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The Estimates of All Homogeneous Expansions for a Subclass of ε Quasi-convex Mappings in Several Complex Variables*

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Abstract The authors obtain the estimates of all homogeneous expansions for a subclass of ε quasi-convex mappings on the unit ball in complex Banach spaces. Moreover, the estimates of all homogeneous expansions for the above generalized mappings on the unit polydisk in $\mathbb{C}^n$ are also obtained. Especially, the above estimates are only sharp for a subclass of starlike mappings, quasi-convex mappings and quasi-convex mappings of type $\mathbb{A}$. The results are the generalization of many known results.

Keywords Homogeneous expansion, ε quasi-convex mapping, ε starlike mapping, Starlike mapping, Quasi-convex mapping, Quasi-convex mappings of type $\mathbb{A}$

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1 Introduction

It is well known that biholomorphic starlike mappings and biholomorphic convex mappings are two extreme significant families of mappings in the geometric function theory of several complex variables. So the family of ε starlike mappings which were originally introduced by Gong and Liu [2] is a meaningful family of mappings in several complex variables in that it is a family of mappings between the family of biholomorphic starlike mappings and the family of biholomorphic convex mappings. After that Liu and Zhu [5] extended the above family of ε starlike mappings to a new family of mappings which is called ε quasi-convex mappings, and $\varepsilon \in [0,1]$ is widened to $\varepsilon \in [-1,1]$. However, there are only a few results for ε starlike mappings and ε quasi-convex mappings, for instance, the generalized Roper-Suffridge extension operator preserved ε starlikeness on some Reinhardt domains in $\mathbb{C}^n$ for $\varepsilon \in [0,1]$ (see [2–3]) and ε quasi-convexity on some domains in complex Banach spaces $\varepsilon \in [-1,1]$ (see [5]), and the estimates of m-th ($m = k+1, k+2, \cdots, 2k$) homogeneous expansions for ε starlike mappings f ($z = 0$ is a zero of order $k+1$ of $f(z) - z$) on the unit polydisk in $\mathbb{C}^n$ and ε quasi-convex mappings

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f ($z = 0$ is a zero of order $k+1$ of $f(z) - z$) on the unit ball in complex Banach spaces were established respectively (see [5, 7]).

We denote by X the complex Banach space with the norm $\|\cdot\|$. Let X^* denote the dual space of X, let B be the open unit ball in X, and let U be the Euclidean open unit disk in $\mathbb{C}$. Also, we denote by U^n the open unit polydisk in $\mathbb{C}^n$, and let $\mathbb{N}^*$ be the set of all positive integers. Let ∂U^n denote the boundary of U^n, $(\partial U)^n$ be the distinguished boundary of U^n. Let the symbol $'$ stand for transpose. For each $x \in X \backslash \{0\}$, we define

$$T(x) = \{T_x \in X^* : \ \|T_x\| = 1, \ T_x(x) = \|x\|\}.$$

Let $H(B)$ be the set of all holomorphic mappings from B into X. It is shown that if $f \in H(B)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{D}^n f(x)((y-x)^n),$$

for all y in some neighborhood of $x \in B$, where $\mathrm{D}^n f(x)$ is the nth-Fréchet derivative of f at x, and for $n \geqslant 1$,

$$\mathrm{D}^n f(x)((y-x)^n) = \mathrm{D}^n f(x)(\underbrace{y-x, \cdots, y-x}_{n}).$$

A holomorphic mapping $f : B \to X$ is called to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. We say that a mapping $f \in H(B)$ is a locally biholomorphic mapping if the Fréchet derivative $\mathrm{D}f(x)$ has a bounded inverse for each $x \in B$. If $f : B \to X$ is a holomorphic mapping, then we say that f is normalized if $f(0) = 0$ and $\mathrm{D}f(0) = \mathrm{I}$, where I stands for the identity operator from X into X.

A normalized biholomorphic mapping $f : B \to X$ is called to be a starlike mapping if $f(B)$ is a starlike domain with respect to the origin.

Now we recall some definitions as follows.

Definition 1.1 (see [3]) *Let $f : B \to X$ be a locally biholomorphic mapping with $0 \in f(B)$. f is said to be an ε starlike mapping on B if there exists a positive number ε, $0 \leqslant \varepsilon \leqslant 1$, such that $f(B)$ is starlike with respect to every point in $\varepsilon f(B)$.*

We denote by $S^*_\varepsilon(B)$ the set of all ε starlike mappings on B.

Definition 1.2 (see [5]) *Let $\varepsilon \in [-1, 1]$, and $f : B \to X$ be a normalized locally biholomorphic mappings. If*

$$\mathrm{Re}\,\{T_x[(\mathrm{D}f(x))^{-1}(f(x) - \varepsilon f(\xi x))]\} \geqslant 0, \quad x \in B, \ \xi \in \overline{U},$$

then f is said to be an ε quasi-convex mapping on B.

Let $Q_\varepsilon(B)$ be the set of all ε quasi-convex mappings on B.

It is obviously known that

$$S_{\varepsilon}^{*}(B) \subsetneqq Q_{\varepsilon}(B), \quad S_{\varepsilon}^{*}(U) = Q_{\varepsilon}(U), \quad \varepsilon \in [0,1]$$

from Definitions 1.1–1.2.

Definition 1.3 (see [9]) *Suppose that $f : B \to X$ is a normalized locally biholomorphic mapping, and denote*

$$G_f(\alpha,\beta) = \frac{2\alpha}{T_u[(\mathrm{D}f(\alpha u))^{-1}(f(\alpha u) - f(\beta u))]} - \frac{\alpha+\beta}{\alpha-\beta}.$$

If

$$\operatorname{Re} G_f(\alpha,\beta) \geqslant 0, \quad u \in \partial B,\ \alpha,\beta \in U,$$

then f is said to be a quasi-convex mapping of type $\mathbb{A}$ on B.

We denote by $Q_{\mathbb{A}}(B)$ the set of all quasi-convex mappings of type $\mathbb{A}$ on B.

Definition 1.4 (see [9]) *Suppose that $f : B \to X$ is a normalized locally biholomorphic mapping. If*

$$\operatorname{Re}\{T_x[(\mathrm{D}f(x))^{-1}(f(x) - f(\xi x))]\} \geqslant 0, \quad x \in B,\ \xi \in \overline{U},$$

then f is said to be a quasi-convex mapping on B.

Let $Q(B)$ be the set of all quasi-convex mappings on B.

Definitions 1.3 and 1.4 are actually the same definitions in one complex variable, and $Q_{\mathbb{A}}(B) = Q(B)$ (see [9]).

In this paper, we shall establish the estimates of all homogeneous expansions for a subclass of ε quasi-convex mappings on the unit ball in complex Banach spaces. Furthermore, we shall also obtain the estimates of all homogeneous expansions for the above generalized mappings on the unit polydisk in $\mathbb{C}^n$. In particular, the above estimates are only sharp for a subclass of starlike mappings, quasi-convex mappings and quasi-convex mappings of type $\mathbb{A}$. It is shown that a weak version of the Bieberbach conjecture in several complex variables (see [1]) will be proved as a corollary, and our results generalize many known results.

2 Estimates of All Homogeneous Expansions for a Subclass of ε Quasi-convex Mappings on the Unit Ball in Complex Banach Spaces

In order to establish the desired theorems in this section, it is necessary to give the lemmas as follows.

Lemma 2.1 *Suppose* $\varepsilon \in [-1,1]$. *If* $f, g : B \to \mathbb{C} \in H(B)$, $f(0) = g(0) = 1$, *and* $(1+\varepsilon)(f(x)+\mathrm{D}f(x)x) = (f(x)+\varepsilon f(-x))g(x)$, *then*

$$\begin{aligned}&(1+3\varepsilon)\mathrm{D}f(0)x = (1+\varepsilon)\mathrm{D}g(0)x,\\ &\frac{((m-1)(1+\varepsilon)+(1+(-1)^m)\varepsilon)\mathrm{D}^{m-1}f(0)(x^{m-1})}{(m-1)!}\\ =&\frac{(1+\varepsilon)\mathrm{D}^{m-1}g(0)(x^{m-1})}{(m-1)!}+\frac{\mathrm{D}^{m-2}g(0)(x^{m-2})}{(m-2)!}(1-\varepsilon)\mathrm{D}f(0)x\\ &+\cdots+\mathrm{D}g(0)x\frac{(1+(-1)^{m-2}\varepsilon)\mathrm{D}^{m-2}f(0)(x^{m-2})}{(m-2)!},\quad x\in B,\ m=3,4,\cdots.\end{aligned}$$

Proof In view of the hypothesis of Lemma 2.1, we have

$$\begin{aligned}&(1+\varepsilon)\Big(1+2\mathrm{D}f(0)x+\frac{3\mathrm{D}^2f(0)(x^2)}{2!}+\cdots+\frac{m\mathrm{D}^{m-1}f(0)(x^{m-1})}{(m-1)!}+\cdots\Big)\\ =&\Big(1+\varepsilon+(1-\varepsilon)\mathrm{D}f(0)x+\frac{(1+\varepsilon)\mathrm{D}^2f(0)(x^2)}{2!}+\cdots\\ &+\frac{(1+(-1)^{m-1}\varepsilon)\mathrm{D}^{m-1}f(0)(x^{m-1})}{(m-1)!}+\cdots\Big)\\ &\cdot\Big(1+\mathrm{D}g(0)x+\frac{\mathrm{D}^2g(0)(x^2)}{2!}+\cdots+\frac{\mathrm{D}^{m-1}g(0)(x^{m-1})}{(m-1)!}+\cdots\Big).\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}&(1+\varepsilon)\Big(1+2\mathrm{D}f(0)x+\frac{3\mathrm{D}^2f(0)(x^2)}{2!}+\cdots+\frac{m\mathrm{D}^{m-1}f(0)(x^{m-1})}{(m-1)!}+\cdots\Big)\\ =&1+\varepsilon+(1-\varepsilon)\mathrm{D}f(0)x+(1+\varepsilon)\mathrm{D}g(0)x+\frac{(1+\varepsilon)\mathrm{D}^2f(0)(x^2)}{2!}+\mathrm{D}g(0)x\cdot(1-\varepsilon)\mathrm{D}f(0)x\\ &+\frac{(1+\varepsilon)\mathrm{D}^2g(0)(x^2)}{2!}+\cdots+\frac{(1+(-1)^{m-1}\varepsilon)\mathrm{D}^{m-1}f(0)(x^{m-1})}{(m-1)!}\\ &+\mathrm{D}g(0)x\frac{(1+(-1)^{m-2}\varepsilon)\mathrm{D}^{m-2}f(0)(x^{m-2})}{(m-2)!}+\cdots\\ &+\frac{\mathrm{D}^{m-2}g(0)(x^{m-2})}{(m-2)!}(1-\varepsilon)\mathrm{D}f(0)x+\frac{(1+\varepsilon)\mathrm{D}^{m-1}g(0)(x^{m-1})}{(m-1)!}+\cdots.\end{aligned}$$

Compare the homogeneous expansions of the two sides in the above equality. We derived the desired result. This completes the proof.

Lemma 2.2 *Suppose* $\varepsilon \in \big(-\frac{1}{3},1\big]$. *Then*

$$\begin{aligned}&2(1+\varepsilon)+2(1-\varepsilon)\Big(1+\frac{1-\varepsilon}{(2-1)(1+\varepsilon)+(1+(-1)^2)\varepsilon}\Big)+\cdots\\ &+2(1+(-1)^{l-1}\varepsilon)\prod_{k=2}^{l}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\\ =&(l(1+\varepsilon)+(1+(-1)^{l+1})\varepsilon+1-\varepsilon)\prod_{k=2}^{l}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\qquad(2.1)\end{aligned}$$

for $l=2,3,\cdots$.

Proof It is easily known that $(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon>0$ for $\varepsilon\in\left(-\frac{1}{3},1\right]$ and $k=2,3,\cdots$. When $l=2$, a simple calculation shows that the left-hand side of (2.1) is $\frac{2(1+\varepsilon)(3+\varepsilon)}{1+3\varepsilon}$, and the right-hand side of (2.1) is also $\frac{2(1+\varepsilon)(3+\varepsilon)}{1+3\varepsilon}$. Hence (2.1) holds for $l=2$. Assume that (2.1) holds for $l=s$, namely

$$\begin{aligned}&2(1+\varepsilon)+2(1-\varepsilon)\Big(1+\frac{1-\varepsilon}{(2-1)(1+\varepsilon)+(1+(-1)^2)\varepsilon}\Big)+\cdots\\&+2(1+(-1)^{s-1}\varepsilon)\prod_{k=2}^{s}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\\&=(s(1+\varepsilon)+(1+(-1)^{s+1})\varepsilon+1-\varepsilon)\prod_{k=2}^{s}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big).\end{aligned}$$

When $l=s+1$, it is shown that

$$\begin{aligned}&2(1+\varepsilon)+2(1-\varepsilon)\Big(1+\frac{1-\varepsilon}{(2-1)(1+\varepsilon)+(1+(-1)^2)\varepsilon}\Big)+\cdots\\&+2(1+(-1)^{s-1}\varepsilon)\prod_{k=2}^{s}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\\&+2(1+(-1)^{s}\varepsilon)\prod_{k=2}^{s+1}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\\&=(s(1+\varepsilon)+(1+(-1)^{s+1})\varepsilon+1-\varepsilon)\prod_{k=2}^{s}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\\&+2(1+(-1)^{s}\varepsilon)\prod_{k=2}^{s+1}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\\&=((s+1)(1+\varepsilon)+(1+(-1)^{s+2})\varepsilon+1-\varepsilon)\prod_{k=2}^{s+1}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\end{aligned}$$

by a direct computation. Hence (2.1) holds for $l=s+1$. This completes the proof.

Theorem 2.1 *Let* $\varepsilon\in\big(-\frac{1}{3},1\big]$, $f:B\to\mathbb{C}\in H(B)$, $F(x)=xf(x)\in Q_\varepsilon(B)$. *Then*

$$\frac{\|\mathrm{D}^m f(0)(x^m)\|}{m!}\leqslant\prod_{k=2}^{m}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\|x\|^m,\quad x\in B,\ m=2,3,\cdots.$$

Proof Let $W(x)=\frac{1}{1+\varepsilon}(\mathrm{D}F(x))^{-1}(F(x)-\varepsilon F(-x))$. A straightforward computation shows that

$$\frac{1}{1+\varepsilon}(\mathrm{D}F(x))^{-1}(F(x)-\varepsilon F(-x))=\frac{x(f(x)+\varepsilon f(-x))}{(1+\varepsilon)(f(x)+\mathrm{D}f(x)x)},\quad x\in B.$$

Since $F(x)=xf(x)\in Q_\varepsilon(B)$, it is shown that

$$\mathrm{Re}\frac{(1+\varepsilon)(f(x)+\mathrm{D}f(x)x)}{f(x)+\varepsilon f(-x)}=\mathrm{Re}\Big(\frac{\|x\|}{T_x(W(x))}\Big)>0,\quad x\in B\setminus\{0\}\tag{2.2}$$

from Definition 1.2. Consider

$$g(x)=\frac{(1+\varepsilon)(f(x)+\mathrm{D}f(x)x)}{f(x)+\varepsilon f(-x)},\quad x\in B.\tag{2.3}$$

Then $g : B \to \mathbb{C} \in H(B)$, $g(0) = f(0) = 1$, and $(1+\varepsilon)(f(x) + \mathrm{D}f(x)x) = (f(x) + \varepsilon f(-x))g(x)$.

We now prove that

$$\begin{aligned}&\frac{|\mathrm{D}^{m-1}f(0)(x^{m-1})|}{(m-1)!}\\ &\leqslant \prod_{k=2}^{m}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\|x\|^{m-1}, \quad x \in B,\ m = 2, 3, \cdots\end{aligned} \tag{2.4}$$

by the induction method. When $m = 2$, we deduce that

$$\frac{|\mathrm{D}f(0)x|}{1!} \leqslant \frac{2(1+\varepsilon)}{1+3\varepsilon}\|x\| = \Big(1+\frac{1-\varepsilon}{(2-1)(1+\varepsilon)+(1+(-1)^2)\varepsilon}\Big)\|x\|, \quad x \in B$$

from Lemma 2.1 and [8, Lemma 2.2]. This implies that (2.4) holds for $m = 2$.

Assume that (2.4) holds for $m = 3, 4, \cdots, l$. That is,

$$\begin{aligned}&\frac{|\mathrm{D}^{m-1}f(0)(x^{m-1})|}{(m-1)!}\\ &\leqslant \prod_{k=2}^{m}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\|x\|^{m-1}, \quad x \in B,\ m = 3, 4, \cdots, l.\end{aligned}$$

When $m = l+1$, by [8, Lemma 2.2] and (2.4), we obtain

$$\begin{aligned}&\frac{|(l(1+\varepsilon)+(1+(-1)^{l+1})\varepsilon)\mathrm{D}^l f(0)(x^l)|}{l!}\\ &= \Big|\frac{(1+\varepsilon)\mathrm{D}^l g(0)(x^l)}{l!} + \frac{\mathrm{D}^{l-1}g(0)(x^{l-1})}{(l-1)!}(1-\varepsilon)\mathrm{D}f(0)x\\ &\quad + \cdots + \mathrm{D}g(0)x\frac{(1+(-1)^{l-1}\varepsilon)\mathrm{D}^{l-1}f(0)(x^{l-1})}{(l-1)!}\Big|\\ &\leqslant 2(1+\varepsilon)\|x\|^l + 2(1-\varepsilon)\Big(1+\frac{1-\varepsilon}{(2-1)(1+\varepsilon)+(1+(-1)^2)\varepsilon}\Big)\|x\|^l\\ &\quad + \cdots + 2(1+(-1)^{l-1}\varepsilon)\prod_{k=2}^{l}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\|x\|^l\\ &= (l(1+\varepsilon)+(1+(-1)^{l+1})\varepsilon+1-\varepsilon)\prod_{k=2}^{l}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\|x\|^l.\end{aligned}$$

This implies that

$$\frac{|\mathrm{D}^l f(0)(x^l)|}{l!} \leqslant \prod_{k=2}^{l+1}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\|x\|^l, \quad x \in B.$$

Hence we see that (2.4) holds for $m = l+1$.

On the other hand, it is shown that

$$\frac{\mathrm{D}^m F(0)(x^m)}{m!} = x\frac{\mathrm{D}^{m-1}f(0)(x^{m-1})}{(m-1)!}, \quad x \in B,\ m = 2, 3, \cdots, \tag{2.5}$$

if $F(x) = xf(x)$. Consequently from (2.4)–(2.5), it follows the result as desired. This completes the proof.

Taking $\varepsilon = 0$ in Theorem 2.1, we get the following corollary immediately.

Corollary 2.1 *Let $f : B \to \mathbb{C} \in H(B)$, $F(x) = xf(x) \in S^*(B)$. Then*

$$\frac{\|\mathrm{D}^m F(0)(x^m)\|}{m!} \leqslant m\|x\|^m, \quad x \in B,\ m = 2, 3, \cdots,$$

and the above estimates are sharp.

Letting $\varepsilon = 1$ in Theorem 2.1, we also obtain the following corollary directly.

Corollary 2.2 *Let $f : B \to \mathbb{C} \in H(B)$, $F(x) = xf(x) \in Q(B)$ (or $Q_{\mathbb{A}}(B)$). Then*

$$\frac{\|\mathrm{D}^m F(0)(x^m)\|}{m!} \leqslant \|x\|^m, \quad x \in B,\ m = 2, 3, \cdots,$$

and the above estimates are sharp.

Remark 2.1 Corollary 2.1 is the same as the case $k = 1$ of [8, Theorem 2.1].

Corollary 2.3 *Let $\varepsilon \in [0, 1]$, $f : B \to \mathbb{C} \in H(B)$, $F(x) = xf(x) \in Q_\varepsilon(B)$. Then*

$$\|F(x)\| \leqslant \frac{\|x\|}{(1-\|x\|)^{\frac{2}{1+\varepsilon}}}, \quad \|\mathrm{D}F(x)x\| \leqslant \frac{\|x\|(1+\frac{1-\varepsilon}{1+\varepsilon}\|x\|)}{(1-\|x\|)^{\frac{2}{1+\varepsilon}+1}}, \quad x \in B,$$

and the above estimates are sharp for $\varepsilon = 0$ and $\varepsilon = 1$.

Proof It is apparent to see that

$$\begin{aligned}\frac{\|\mathrm{D}^m f(0)(x^m)\|}{m!} &\leqslant \prod_{k=2}^{m}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)}\Big)\|x\|^m \\ &= \prod_{k=2}^{m}\frac{k+(k-2)\varepsilon}{(k-1)(1+\varepsilon)}\|x\|^m \\ &= \frac{\prod\limits_{k=2}^{m}(k+(k-2)\varepsilon)}{(m-1)!(1+\varepsilon)^{m-1}}\|x\|^m, \quad x \in B,\ m = 2, 3, \cdots\end{aligned}$$

from Theorem 2.1. We also know that

$$F(x) = x + \sum_{m=2}^{\infty}\frac{\mathrm{D}^m f(0)(x^m)}{m!}, \quad \mathrm{D}F(x)x = x + \sum_{m=2}^{\infty}\frac{m\mathrm{D}^m f(0)(x^m)}{m!}, \quad x \in B.$$

In view of the triangle inequality of the norm, it follows the result as desired. This completes the proof.

Remark 2.2 When $X = \mathbb{C}, B = U$, Corollary 2.3 reduces to [6, Theorem 4.1].

Corollary 2.4 *Let $\varepsilon \in [0, 1]$, $f : B \to \mathbb{C} \in H(B)$, $F(x) = xf(x) \in Q_\varepsilon(B)$, where B is the unit ball of a complex Hilbert space X. Then*

$$\|\mathrm{D}F(x)\xi\| \leqslant \frac{\|\xi\|(1+\frac{1-\varepsilon}{1+\varepsilon}\|x\|)}{(1-\|x\|)^{\frac{2}{1+\varepsilon}+1}}, \quad x \in B,\ \xi \in X,$$

and the above estimates are sharp for $\varepsilon = 0$ and $\varepsilon = 1$.

Proof It is shown that

$$\sup_{\|x\|=\|\xi\|=1}\frac{\|\mathrm{D}^m F(0)(x^{m-1},\xi)\|}{m!}=\sup_{\|x\|=1}\frac{\|\mathrm{D}^m F(0)(x^m)\|}{m!}$$

(see [4]). Making use of triangle inequalities with respect to the norm in complex Banach spaces, it follows the result as desired. It is not difficult to verify that

$$F_1(x)=\frac{x}{(1-\langle x,e\rangle)^2}\in S^*(B)$$

and

$$F_2(x)=\frac{x}{1-\langle x,e\rangle}\in Q(B)\ (\text{or}\, Q_{\mathbb{A}}(B)),$$

where $\|e\|=1$. It yields that

$$\mathrm{D}F_1(x)\xi=\frac{\xi}{(1-\langle x,e\rangle)^2}+\frac{2\langle \xi,e\rangle x}{(1-\langle x,e\rangle)^3},\quad x\in B,\ \xi\in X$$

and

$$\mathrm{D}F_2(x)\xi=\frac{\xi}{1-\langle x,e\rangle}+\frac{\langle \xi,e\rangle x}{(1-\langle x,e\rangle)^2},\quad x\in B,\ \xi\in X$$

by a simple calculation. Put $x=re,\ \xi=Re\ (0\leqslant r<1,\ R\geqslant 0)$. Then

$$\|\mathrm{D}F_1(x)\xi\|=\frac{R(1+r)}{(1-r)^3},\quad \|\mathrm{D}F_2(x)\xi\|=\frac{R}{(1-r)^2}.$$

Therefore, it is shown that the estimate of Corollary 2.4 is sharp for $\varepsilon=0$ and $\varepsilon=1$. This completes the proof.

3 Sharp Estimates of All Homogeneous Expansions for a Subclass of ε Quasi-convex Mappings on the Unit Polydisk in $\mathbb{C}^n$

Let each m_j be a non-negative integer, $N=m_1+m_2+\cdots+m_n\in\mathbb{N}^*$, and $m_j=0$ imply that the corresponding components in Z and $F(Z)$ are omitted. Let U^{m_l} (resp. U^N) denote the unit polydisk of $\mathbb{C}^{m_l}$ $(l=1,2,\cdots,n)$ (resp. $\mathbb{C}^N$).

Theorem 3.1 *Let* $\varepsilon\in\big(-\frac{1}{3},1\big],\ f_l:U^{m_l}\to\mathbb{C}\in H(U^{m_l}),\ l=1,2,\cdots,n,\ F(Z)=(Z_1f_1(Z_1),\ Z_2f_2(Z_2),\cdots,Z_nf_n(Z_n))'\in Q_\varepsilon^*(U^N)$. *Then*

$$\frac{\|\mathrm{D}^m F(0)(Z^m)\|}{m!}\leqslant\prod_{k=2}^{m}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\|Z\|^m,\quad z\in U^N,\ m=2,3,\cdots.\tag{3.1}$$

Proof Let $F(Z)=(F_1(Z_1),F_2(Z_2),\cdots,F_n(Z_n))'$. According to the hypothesis of Theorem 3.1, for any $\zeta\in\overline{U},\ Z=(Z_1,Z_2,\cdots,Z_n)'\in U^N$, it yields that

$$\begin{aligned}&(\mathrm{D}F(Z))^{-1}(F(Z)-\varepsilon F(\zeta Z))\\ =&((\mathrm{D}F_1(Z_1))^{-1}(F_1(Z_1)-\varepsilon F_1(\zeta Z_1)),(\mathrm{D}F_2(Z_2))^{-1}\\ &\cdot(F_2(Z_2)-\varepsilon F_2(\zeta Z_2)),\cdots,(\mathrm{D}F_n(Z_n))^{-1}(F_n(Z_n)-\varepsilon F_n(\zeta Z_n)))'\end{aligned}$$

by a simple computation. Note that

$$\begin{aligned}&(\mathrm{D}F(Z))^{-1}(F(Z)-\varepsilon F(\zeta Z))\\ =&(0,\cdots,(\mathrm{D}F_l(Z_l))^{-1}(F_1(Z_1)-\varepsilon F_1(\zeta Z_1)),\cdots,0)',\end{aligned}$$

if $Z=(0,\cdots,Z_l,\cdots,0)'\in U^N,\ l=1,2,\cdots,n$. We set

$$\begin{aligned}W(Z)&=(W_1,W_2,\cdots,W_n)'=(W_{11},\cdots,W_{1m_1},W_{21},\cdots,W_{2m_2},\cdots,W_{n1},\cdots,W_{nm_n})'\\ &=(\mathrm{D}F(Z))^{-1}(F(Z)-\varepsilon F(\zeta Z)).\end{aligned}$$

Then it yields that

$$F\in Q_\varepsilon^*(U^N)\Leftrightarrow F_l\in Q_\varepsilon^*(U^{m_l}),\quad l=1,2,\cdots,n$$

from Definition 1.2. Taking into account the facts $\|\mathrm{D}^mF(0)(Z^m)\|=\max\limits_{1\leqslant l\leqslant n}\{\|\mathrm{D}^mF_l(0)(Z_l^m)\|\}$ and $\|Z\|=\max\limits_{1\leqslant l\leqslant n}\{\|Z_l\|\}$, here $\|Z_l\|_{m_l}$ (resp. $\|Z\|_N$) is written as $\|Z_l\|$ (resp. $\|Z\|$) for simplicity, it is shown that (3.1) holds. This completes the proof.

Setting $\varepsilon=0$ in Theorem 3.1, we get the following corollary readily.

Corollary 3.1 *Let* $f_l:U^{m_l}\to\mathbb{C}\in H(U^{m_l}),\ l=1,2,\cdots,n,\ F(Z)=(Z_1f_1(Z_1),Z_2f_2(Z_2),\cdots,Z_nf_n(Z_n))'\in S^*(U^N)$. *Then*

$$\frac{\|\mathrm{D}^mF(0)(Z^m)\|}{m!}\leqslant m\|Z\|^m,\quad z\in U^N,\ m=2,3,\cdots.$$

Taking $\varepsilon=1$ in Theorem 3.1, we also obtain the following corollary immediately.

Corollary 3.2 *Let* $f_l:U^{m_l}\to\mathbb{C}\in H(U^{m_l}),\ l=1,2,\cdots,n,\ F(Z)=(Z_1f_1(Z_1),Z_2f_2(Z_2),\cdots,Z_nf_n(Z_n))'\in Q(U^N)$ *(or* $Q_A(U^N)$*). Then*

$$\frac{\|\mathrm{D}^mF(0)(Z^m)\|}{m!}\leqslant \|Z\|^m,\quad z\in U^N,\ m=2,3,\cdots.$$

Similar to that in the proof of Corollary 2.3, it is not difficult to conclude the following corollary (the details of the proof are omitted here).

Corollary 3.3 *Let* $\varepsilon\in[0,1],\ f_l:U^{m_l}\to\mathbb{C}\in H(U^{m_l}),\ l=1,2,\cdots,n,\ F(Z)=(Z_1f_1(Z_1),Z_2f_2(Z_2),\cdots,Z_nf_n(Z_n))'\in Q_\varepsilon(U^N)$. *Then*

$$\|F(Z)\|\leqslant\frac{\|Z\|}{(1-\|Z\|)^{\frac{2}{1+\varepsilon}}},\quad \|\mathrm{D}F(Z)Z\|\leqslant\frac{\|Z\|(1+\frac{1-\varepsilon}{1+\varepsilon}\|Z\|)}{(1-\|Z\|)^{\frac{2}{1+\varepsilon}+1}},\quad Z\in U^N,$$

and the above estimates are sharp for $\varepsilon=0$ *and* $\varepsilon=1$.

Theorem 3.2 *Suppose* $\varepsilon \in \big(-\frac{1}{3},1\big]$, *and* $F(z)=(F_1(z),F_2(z),\cdots,F_n(z))'\in H(U^n)$. *If* $\mathrm{Re}\,\frac{\mathrm{D}F_j(z)z}{F_j(z)-\varepsilon F_j(\zeta z)}>0,\ \zeta\in U,\ z\in U^n\setminus\{0\}$, *where* j *satisfies the condition* $|z_j|=\|z\|=\max\limits_{1\leqslant l\leqslant n}\{|z_l|\}$, *then*

$$\frac{\|\mathrm{D}^m F(0)(z^m)\|}{m!}\leqslant\prod_{k=2}^{m}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\|z\|^m,\quad z\in U^n,\ m=2,3,\cdots.$$

Proof Fix $z\in U^n\setminus\{0\}$, and set $z_0=\frac{z}{\|z\|}$. Let

$$h_j(\xi)=\frac{\|z\|}{z_j}F_j(\xi z_0),\quad \xi\in U, \tag{3.2}$$

where j satisfies the condition $|z_j|=\|z\|=\max\limits_{1\leqslant l\leqslant n}\{|z_l|\}$. It yields that

$$\frac{h_j'(\xi)\xi}{h_j(\xi)-\varepsilon h_j(\zeta\xi)}=\frac{\mathrm{D}F_j(\xi z_0)\xi z_0}{F_j(\xi z_0)-\varepsilon F_j(\zeta\xi z_0)},\quad \zeta\in\overline{U},\ \xi\in U\setminus\{0\}$$

by a direct calculation. Therefore it is shown that

$$\mathrm{Re}\Big(\frac{h_j'(\xi)\xi}{h_j(\xi)-\varepsilon h_j(\zeta\xi)}\Big)=\mathrm{Re}\Big(\frac{\mathrm{D}F_j(\xi z_0)\xi z_0}{F_j(\xi z_0)-\varepsilon F_j(\zeta\xi z_0)}\Big)>0,\quad \zeta\in\overline{U},\ \xi\in U\setminus\{0\}$$

from the condition $\mathrm{Re}\frac{\mathrm{D}F_j(z)z}{F_j(z)-\varepsilon F_j(\zeta z)}>0,\ \zeta\in U,\ z\in U^n\setminus\{0\}$. That is, $h_j\in S_\varepsilon^*(U)$.

On the other hand, it is readily shown that

$$\xi+\sum_{m=2}^{\infty}a_m\xi^m=\xi+\frac{\|z\|}{z_j}\sum_{m=2}^{\infty}\frac{\mathrm{D}^m F_j(0)(z_0^m)}{m!}\xi^m$$

from (3.2). Comparing the coefficients of the two sides in the above equality, it yields that

$$\frac{\|z\|}{z_j}\frac{\mathrm{D}^m F_j(0)(z_0^m)}{m!}=a_m,\quad m=2,3,\cdots.$$

Consequently by Theorem 2.1 (the case of $X=\mathbb{C},\ B=U$), we have

$$\frac{|\mathrm{D}^m F_j(0)(z_0^m)|}{m!}\leqslant\prod_{k=2}^{m}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big),\quad z_0\in\partial U^n.$$

When $z_0\in(\partial U)^n$, we have

$$\frac{|\mathrm{D}^m F_l(0)(z_0^m)|}{m!}\leqslant\prod_{k=2}^{m}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big),\quad l=1,2,\cdots,n.$$

Also noticing that $\mathrm{D}^m F_l(0)(z^m)$ is a holomorphic function on $\overline{U^n}$, in view of the maximum modulus theorem of holomorphic functions on the unit polydisk, we conclude that

$$\frac{|\mathrm{D}^m F_l(0)(z_0^m)|}{m!}\leqslant\prod_{k=2}^{m}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big),\quad z_0\in\partial U^n,\ l=1,2,\cdots,n.$$

This implies that

$$\frac{|\mathrm{D}^m F_l(0)(z^m)|}{m!}\leqslant\prod_{k=2}^{m}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\|z\|^m,\quad z\in U^n,\ l=1,2,\cdots,n.$$

Therefore it follows that

$$\frac{\|\mathrm{D}^m F(0)(z^m)\|}{m!}\leqslant\prod_{k=2}^{m}\Big(1+\frac{1-\varepsilon}{(k-1)(1+\varepsilon)+(1+(-1)^k)\varepsilon}\Big)\|z\|^m,\quad z\in U^n.$$

This completes the proof.

Putting $\varepsilon=0$ in Theorem 3.2, we obtain the following corollary directly.

Corollary 3.4 *Suppose* $F(z)=(F_1(z),F_2(z),\cdots,F_n(z))'\in H(U^n)$*. If* $\mathrm{Re}\frac{\mathrm{D}F_j(z)z}{F_j(z)}>0,\ z\in U^n\setminus\{0\}$*, where* j *satisfies the condition* $|z_j|=\|z\|=\max\limits_{1\leqslant l\leqslant n}\{|z_l|\}$*, then*

$$\frac{\|\mathrm{D}^m F(0)(z^m)\|}{m!}\leqslant m\|z\|^m,\quad z\in U^n,\ m=2,3,\cdots.$$

Letting $\varepsilon=1$ in Theorem 3.2, we also get the following corollary easily.

Corollary 3.5 *Suppose* $F(z)=(F_1(z),F_2(z),\cdots,F_n(z))'\in H(U^n)$*. If* $\mathrm{Re}\frac{\mathrm{D}F_j(z)z}{F_j(z)-F_j(\zeta z)}>0,\ \zeta\in U,\ z\in U^n\setminus\{0\}$*, where* j *satisfies the condition* $|z_j|=\|z\|=\max\limits_{1\leqslant l\leqslant n}\{|z_l|\}$*, then*

$$\frac{\|\mathrm{D}^m F(0)(z^m)\|}{m!}\leqslant\|z\|^m,\quad z\in U^n,\ m=2,3,\cdots.$$

With the analogous arguments as in the proof of Corollary 2.3, it is easy to deduce the following corollary (the details of the proof are omitted here).

Corollary 3.6 *Suppose that* $\varepsilon\in[0,1],\ F(z)=(F_1(z),F_2(z),\cdots,F_n(z))'\in H(U^n)$*. If* $\mathrm{Re}\frac{\mathrm{D}F_j(z)z}{F_j(z)-\varepsilon F_j(\zeta z)}>0,\ \zeta\in U,\ z\in U^n\setminus\{0\}$*, where* j *satisfies the condition* $|z_j|=\|z\|=\max\limits_{1\leqslant l\leqslant n}\{|z_l|\}$*, then*

$$\|F(z)\|\leqslant\frac{\|z\|}{(1-\|z\|)^{\frac{2}{1+\varepsilon}}},\quad \|\mathrm{D}F(z)z\|\leqslant\frac{\|z\|(1+\frac{1-\varepsilon}{1+\varepsilon}\|z\|)}{(1-\|z\|)^{\frac{2}{1+\varepsilon}+1}},$$

and the above estimates are sharp for $\varepsilon=0$ *and* $\varepsilon=1$*.*

Remark 3.1 We readily see that Theorem 2.1 is the special case of Theorem 3.2 when $X=\mathbb{C}^n, B=U^n$, and Theorem 3.1 is the special case of Theorem 3.2 for $m_1=n,\ m_l=0,\ l=2,\cdots,n$ or $m_l=1,\ l=1,2,\cdots,n$ as well.

Remark 3.2 For the estimates of all homogeneous expansions for a subclass of ε quasi-convex mappings, $\varepsilon\in\big(-\frac{1}{3},1\big]$ is a sufficient condition in Theorems 2.1, 3.1–3.2. We do not know the corresponding results for the case $\varepsilon\in\big[-1,-\frac{1}{3}\big]$ nowadays. However, concerning the growth theorem and the upper bounds of the distortion theorem for a subclass of ε quasi-convex mappings, $\varepsilon \in [0,1]$ seems to be a necessary condition in Corollaries 2.3–2.4, 3.3 and 3.6. This implies that the condition of growth theorem and the upper bounds of the distortion theorem for a subclass of ε quasi-convex mappings is stronger than the condition of the estimates of all homogeneous expansions for a subclass of ε quasi-convex mappings.

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