2}\log\frac{1}{c} \tag{2.10}$$

to get for

$$2\beta = \frac{1}{2}\log\frac{1}{c} - 2\sigma, \tag{2.11}$$

$$|u_\ell - u_{\ell+r}|_{V_{\frac{\ell}{2}}} \le C\mathrm{e}^{-\beta\ell} \tag{2.12}$$

for some constant C independent of ℓ. Recall that the norm in V_ℓ is just the induced $H^1(\Omega_\ell)$-norm.

The estimate above holds for any $r \in [0,1]$. For any $t > 0$ one deduces then by the triangular inequality

$$\begin{aligned}
|u_\ell - u_{\ell+t}|_{V_{\frac{\ell}{2}}} &\le |u_\ell - u_{\ell+1}|_{V_{\frac{\ell}{2}}} + |u_{\ell+1} - u_{\ell+2}|_{V_{\frac{\ell}{2}}} + \cdots + |u_{\ell+[t]} - u_{\ell+t}|_{V_{\frac{\ell}{2}}} \\
&\le |u_\ell - u_{\ell+1}|_{V_{\frac{\ell}{2}}} + |u_{\ell+1} - u_{\ell+2}|_{V_{\frac{\ell+1}{2}}} + \cdots + |u_{\ell+[t]} - u_{\ell+t}|_{V_{\frac{\ell+[t]}{2}}} \\
&\le C\mathrm{e}^{-\beta\ell} + C\mathrm{e}^{-\beta(\ell+1)} + \cdots + C\mathrm{e}^{-\beta(\ell+[t])} \\
&\le C\frac{1}{1-\mathrm{e}^{-\beta}}\mathrm{e}^{-\beta\ell}.
\end{aligned} \tag{2.13}$$

Thus for any $\ell_0 < \frac{\ell}{2}$ we see that u_ℓ is a Cauchy sequence in $H^1(\Omega_{\ell_0})$. This completes the proof of the lemma.

End of the Proof of Theorem 1.1 Let us fix ℓ_0. For ℓ large enough u_ℓ is a Cauchy sequence in $H^1(\Omega_{\ell_0})$ and thus converges toward some $u_\infty \in V_{\ell_0}$ (see (1.9)). Since by (1.10),

$$\int_{\Omega_{\ell_0}} A(x)\nabla u_\ell(x)\cdot\nabla v(x) + \beta(x, u_\ell)v\mathrm{d}x = \langle f, v\rangle, \quad \forall v \in V_{\ell_0}^0, \tag{2.14}$$

passing to the limit in ℓ we obtain the two first properties of (1.18). Next, going back to (2.13) written in replacing ℓ by 2ℓ we get for some other constant C

$$|u_{2\ell} - u_{2\ell+t}|_{V_\ell} \leq C\mathrm{e}^{-2\beta\ell}. \tag{2.15}$$

Letting $t \to \infty$ it comes

$$|u_{2\ell} - u_\infty|_{V_\ell} \leq C\mathrm{e}^{-2\beta\ell} \tag{2.16}$$

and thus by (2.8)

$$|u_\infty|_{V_\ell} \leq C\mathrm{e}^{-2\beta\ell} + |u_{2\ell}|_{V_{2\ell}} \leq C\mathrm{e}^{-2\beta\ell} + C\mathrm{e}^{2\sigma\ell} \leq C\mathrm{e}^{2\sigma\ell} \tag{2.17}$$

for some constant C. This completes the proof of (1.8). Passing to the limit in t in (2.13) leads to (1.19). To show now that the solution to (1.18) is unique if u'_∞ is another solution one has

$$\int_{\Omega_\ell} A(x)\nabla(u_\infty - u'_\infty)\cdot\nabla v + \{\beta(x,u_\infty) - \beta(x,u'_\infty)\}v\mathrm{d}x = 0, \quad \forall v \in V_\ell^0,\ \forall\ell. \tag{2.18}$$

It is clear that $(u_\infty - u'_\infty)\rho_{\ell_1} \in V_\ell^0$, then, using the arguments used for $u_\ell - u_{\ell+r}$ for $u_\infty - u'_\infty$ (see (2.7)), leads to

$$\begin{aligned} &\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\infty - u'_\infty)|^2 + (u_\infty - u'_\infty)^2\mathrm{d}x \\ &\leq \frac{2}{c}\mathrm{e}^{-\frac{\ell}{2}\log\frac{1}{c}} \int_{\Omega_\ell} |\nabla u_\infty|^2 + u_\infty^2 + |\nabla u'_\infty|^2 + (u'_\infty)^2\mathrm{d}x. \end{aligned} \tag{2.19}$$

Using the last property of (1.18) and (2.10) one deduces easily that $u_\infty = u'_\infty$. This completes the proof of the theorem.

Remark 2.1 A priori u_∞ depends on the choice of ω. However when σ is chosen small enough it leads to the same solution for two different ω. Indeed suppose that ω and ω' are two bounded open convex subsets of $\mathbb{R}^n$ containing 0. For some positive constant c_1 one has

$$\omega \subset c_1\omega'.$$

Suppose now R small enough in such a way that $B(0,R)$ is included in ω and ω' (see (2.1)). Then Theorem 1.1 is true for $\sigma < \sigma_0(\omega)$ where $\sigma_0(\omega)$ is some constant depending on ω (see (2.10)). Then choosing

$$\sigma < \sigma_0(\omega) \wedge \sigma_0(\omega')$$

one gets solutions u_∞, u'_∞ to (1.18) corresponding to ω, ω' respectively. Let us denote by $V_\ell(\omega)$ (respectively $V_\ell^0(\omega)$) the spaces V_ℓ (respectively V_ℓ^0) corresponding to ω. Due to the inclusion above one has

$$V_\ell^0(\omega) \subset V_{\ell c_1}^0(\omega'),$$

and thus by (1.18) corresponding to ω and ω' one has

$$\int_{\Omega_\ell} A(x)\nabla(u_\infty - u'_\infty)\cdot\nabla v + \{\beta(x,u_\infty) - \beta(x,u'_\infty)\}v\mathrm{d}x = 0, \quad \forall v \in V_\ell^0(\omega).$$

Then since $(u_\infty - u'_\infty)\rho_{\ell_1} \in V_\ell^0(\omega)$ for any $\ell_1 \le \ell - 1$ one deduces as above the inequality (2.19) and the equality of u_∞ and u'_∞ follows. Different choices of ω might be useful for computing approximate solutions since a cube is more simple to discretise than a ball or an ellipse.

Note that if (1.17) is replaced by

$$|f|_{V_\ell^*} = O(\ell^\gamma)$$

for some positive constant γ then the solution u_∞ obtained above is independent of ω, the third condition of (2.18) being replaced by $|u_\infty|_{V_\ell} = O(\ell^\gamma)$.

3 Remarks and Applications

If $a(x) \ge \lambda > 0$ or, more precisely, $a(x) \ge c > 0$ for a constant c that without loss of generality we can take equal to λ, then clearly it follows that (1.16) holds and thus Theorem 1.1 applies.

Note that in constructing a solution to (1.18) one can possibly consider only the ℓ's such that $\ell \in \mathbb{N}$. Then — this is common practice in numerical analysis — suppose that Ω_ℓ can be covered by similar triangles, rectangles, such that on each of them one has

$$0 \le a \le \mu, \qquad \int_Q a(x)\ \mathrm{d}x \ge \epsilon$$

for some constants μ and ϵ. Then one can show (see [1]) that there exists $\delta = \delta(\mu, \epsilon)$ such that

$$\int_Q |\nabla u|^2 + au^2 \mathrm{d}x \ge \delta \int_Q u^2\ \mathrm{d}x. \tag{3.1}$$

It is clear then that (1.16) holds. To convince the reader by a more simple case suppose that on each $Q = Q_i$ covering Ω_ℓ the function a is the same up perhaps to a rigid motion. Then on each Q_i one has for some δ,

$$\int_{Q_i} |\nabla u|^2 + au^2 \mathrm{d}x \ge \delta \int_{Q_i} u^2\ \mathrm{d}x. \tag{3.2}$$

Summing up on the different Q_i allows to get (1.16). Such a situation arises for instance in the case of a periodic a. We refer to [4] for details.

One should notice that the third condition in (1.18) is necessary in order to be able to state an existence and uniqueness result in all generality. Indeed suppose for instance that Ω is a domain in $\mathbb{R}^2$ containing the strip,

$$S = \mathbb{R} \times (-\alpha, \alpha), \quad \alpha > 0.$$

Suppose in addition that

$$A = \mathrm{I}d, \quad \beta(x, u) = a(x)u, \quad a(x) = 0 \quad \text{on } S,$$

i.e., the principal part of the operator is the usual Laplace operator. Then, since a vanishes on S, it is clear that for any n the function defined by

$$v_n = \mathrm{e}^{\sqrt{\lambda_n} x_1} \cos(\sqrt{\lambda_n} x_2), \quad \sqrt{\lambda_n} = \frac{(2n+1)\pi}{2\alpha}$$

satisfies

$$-\Delta v_n + a v_n = 0 \quad \text{in } S, \quad v_n = 0 \quad \text{on } \partial S,$$

where ∂S denotes the boundary of S. Thus adding to a solution to (1.18) a combination of these v_n's would lead to another solution to the first part of (1.18) in such a way that uniqueness is lost — even though (1.16) could be satisfied (see above). Of course for such solutions the third property of (1.18) is not satisfied. We refer the reader to [3] for the connection between σ and λ_1.

Note that the assumption (1.6) can be relaxed at the expense of changing $\beta(x,u)$ into $\beta(x,u) - \beta(x,0)$ and f into $f - \beta(x,0)$.

In the case of $f = 0$ Theorem 1.1 provides a Liouville type result, namely, if u is a function satisfying (1.18) with $f = 0$ then $u = 0$. This is clear since $f = 0$ satisfies (1.17) and then the only solution to (1.18) is 0.

Theorem 1.1 could be extended to nonlinear operators when the operator

$$-\nabla \cdot (A(x)\nabla u)$$

is replaced by an operator of the type

$$-\nabla \cdot (A(x, \nabla u))$$

equipped with the ad hoc structural assumptions.

Acknowledgements The author would like to thank the referee for some constructive remarks which allow him to improve this paper. This work was performed when the author visiting the City University of Hong Kong and the USTC in Hefei. He is very greatful to these institutions for their support. This article was also written during a part time employment at the S. M. Nikolskii Mathematical Institute of RUDN University, 6 Miklukho-Maklay St, Moscow, 117198.

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