Chin. Ann. Math. Ser. B
39(4), 2018, 633–642
DOI: 10.1007/s11401-018-0086-4

Chinese Annals of Mathematics, Series B

Nonlinear Maps Preserving the Jordan Triple ∗-Product on Factor von Neumann Algebras*

Changjing LI[1] Quanyuan CHEN[2] Ting WANG[3]

Abstract Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras. For $A, B \in \mathcal{A}$, define by $[A, B]_* = AB - BA^*$ the skew Lie product of A and B. In this article, it is proved that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([[A, B]_*, C]_*) = [[\Phi(A), \Phi(B)]_*, \Phi(C)]_*$ for all $A, B, C \in \mathcal{A}$ if and only if Φ is a linear ∗-isomorphism, or a conjugate linear ∗-isomorphism, or the negative of a linear ∗-isomorphism, or the negative of a conjugate linear ∗-isomorphism.

Keywords Jordan triple ∗-product, Isomorphism, von Neumann algebras
2000 MR Subject Classification 47B48, 46L10

1 Introduction

Let $\mathcal{A}$ be a ∗-algebra and η be a non-zero scalar. For $A, B \in \mathcal{A}$, define the Jordan η-∗-product of A and B by $A\diamondsuit_\eta B = AB + \eta BA^*$. The Jordan (-1)-∗-product, which is customarily called the skew Lie product, was extensively studied because it naturally arises in the problem of representing quadratic functionals with sesquilinear functionals (see [9–11]) and in the problem of characterizing ideals (see [2, 8]). We often write the Jordan (-1)-∗-product by $[A, B]_*$, that is $[A, B]_* = AB - BA^*$. A not necessarily linear map Φ between ∗-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to preserve the Jordan η-∗-product if $\Phi(A\diamondsuit_\eta B) = \Phi(A)\diamondsuit_\eta \Phi(B)$ for all $A, B \in \mathcal{A}$. Recently, many authors have started to pay more attention to the maps preserving the Jordan η-∗-product between ∗-algebras (see [1, 3, 6–7]). In [3], Dai and Lu proved that if Φ is a bijective map preserving the Jordan η-∗-product between two von Neumann algebras, one of which has no central abelian projections, then Φ is a linear ∗-isomorphism if η is not real and Φ is a sum of a linear ∗-isomorphism and a conjugate linear ∗-isomorphism if η is real.

Recently, Huo et al. [5] studied a more general problem. They considered the Jordan triple η-∗-product of three elements A, B and C in a ∗-algebra $\mathcal{A}$ defined by $A\diamondsuit_\eta B\diamondsuit_\eta C = (A\diamondsuit_\eta B)\diamondsuit_\eta C$ (we should be aware that $\diamondsuit_\eta$ is not necessarily associative). A map Φ between ∗-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to preserve the Jordan triple η-∗-product if $\Phi(A\diamondsuit_\eta B\diamondsuit_\eta C) = \Phi(A)\diamondsuit_\eta \Phi(B)\diamondsuit_\eta \Phi(C)$ for all $A, B, C \in \mathcal{A}$. Clearly a map between ∗-algebras preserving the Jordan η-∗-product also

Manuscript received October 25, 2015. Revised March 3, 2016.
[1]Corresponding author. School of Mathematical Sciences, Shandong Normal University, Jinan 250014, China. E-mail: lcjbxh@163.com
[2]College of Information, Jingdezhen Ceramic Institute, Jingdezhen 333403, Jiangxi, China. E-mail: cqy0798@163.com
[3]Department of Mathematics and Statistics, Nanyang Normal University, Nanyang 473061, Henan, China. E-mail: tingwang526@126.com
*The work was supported by the National Natural Science Foundation of China (No. 11526123, No. 11401273) and the Natural Science Foundation of Shandong Province of China (No. ZR2015PA010).

preserves the Jordan triple η-$*$-product. However, the map $\Phi : \mathbb{C} \to \mathbb{C}$, $\Phi(\alpha+\beta\mathrm{i}) = -4(\alpha^3+\beta^3\mathrm{i})$ is a bijection which preserves the Jordan triple (-1)-$*$-product and Jordan triple 1-$*$-product, but it does not preserve the Jordan (-1)-$*$-product or Jordan 1-$*$-product. So, the class of those maps preserving the Jordan triple η-$*$-product is, in principle wider than the class of maps preserving the Jordan η-$*$-product.

In [5], let $\eta \neq -1$ be a non-zero complex number, and let Φ be a bijection between two von Neumann algebras, one of which has no central abelian projections, satisfying $\Phi(I) = I$ and preserving the Jordan triple η-$*$-product. Huo et al. showed that Φ is a linear $*$-isomorphism if η is not real and Φ is the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism if η is real. On the one hand, Huo et al. did not consider the case $\eta = -1$. However, the Jordan (triple) (-1)-$*$-product is the most meaningful and important in Jordan (triple) η-$*$-products. On the other hand, it is easy to see that a map Φ preserving the Jordan triple η-$*$-product does not need to satisfy $\Phi(I) = I$. Indeed, let $\Phi(A) = -A$ for all $A \in \mathcal{A}$. Then Φ preserves the Jordan triple η-$*$-product but $\Phi(I) = -I$. Because of the above two reasons, in this paper, we will discuss maps preserving the Jordan triple (-1)-$*$-product without the assumption $\Phi(I) = I$. We mainly prove that a bijective map Φ between two factor von Neumann algebras preserves the Jordan triple (-1)-$*$-product if and only if Φ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.

As usual, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the real field and complex field. Throughout, algebras and spaces are over $\mathbb{C}$. A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space H containing the identity operator I. $\mathcal{A}$ is a factor von Neumann algebra means that its center contains only the scalar operators. It is clear that if $\mathcal{A}$ is a factor von Neumann algebra, then $\mathcal{A}$ is prime, that is, for $A, B \in \mathcal{A}$ if $A\mathcal{A}B = \{0\}$, then $A = 0$ or $B = 0$.

2 The Main Result and Its Proof

To complete the proof of main theorem, we need two lemmas.

Lemma 2.1 *Let $\mathcal{A}$ be an arbitrary factor von Neumann algebra with the identity operator I and $A \in \mathcal{A}$. If $AB = BA^*$ for all $B \in \mathcal{A}$, then $A \in \mathbb{R}I$.*

Proof In fact, take $B = I$, then $A = A^*$. So $AB = BA$ for all $B \in \mathcal{A}$, which implies A belongs to the center of $\mathcal{A}$. Note that $\mathcal{A}$ is a factor, it follows that $A \in \mathbb{R}I$.

Lemma 2.2 (see [4, Problem 230]) *Let $\mathcal{A}$ be a Banach algebra with the identity I. If $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ are such that $[A, B] = \lambda I$, where $[A, B] = AB - BA$, then $\lambda = 0$.*

The main result in this paper is as follows.

Theorem 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras. Then a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([[A, B]_*, C]_*) = [[\Phi(A), \Phi(B)]_*, \Phi(C)]_*$ for all $A, B, C \in \mathcal{A}$ if and only if Φ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.*

Proof Clearly, we only need to prove the necessity. First we give a key technique. Suppose that $A_1, A_2, \cdots, A_n$ and T are in $\mathcal{A}$ such that $\Phi(T) = \sum_{i=1}^{n} \Phi(A_i)$. Then for all $S_1, S_2 \in \mathcal{A}$, we have

$$\Phi([[S_1, S_2]_*, T]_*) = [[\Phi(S_1), \Phi(S_2)]_*, \Phi(T)]_* = \sum_{i=1}^{n} \Phi([[S_1, S_2]_*, A_i]_*), \tag{2.1}$$

$$\Phi([[S_1, T]_*, S_2]_*) = [[\Phi(S_1), \Phi(T)]_*, \Phi(S_2)]_* = \sum_{i=1}^{n} \Phi([[S_1, A_i]_*, S_2]_*) \tag{2.2}$$

and

$$\Phi([[T, S_1]_*, S_2]_*) = [[\Phi(T), \Phi(S_1)]_*, \Phi(S_2)]_* = \sum_{i=1}^{n} \Phi([[A_i, S_1]_*, S_2]_*). \tag{2.3}$$

In the following, we will complete the proof of Theorem 2.1 by proving several claims.

Claim 1 $\Phi(0) = 0$.

Since Φ is surjective, there exists $A \in \mathcal{A}$ such that $\Phi(A) = 0$. Then we obtain $\Phi(0) = \Phi([[0, A]_*, A]_*) = [[\Phi(0), \Phi(A)]_*, \Phi(A)]_* = 0$.

Claim 2 $\Phi(\mathbb{R}I) = \mathbb{R}I$, $\Phi(\mathbb{C}I) = \mathbb{C}I$ and Φ preserves self-adjoint elements in both directions.

Let $\lambda \in \mathbb{R}$ be arbitrary. Since Φ is surjective, there exists $B \in \mathcal{A}$ such that $\Phi(B) = I$. By Claim 1, we have that

$$\begin{aligned} 0 &= \Phi([[\lambda I, A]_*, B]_*) \\ &= [[\Phi(\lambda I), \Phi(A)]_*, I]_* \\ &= \Phi(\lambda I)\Phi(A) - \Phi(A)\Phi(\lambda I)^* - \Phi(A)^*\Phi(\lambda I)^* + \Phi(\lambda I)\Phi(A)^* \end{aligned}$$

holds true for all $A \in \mathcal{A}$. That is,

$$\Phi(\lambda I)(\Phi(A) + \Phi(A)^*) = (\Phi(A) + \Phi(A)^*)\Phi(\lambda I)^*$$

holds true for all $A \in \mathcal{A}$. So

$$\Phi(\lambda I)B = B\Phi(\lambda I)^*$$

holds true for all $B = B^* \in \mathcal{B}$. Since for every $B \in \mathcal{B}$, $B = B_1 + \mathrm{i}B_2$ with $B_1 = \frac{B+B^*}{2}$ and $B_2 = \frac{B-B^*}{2\mathrm{i}}$, it follows that

$$\Phi(\lambda I)B = B\Phi(\lambda I)^*$$

holds true for all $B \in \mathcal{B}$. It follows from Lemma 2.1 that $\Phi(\lambda I) \in \mathbb{R}I$. Note that Φ^{-1} has the same properties as Φ. Similarly, if $\Phi(A) \in \mathbb{R}I$, then $A \in \mathbb{R}I$. Therefore, $\Phi(\mathbb{R}I) = \mathbb{R}I$.

Let $A = A^* \in \mathcal{A}$. Since $\Phi(\mathbb{R}I) = \mathbb{R}I$, there exists $\lambda \in \mathbb{R}$ such that $\Phi(\lambda I) = I$. Then

$$\begin{aligned} 0 &= \Phi([[A, \lambda I]_*, \lambda I]_*) = [[\Phi(A), I]_*, I]_* \\ &= 2\Phi(A) - 2\Phi(A)^*. \end{aligned}$$

Hence $\Phi(A) = \Phi(A)^*$. Similarly, if $\Phi(A) = \Phi(A)^*$, then $A = A^* \in \mathcal{A}$. Therefore Φ preserves self-adjoint elements in both directions.

Let $\lambda \in \mathbb{C}$ be arbitrary. For every $A = A^* \in \mathcal{A}$, we obtain that

$$\begin{aligned} 0 &= \Phi([[A, \lambda I]_*, C]_*) \\ &= [[\Phi(A), \Phi(\lambda I)]_*, \Phi(C)]_* \end{aligned}$$

holds true for all $C \in \mathcal{A}$. It follows from Lemma 2.1 that

$$[\Phi(A), \Phi(\lambda I)]_* \in \mathbb{R}I.$$

Since $A = A^*$, we have $\Phi(A) = \Phi(A)^*$. Hence

$$[\Phi(A), \Phi(\lambda I)] \in \mathbb{R}I.$$

It follows from Lemma 2.2 that

$$[\Phi(A), \Phi(\lambda I)] = 0,$$

and then

$$B\Phi(\lambda I) = \Phi(\lambda I)B$$

for all $B = B^* \in \mathcal{B}$. Thus for every $B \in \mathcal{B}$, since $B = B_1 + \mathrm{i}B_2$ with $B_1 = \frac{B+B^*}{2}$ and $B_2 = \frac{B-B^*}{2\mathrm{i}}$, we get

$$\Phi(\lambda I)B = B\Phi(\lambda I)$$

holds true for all $B \in \mathcal{B}$. Hence $\Phi(\lambda I) \in \mathbb{C}I$. Similarly, if $\Phi(A) \in \mathbb{C}I$, then $A \in \mathbb{C}I$. Therefore, $\Phi(\mathbb{C}I) = \mathbb{C}I$.

Claim 3 $\Phi\big(\frac{1}{2}I\big) = \pm\frac{1}{2}I$, $\Phi\big(\frac{1}{2}\mathrm{i}I\big) = \pm\frac{1}{2}\mathrm{i}I$, $\Phi(iA) = \mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$ or $\Phi(\mathrm{i}A) = -\mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$, where i is the imaginary unit.

By Claim 2, we have

$$\Phi\Big(-\frac{1}{2}I\Big) = \alpha I, \quad \Phi\Big(\frac{1}{2}I\Big) = \beta I$$

and

$$\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big) = (\gamma_1 + \gamma\mathrm{i})I, \quad \Phi\Big(\frac{1}{2}\mathrm{i}I\Big) = (\omega_1 + \omega\mathrm{i})I,$$

where $\alpha, \beta, \gamma, \omega, \gamma_1, \omega_1 \in \mathbb{R}$ and $\alpha\beta\gamma\omega \neq 0$. It follows from $0 = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}\mathrm{i}I\big]_*, -\frac{1}{2}I\big]_*$ that

$$0 = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}\mathrm{i}I\Big)\Big]_*, \Phi\Big(-\frac{1}{2}I\Big)\Big]_* = [[(\gamma_1 + \gamma\mathrm{i})I, (\gamma_1 + \gamma\mathrm{i})I]_*, \alpha I]_* = 4\alpha\gamma\gamma_1\mathrm{i}I.$$

So $\gamma_1 = 0$. Similarly, by the equality $0 = \big[\big[\frac{1}{2}\mathrm{i}I, \frac{1}{2}\mathrm{i}I\big]_*, -\frac{1}{2}I\big]_*$, we get that $\omega_1 = 0$.

Now we get

$$\Phi\Big(-\frac{1}{2}I\Big) = \alpha I, \quad \Phi\Big(\frac{1}{2}I\Big) = \beta I, \quad \Phi\Big(-\frac{1}{2}\mathrm{i}I\Big) = \gamma\mathrm{i}I, \quad \Phi\Big(\frac{1}{2}\mathrm{i}I\Big) = \omega\mathrm{i}I.$$

It follows from $-\frac{1}{2}\mathrm{i}I = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}I\big]_*, -\frac{1}{2}I\big]_*$ that

$$\gamma\mathrm{i}I = \Phi\Big(-\frac{1}{2}\mathrm{i}I\Big) = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}I\Big)\Big]_*, \Phi\Big(-\frac{1}{2}I\Big)\Big]_* = 4\alpha^2\gamma\mathrm{i}I. \tag{2.4}$$

Also the equality $\frac{1}{2}\mathrm{i}I = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}I\big]_*, \frac{1}{2}I\big]_*$ implies

$$\omega\mathrm{i}I = \Phi\Big(\frac{1}{2}\mathrm{i}I\Big) = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}I\Big)\Big]_*, \Phi\Big(\frac{1}{2}I\Big)\Big]_* = 4\alpha\beta\gamma\mathrm{i}I, \tag{2.5}$$

the equality $\frac{1}{2}I = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}I\big]_*, -\frac{1}{2}\mathrm{i}I\big]_*$ implies

$$\beta I = \Phi\Big(\frac{1}{2}I\Big) = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}I\Big)\Big]_*, \Phi\Big(-\frac{1}{2}\mathrm{i}I\Big)\Big]_* = -4\alpha\gamma^2 I \tag{2.6}$$

and $-\frac{1}{2}I_{\mathcal{A}} = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}I\big]_*, \frac{1}{2}\mathrm{i}I\big]_*$ ensures that

$$\alpha I = \Phi\Big(-\frac{1}{2}I\Big) = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}I\Big)\Big]_*, \Phi\Big(\frac{1}{2}\mathrm{i}I\Big)\Big]_* = -4\alpha\gamma\omega I. \tag{2.7}$$

Now (2.4)–(2.7) ensures that

$$\alpha^2 = \gamma^2 = \frac{1}{4}, \quad \alpha = -\beta, \quad \gamma = -\omega. \tag{2.8}$$

For every $A \in \mathcal{A}$, it follows from $\mathrm{i}A = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}I\big]_*, A\big]_*$ that

$$\Phi(\mathrm{i}A) = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}I\Big)\Big]_*, \Phi(A)\Big]_* = 4\alpha\gamma\mathrm{i}\Phi(A), \tag{2.9}$$

which together with (2.8) implies that $\Phi(\mathrm{i}A) = \mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$ or $\Phi(\mathrm{i}A) = -\mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$.

Choose an arbitrary nontrivial projection $P_1 \in \mathcal{A}$, and write $P_2 = I - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j,\ i,j = 1,2$, then $\mathcal{A} = \sum\limits_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$, we may write $A = \sum\limits_{i,j=1}^{2} A_{ij}$. In all that follows, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. The following Claims 4–9 are devoted to the additivity of Φ.

Claim 4 For every $A_{11} \in \mathcal{A}_{11}$ and $B_{22} \in \mathcal{A}_{22}$, we have

$$\Phi(A_{11} + B_{22}) = \Phi(A_{11}) + \Phi(B_{22}).$$

Since Φ is surjective, we may find an element $T = \sum\limits_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(B_{22}).$$

Since $[[\mathrm{i}P_1, I]_*, A_{22}]_* = 0$, it follows from (2.1) and Claim 1 that

$$\Phi([[\mathrm{i}P_1, I]_*, T]_*) = \Phi([[\mathrm{i}P_1, I]_*, A_{11}]_*).$$

By the injectivity of Φ, we obtain that

$$2\mathrm{i}(P_1T + TP_1) = [[\mathrm{i}P_1, I]_*, T]_* = [\mathrm{i}P_1, I]_*, A_{11}]_* = 4\mathrm{i}A_{11},$$

and then we get $T_{11} = A_{11},\ T_{12} = T_{21} = 0$. Similarly, $T_{22} = B_{22}$, proving the claim.

Claim 5 For every $A_{12} \in \mathcal{A}_{12},\ B_{21} \in \mathcal{A}_{21}$, we have

$$\Phi(A_{12} + B_{21}) = \Phi(A_{12}) + \Phi(B_{21}).$$

Choose $T = \sum\limits_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{12}) + \Phi(B_{21}).$$

Since

$$[[\mathrm{i}(P_2 - P_1), I]_*, A_{12}]_* = [[\mathrm{i}(P_2 - P_1), I]_*, B_{21}]_* = 0,$$

it follows from (2.1) that

$$\Phi([[\mathrm{i}(P_2 - P_1), I]_*, T]_*) = 0.$$

From this, we get $[[\mathrm{i}(P_2-P_1),I]_*,T]_*=0$. So $T_{11}=T_{22}=0$.

Since $[[A_{12},P_1]_*,I]_*=0$, it follows from (2.3) that

$$\Phi([[T,P_1]_*,I]_*)=\Phi([[B_{21},P_1]_*,I]_*).$$

By the injectivity of Φ, we obtain that

$$2(TP_1-P_1T^*)=[[T,P_1]_*,I]_*=[[B_{21},P_1]_*,I]_*=2(B_{21}-B_{21}^*).$$

Hence $T_{21}=B_{21}$. Similarly, $T_{12}=A_{12}$, proving the claim.

Claim 6 For every $A_{11}\in\mathcal{A}_{11},\ B_{12}\in\mathcal{A}_{12},\ C_{21}\in\mathcal{A}_{21},\ D_{22}\in\mathcal{A}_{22}$, we have

$$\Phi(A_{11}+B_{12}+C_{21})=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})$$

and

$$\Phi(B_{12}+C_{21}+D_{22})=\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22}).$$

Let $T=\sum\limits_{i,j=1}^{2}T_{ij}\in\mathcal{A}$ be such that

$$\Phi(T)=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21}).$$

It follows from (2.1) and Claim 5 that

$$\begin{aligned}
&\Phi(2\mathrm{i}(P_2T+TP_2))\\
&=\Phi([[\mathrm{i}P_2,I]_*,T]_*)\\
&=\Phi([[\mathrm{i}P_2,I]_*,A_{11}]_*)+\Phi([[\mathrm{i}P_2,I]_*,B_{12}]_*)+\Phi([[\mathrm{i}P_2,I]_*,C_{21}]_*)\\
&=\Phi(2\mathrm{i}B_{12})+\Phi(2\mathrm{i}C_{21})\\
&=\Phi(2\mathrm{i}(B_{12}+C_{21})).
\end{aligned}$$

Thus $P_2T+TP_2=B_{12}+C_{21}$, which implies $T_{22}=0,\ T_{12}=B_{12},\ T_{21}=C_{21}$. Now we get $T=T_{11}+B_{12}+C_{21}$.

Since

$$[[\mathrm{i}(P_2-P_1),I]_*,B_{12}]_*=[[\mathrm{i}(P_2-P_1),I]_*,C_{21}]_*=0,$$

it follows from (2.1) that

$$\Phi([[\mathrm{i}(P_2-P_1),I]_*,T]_*)=\Phi([[\mathrm{i}(P_2-P_1),I]_*,A_{11}]_*),$$

from which we get $T_{11}=A_{11}$. Consequently, $\Phi(A_{11}+B_{12}+C_{21})=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})$.

Similarly, we can get that $\Phi(B_{12}+C_{21}+D_{22})=\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22})$.

Claim 7 For every $A_{11}\in\mathcal{A}_{11},\ B_{12}\in\mathcal{A}_{12},\ C_{21}\in\mathcal{A}_{21},\ D_{22}\in\mathcal{A}_{22}$, we have

$$\Phi(A_{11}+B_{12}+C_{21}+D_{22})=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22}).$$

Let $T=\sum\limits_{i,j=1}^{2}T_{ij}\in\mathcal{A}$ be such that

$$\Phi(T)=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22}).$$

It follows from (2.1) and Claim 6 that

$$\begin{aligned}\Phi(2\mathrm{i}P_1T+2\mathrm{i}TP_1)&=\Phi([[\mathrm{i}P_1,I]_*,T]_*)\\&=\Phi([[\mathrm{i}P_1,I]_*,A_{11}]_*)+\Phi([[\mathrm{i}P_1,I]_*,B_{12}]_*)\\&\quad+\Phi([[\mathrm{i}P_1,I]_*,C_{21}]_*)+\Phi([[\mathrm{i}P_1,I]_*,D_{22}]_*)\\&=\Phi(4\mathrm{i}A_{11})+\Phi(2\mathrm{i}B_{12})+\Phi(2\mathrm{i}C_{21})\\&=\Phi(4\mathrm{i}A_{11}+2\mathrm{i}B_{12}+2\mathrm{i}C_{21}).\end{aligned}$$

Thus

$$P_1T+TP_1=2A_{11}+B_{12}+C_{21},$$

it follows that $T_{11}=A_{11},\ T_{12}=B_{12},\ T_{21}=C_{21}$. Similarly, we can get

$$\Phi(2\mathrm{iP_2T}+2\mathrm{iTP_2})=\Phi(4\mathrm{iD_{22}}+2\mathrm{iB_{12}}+2\mathrm{iC_{21}}).$$

From this, we get $T_{22}=D_{22}$, proving the claim.

Claim 8 For every $A_{jk},B_{jk}\in\mathcal{A}_{jk},\ 1\le j\ne k\le 2$, we have

$$\Phi(A_{jk}+B_{jk})=\Phi(A_{jk})+\Phi(B_{jk}).$$

Since

$$\Big[\Big[\frac{\mathrm{i}}{2}I,P_j+A_{jk}\Big]_*,P_k+B_{jk}\Big]_*=\mathrm{i}(A_{jk}+B_{jk})+\mathrm{i}(A_{jk}^*)+\mathrm{i}(B_{jk}A_{jk}^*),$$

we get from Claim 7 that

$$\begin{aligned}&\Phi(\mathrm{i}(A_{jk}+B_{jk}))+\Phi(\mathrm{i}A_{jk}^*)+\Phi(\mathrm{i}(B_{jk}A_{jk}^*))\\&=\Phi\Big(\Big[\Big[\frac{\mathrm{i}}{2}I,P_j+A_{jk}\Big]_*,P_k+B_{jk}\Big]_*\Big)\\&=\Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big),\Phi(P_j+A_{jk})\Big]_*,\Phi(P_k+B_{jk})\Big]_*\\&=\Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big),\Phi(P_j)+\Phi(A_{jk})\Big]_*,\Phi(P_k)+\Phi(B_{jk})\Big]_*\\&=\Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big),\Phi(P_j)\Big]_*,\Phi(P_k)\Big]_*+\Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big),\Phi(P_j)\Big]_*,\Phi(B_{jk})\Big]_*\\&\quad+\Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big),\Phi(A_{jk})\Big]_*,\Phi(P_k)\Big]_*+\Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big),\Phi(A_{jk})\Big]_*,\Phi(B_{jk})\Big]_*\\&=\Phi(\mathrm{i}B_{jk})+\Phi(\mathrm{i}(A_{jk}+A_{jk}^*))+\Phi(\mathrm{i}(B_{jk}A_{jk}^*))\\&=\Phi(\mathrm{i}B_{jk})+\Phi(\mathrm{i}A_{jk})+\Phi(\mathrm{i}A_{jk}^*)+\Phi(\mathrm{i}(B_{jk}A_{jk}^*)),\end{aligned}$$

which implies $\Phi(\mathrm{i}(A_{jk}+B_{jk}))=\Phi(\mathrm{i}B_{jk})+\Phi(\mathrm{i}A_{jk})$. By Claim 3, we obtain that $\Phi(A_{jk}+B_{jk})=\Phi(A_{jk})+\Phi(B_{jk})$.

Claim 9 For every $A_{jj}\in\mathcal{A}_{jj}$ and $B_{jj}\in\mathcal{A}_{jj},\ 1\le j\le 2$, we have

$$\Phi(A_{jj}+B_{jj})=\Phi(A_{jj})+\Phi(B_{jj}).$$

Let $T=\sum\limits_{i,j=1}^{2}T_{ij}\in\mathcal{A}$ be such that

$$\Phi(T)=\Phi(A_{jj})+\Phi(B_{jj}).$$

For $1 \leq j \neq k \leq 2$, it follows from (2.1) that

$$\Phi([[\mathrm{i}P_k, I]_*, T]_*) = \Phi([[\mathrm{i}P_k, I]_*, A_{jj}]_*) + \Phi([[\mathrm{i}P_k, I_{\mathcal{A}}]_*, B_{jj}]_*) = 0.$$

Hence $P_kT + TP_k = 0$, which implies $T_{jk} = T_{kj} = T_{kk} = 0$. Now we get $T = T_{jj}$.

For every $C_{jk} \in \mathcal{A}_{jk},\ j \neq k$, it follows from (2.2) and Claim 8 that

$$\begin{aligned}\Phi(2\mathrm{i}T_{jj}C_{jk}) &= \Phi([[\mathrm{i}P_j, T_{jj}]_*, C_{jk}]_*)\\ &= \Phi([[\mathrm{i}P_j, A_{jj}]_*, C_{jk}]_*) + \Phi([[\mathrm{i}P_j, B_{jj}]_*, C_{jk}]_*)\\ &= \Phi(2\mathrm{i}A_{jj}C_{jk}) + \Phi(2\mathrm{i}B_{jj}C_{jk})\\ &= \Phi(2\mathrm{i}(A_{jj}C_{jk} + B_{jj}C_{jk})).\end{aligned}$$

Hence

$$(T_{jj} - A_{jj} - B_{jj})C_{jk} = 0$$

for all $C_{jk} \in \mathcal{A}_{jk}$. By the primeness of $\mathcal{A}$, we get that $T_{jj} = A_{jj} + B_{jj}$, proving the claim.

Claim 10 Φ is $*$-additive.

The additivity of Φ is an immediate consequence of Claims 7–9. For every $A \in \mathcal{A},\ A = A_1 + \mathrm{i}A_2$, where $A_1 = \frac{A+A^*}{2}$ and $A_2 = \frac{A-A^*}{2\mathrm{i}}$ are self-adjoint elements. By Claims 2–3 , if for every $A \in \mathcal{A}$, $\Phi(\mathrm{i}A) = \mathrm{i}\Phi(A)$, then

$$\begin{aligned}\Phi(A^*) &= \Phi(A_1 - \mathrm{i}A_2) = \Phi(A_1) - \Phi(\mathrm{i}A_2)\\ &= \Phi(A_1) - \mathrm{i}\Phi(A_2) = \Phi(A_1)^* - \mathrm{i}\Phi(A_2)^*\\ &= \Phi(A_1)^* + (\mathrm{i}\Phi(A_2))^* = \Phi(A_1 + \mathrm{i}A_2)^*\\ &= \Phi(A)^*.\end{aligned}$$

Similarly, if $\Phi(\mathrm{i}A) = -\mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$, we also have $\Phi(A^*) = \Phi(A)^*$.

By Claims 3 and 10, we get that $\Phi(I) = I$ or $\Phi(I) = -I$. In the rest of this section, we deal with these two cases respectively.

Case 1 If $\Phi(I) = I$, then Φ is either a linear $*$-isomorphism or a conjugate linear $*$-isomorphism.

If $\Phi(I) = I$, by (2.8)–(2.9) and Claim 10, then $\alpha = -\frac{1}{2},\ \beta = \frac{1}{2},\ \gamma = -\omega,\ \Phi(\mathrm{i}I) = 2\omega\mathrm{i}I$ and $\Phi(\mathrm{i}A) = -2\gamma\mathrm{i}\Phi(A)$ for all $A \in \mathcal{A}$. For all $A, B \in \mathcal{A}$, we can obtain that

$$\begin{aligned}-4\gamma\mathrm{i}\Phi(AB + BA^*) &= 2\Phi(\mathrm{i}(AB + BA^*)) = \Phi(2\mathrm{i}(AB + BA^*))\\ &= \Phi([[\mathrm{i}I, A]_*, B]_*) = [[\Phi(\mathrm{i}I), \Phi(A)]_*, \Phi(B)]_*\\ &= 4\omega\mathrm{i}(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)^*).\end{aligned}$$

From this, we get

$$\Phi(AB + BA^*) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)^*. \tag{2.10}$$

For all $A, B \in \mathcal{A}$, it follows from Claim 3 that

$$\begin{aligned}\Phi(AB - BA^*) &= \Phi((\mathrm{i}A)(-\mathrm{i}B) + (-\mathrm{i}B)(\mathrm{i}A)^*)\\ &= \Phi(\mathrm{i}A)\Phi(-\mathrm{i}B) + \Phi(-\mathrm{i}B)\Phi(\mathrm{i}A)^*\\ &= -\Phi(\mathrm{i}A)\Phi(\mathrm{i}B) - \Phi(\mathrm{i}B)\Phi(\mathrm{i}A)^*\\ &= \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^*.\end{aligned} \tag{2.11}$$

Summing (2.10) with (2.11), we get that $\Phi(AB) = \Phi(A)\Phi(B)$.

For every rational number q, we have $\Phi(qI) = qI$. Indeed, since q is a rational number, there exist two integers r and s such that $q = \frac{r}{s}$. Since $\Phi(I) = I$ and Φ is additive, we get that

$$\Phi(qI) = \Phi\Big(\frac{r}{s}I\Big) = r\Phi\Big(\frac{1}{s}I\Big) = \frac{r}{s}\Phi(I) = qI.$$

Let A be a positive element in $\mathcal{A}$. Then $A = B^2$ for some self-adjoint element $B \in \mathcal{A}$. It follows from Claim 11 that $\Phi(A) = \Phi(B)^2$. By Claim 2, we get that $\Phi(B)$ is self-adjoint. So $\Phi(A)$ is positive. This shows that Φ preserves positive elements.

Let $\lambda \in \mathbb{R}$. Choose sequence $\{a_n\}$ and $\{b_n\}$ of rational numbers such that $a_n \le \lambda \le b_n$ for all n and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \lambda$. It follows from

$$a_n I \le \lambda I \le b_n I$$

that

$$a_n I_{\mathcal{B}} \le \Phi(\lambda I) \le b_n I.$$

Taking the limit, we get that $\Phi(\lambda I_{\mathcal{A}}) = \lambda I_{\mathcal{B}}$. Hence for all $A \in \mathcal{A}$,

$$\Phi(\lambda A) = \Phi((\lambda I)A) = \Phi(\lambda I)\Phi(A) = \lambda\Phi(A).$$

Hence Φ is real linear. Therefore, if $\Phi(\mathrm{i}A) = \mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$, then Φ is a linear $*$-isomorphism. If $\Phi(\mathrm{i}A) = -\mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$, then Φ is a conjugate linear $*$-isomorphism.

Case 2 If $\Phi(I) = -I$, then Φ is either the negative of a linear $*$-isomorphism or the negative of a conjugate linear $*$-isomorphism.

Consider that the map $\Psi : \mathcal{A} \to \mathcal{B}$ defined by $\Psi(A) = -\Phi(A)$ for all $A \in \mathcal{A}$. It is easy to see that Ψ satisfies $\Psi([[A,B]_*, C]_*) = [[\Psi(A), \Psi(B)]_*, \Psi(C)]_*$ for all $A, B, C \in \mathcal{A}$ and $\Psi(I) = I$. Then the arguments for Case 1 ensure that Ψ is either a linear $*$-isomorphism or a conjugate linear $*$-isomorphism. So Φ is either the negative of a linear $*$-isomorphism or the negative of a conjugate linear $*$-isomorphism.

Combining Cases 1–2, the proof of Theorem 2.1 is finished.

Acknowledgement The authors would like to thank the referee for his valuable comments and suggestions.

References

[1] Bai, Z. and Du, S., Maps preserving products $XY - YX^*$ on von Neumann algebras, *J. Math. Anal. Appl.,* **386**, 2012, 103–109.

[2] Brešar, M. and Fošner, A., On ring with involution equipped with some new product, *Publ. Math. Debrecen,* **57**, 2000, 121–134.

[3] Dai, L. and Lu, F., Nonlinear maps preserving Jordan ∗-products, *J. Math. Anal. Appl.,* **409**, 2014, 180–188.

[4] Halmos, P. R., A Hilbert Space Problem Book, 2nd ed, Springer-Verlag, New York, Heideberg, Berlin, 1982.

[5] Huo, D., Zheng, B. and Liu, H., Nonlinear maps preserving Jordan triple η-∗-products, *J. Math. Anal. Appl.,* **430**, 2015, 830–844.

[6] Li, C., Lu, F. and Fang, X., Mappings preserving new product $XY + YX^*$ on factor von Neumann algebras, *Linear Algebra Appl.,* **438**, 2013, 2339–2345.

[7] Li, C., Lu, F. and Fang, X., Nonlinear ξ-Jordan $*$-derivations on von Neumann algebras, *Linear Multilinear Algebra,* **62**, 2014, 466–473.

[8] Molnár, L., A condition for a subspace of $\mathcal{B}$(H) to be an ideal, *Linear Algebra Appl.,* **235**, 1996, 229–234.

[9] Šemrl, P., On Jordan $*$-derivations and an application, *Colloq. Math.,* **59**, 1990, 241–251.

[10] Šemrl, P., Quadratic functionals and Jordan $*$-derivations, *Studia Math.,* **97**, 1991, 157–165.

[11] Šemrl, P., Quadratic and quasi-quadratic functionals, *Proc. Amer. Math. Soc.,* **119**, 1993, 1105–1113.