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On Bounded Positive (m,p)-Circle Domains*

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Abstract Let D be a bounded positive (m,p)-circle domain in $\mathbb{C}^2$. The authors prove that if $\dim(\mathrm{Iso}(D)^0)=2$, then D is holomorphically equivalent to a Reinhardt domain; if $\dim(\mathrm{Iso}(D)^0)=4$, then D is holomorphically equivalent to the unit ball in $\mathbb{C}^2$. Moreover, the authors prove the Thullen's classification on bounded Reinhardt domains in $\mathbb{C}^2$ by the Lie group technique.

Keywords (m,p)-Circular domain, Reinhardt domain, Holomorphically equivalent
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1 Introduction

Let D be a bounded domain containing the origin in $\mathbb{C}^2$. Denote by $\mathrm{Aut}(D)$ the holomorphic automorphism group of the domain D, $\mathrm{Aut}(D)^0$ the unit connected component of $\mathrm{Aut}(D)$, and $\mathrm{Iso}(D)$ the isotropic subgroup with the origin of $\mathrm{Aut}(D)$ and $\mathrm{Iso}(D)^0$ the unit connected component of $\mathrm{Iso}(D)$. Cartan [1] proved that $\mathrm{Aut}(D)$ is a real Lie group and $\mathrm{Iso}(D)$ is a compact Lie subgroup of $\mathrm{Aut}(D)$. In this paper, $\mathfrak{L}$ is called the Lie algebra of a Lie group G, if $\mathfrak{L}$ consists of all left invariant vector fields of the Lie group G. Moreover, $\mathrm{aut}(D)$ and $\mathrm{iso}(D)$ denote the Lie algebras of the Lie group $\mathrm{Aut}(D)$ and $\mathrm{Iso}(D)$, respectively.

If $\mathrm{Iso}(D)^0$ has a 1-dimensional real Lie subgroup

$$x'=x\mathrm{e}^{\mathrm{i}\theta},\quad y'=y\mathrm{e}^{\mathrm{i}\theta},\quad 0\le\theta<2\pi,$$

we call D a circle domain; if $\mathrm{Iso}(D)^0$ has a 2-dimensional real Lie subgroup

$$x'=x\mathrm{e}^{\mathrm{i}\theta},\quad y'=y\mathrm{e}^{\mathrm{i}\varphi},\quad 0\le\theta,\varphi<2\pi,$$

we call D a Reinhardt domain; if $\mathrm{Iso}(D)^0$ has a 1-dimensional real Lie subgroup

$$x'=x\mathrm{e}^{\mathrm{i}m\theta},\quad y'=y\mathrm{e}^{\mathrm{i}p\theta},\quad 0\le\theta<2\pi,$$

where $m,p\in\mathbb{Z}$, $m>p\ge 1$, g.c.d.$(m,p)=1$, we call D a positive (m,p)-circle domain; if $\mathrm{Iso}(D)^0$ has a 1-dimensional real Lie subgroup

$$x'=x\mathrm{e}^{\mathrm{i}\theta},\quad y'=y,\quad 0\le\theta<2\pi,$$

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we call D a semi-circle domain.

By using the Thullen condition: $\dim(\mathrm{Aut}(D)) > \dim(\mathrm{Iso}(D))$, Thullen [2] succeeded in classifying bounded Reinhardt domains in $\mathbb{C}^2$. Any bounded Reinhardt domain with the Thullen condition in $\mathbb{C}^2$ is linearly equivalent to one of the three classes:

(1) Polydisc $\Delta^2 = \{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}$;

(2) Thullen domain $D_\lambda = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^\lambda < 1\}$ $(\lambda > 0, \lambda \neq 2)$;

(3) Unit ball $B^2 = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 < 1\}$.

By adding the condition: $\mathrm{Iso}(D)^0 \subset \mathrm{U}_2$, where U_2 is a unitary group of degree 2, Cartan [3] gave some classification and realization of domains in $\mathbb{C}^2$. For circular domains and a part of semi-circular domains in $\mathbb{C}^2$, Cartan [4] gave the classification and the realization.

Xu and Wang [5–6] gave the classification and the realization of bounded positive (m, p)-circle domains and bounded semi-circle domains when the isotropic subgroup with the origin is a 1-dimensional real Lie group. The holomorphic automorphism groups are also determined. Clearly, all Reinhardt domains in $\mathbb{C}^2$ are semi-circle domains and positive (m, p)-circle domains.

Recently, Yamamori [7] proved that the isotropic subgroup with the origin of bounded positive (m, p)-circle domains is a linear group when $p \geq 2$. Yamamori [8] and Rong [9] also obtained some more general results in $\mathbb{C}^n$.

In this paper, we only consider bounded domains containing the origin with the Thullen condition: $\dim(\mathrm{Aut}(D)) > \dim(\mathrm{Iso}(D))$.

2 The Isotropic Subgroup of Bounded Positive (m, p)-Circle Domains

In this section, we consider a bounded positive (m, p)-circle domain D with $\dim(\mathrm{Iso}(D)^0) = 2$ or 4. By the definition, if D is a bounded positive (m, p)-circle domain, then

$$A = \mathrm{i}mx\frac{\partial}{\partial x} + \mathrm{i}py\frac{\partial}{\partial y} \in \mathrm{iso}(D).$$

Firstly, we introduce some results of Cartan.

Lemma 2.1 (cf. [1]) *Let D be a bounded domain in $\mathbb{C}^n$. If the Taylor expansions at $z = 0$ of $\sigma, \tau \in \mathrm{Iso}(D)$ are*

$$\begin{aligned} \sigma: &\quad w = zA + \textit{terms of } z \textit{ with higher degrees}, \\ \tau: &\quad w = zB + \textit{terms of } z \textit{ with higher degrees}, \end{aligned}$$

respectively, then $\sigma = \tau$ if and only if $A = B$, where $A, B \in \mathrm{GL}(n, \mathbb{C})$.

Lemma 2.2 (cf. [1]) *Let D be a bounded domain in $\mathbb{C}^n$. If the Taylor expansions at $z = 0$ of $X = \xi(z)\frac{\partial^{\mathrm{T}}}{\partial z}$, $Y = \eta(z)\frac{\partial^{\mathrm{T}}}{\partial z} \in \mathrm{iso}(D)$ are*

$$\begin{aligned} X &= zA\frac{\partial^{\mathrm{T}}}{\partial z} + \textit{terms of } z \textit{ with higher degrees}, \\ Y &= zB\frac{\partial^{\mathrm{T}}}{\partial z} + \textit{terms of } z \textit{ with higher degrees}, \end{aligned}$$

respectively, where $\frac{\partial^{\mathrm{T}}}{\partial z} = \big(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\big)^{\mathrm{T}}$, then $X = Y$ if and only if $A = B$, where $A, B \in \mathrm{gl}(n, \mathbb{C})$. In particular, $X = 0$ if and only if $A = 0$.

Lemma 2.3 (cf. [1]) *Let D be a bounded domain in $\mathbb{C}^n$. Given a vector field $X = \xi(z)\frac{\partial^{\mathrm{T}}}{\partial z} \in \mathrm{aut}(D)$, then X, $\mathrm{i}X \in \mathrm{aut}(D)$ if and only if $X = 0$. Moreover, $X \in \mathrm{iso}(D)$ if and only if $\xi(0) = 0$.*

Lemma 2.4 (cf. [3]) *Let D be a bounded positive (m,p)-circle domain, then $\dim(\mathrm{Iso}(D)^0) = 1, 2, 4$. When $\dim(\mathrm{Iso}(D)^0) = 2$ or 4, there exists a homogeneous complex affine transformation $(u,v) = (x,y)Q$, $Q \in \mathrm{GL}(n,\mathbb{C})$, which maps D onto a bounded domain D_1, and $\mathrm{Iso}(D_1)^0$ is*

$$(u',v') = (u,v)\begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix} + \textit{ terms of } (u,v) \textit{ with higher degrees}, \quad 0 \le \theta, \varphi < 2\pi,$$

or

$$(u',v') = (u,v)U + \textit{ terms of } (u,v) \textit{ with higher degrees},$$

where U is the traversal of all unitary matrices of degree 2.

Now we can get the following theorem.

Theorem 2.1 *Suppose that D is a bounded positive (m,p)-circle domain, and Q is defined in Lemma 2.4. If $\dim(\mathrm{Iso}(D)^0) = 2$, then*

$$Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_4 \end{pmatrix}, \quad q_1 q_4 \ne 0$$

or

$$Q = \begin{pmatrix} 0 & q_2 \\ q_3 & 0 \end{pmatrix}, \quad q_2 q_3 \ne 0.$$

If $\dim(\mathrm{Iso}(D)^0) = 4$, then

$$Q = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} U_0,$$

where $\lambda_1, \lambda_2 > 0$, U_0 is a unitary matrix of degree 2.

Proof Since $\mathrm{Iso}(D)^0$ has a 1-dimensional real Lie subgroup

$$x' = x\mathrm{e}^{\mathrm{i}m\theta}, \quad y' = y\mathrm{e}^{\mathrm{i}p\theta}, \quad 0 \le \theta < 2\pi,$$

$\mathrm{Iso}(D_1)^0$ has a 1-dimensional real Lie subgroup

$$(u',v') = (u,v)Q^{-1}\begin{pmatrix} \mathrm{e}^{\mathrm{i}m\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}p\theta} \end{pmatrix} Q, \quad 0 \le \theta < 2\pi.$$

(I) The case of $\dim(\mathrm{Iso}(D)^0) = 2$

Since $\mathrm{Iso}(D_1)^0$ is

$$(u',v') = (u,v)\begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix} + \text{ terms of } (u,v) \text{ with higher degrees}, \quad 0 \le \theta, \varphi < 2\pi,$$

by Lemma 2.1, for any fixed $0 \le \theta < 2\pi$, there exists a unique matrix $\left(\begin{smallmatrix} \mathrm{e}^{\mathrm{i}\theta_1} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta_2} \end{smallmatrix}\right)$, such that

$$Q^{-1}\begin{pmatrix} \mathrm{e}^{\mathrm{i}m\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}p\theta} \end{pmatrix} Q = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_1} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta_2} \end{pmatrix},$$

where θ_1, θ_2 are real continuous functions on $t \in \mathbb{R}$. Denote $Q = \left(\begin{smallmatrix} q_1 & q_2 \\ q_3 & q_4 \end{smallmatrix}\right)$, so we have

$$\begin{pmatrix} \mathrm{e}^{im\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}p\theta} \end{pmatrix} \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_1} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta_2} \end{pmatrix}.$$

By a direct computation, we can get

$$\mathrm{e}^{im\theta} q_1 = \mathrm{e}^{\mathrm{i}\theta_1} q_1, \quad \mathrm{e}^{im\theta} q_2 = \mathrm{e}^{\mathrm{i}\theta_2} q_2, \quad \mathrm{e}^{\mathrm{i}p\theta} q_3 = \mathrm{e}^{\mathrm{i}\theta_1} q_3, \quad \mathrm{e}^{\mathrm{i}p\theta} q_4 = \mathrm{e}^{\mathrm{i}\theta_2} q_4.$$

As we know, $\det Q \neq 0$, i.e., $q_1 q_4 - q_2 q_3 \neq 0$.

(1) If $q_4 = 0$, then $q_2 q_3 \neq 0$, i.e., $q_2 \neq 0$ and $q_3 \neq 0$. So $\mathrm{e}^{im\theta} = \mathrm{e}^{\mathrm{i}\theta_2}$, $\mathrm{e}^{\mathrm{i}p\theta} = \mathrm{e}^{\mathrm{i}\theta_1}$. If $q_1 \neq 0$, then we have $\mathrm{e}^{im\theta} = \mathrm{e}^{\mathrm{i}\theta_1} = \mathrm{e}^{\mathrm{i}p\theta}$. Clearly, $\mathrm{e}^{im\theta} = \mathrm{e}^{\mathrm{i}p\theta}$ does not hold for any $\theta \in [0, 2\pi)$. So $q_1 = 0$. As the same reason, if $q_1 = 0$, then $q_4 = 0$.

(2) If $q_2 = 0$, then $q_1 q_4 \neq 0$, i.e., $q_1 \neq 0$ and $q_4 \neq 0$. As the same reason as (1), we can get $q_2 = q_3 = 0$.

(3) If $q_1 q_2 \neq 0$, according to (1) and (2), we can get $q_3 q_4 \neq 0$. So $\mathrm{e}^{im\theta} = \mathrm{e}^{\mathrm{i}p\theta}$ holds for any $\theta \in [0, 2\pi)$, which gives a contradiction.

(II) The case of $\dim(\mathrm{Iso}(D)^0) = 4$.

Since $\mathrm{Iso}(D_1)^0$ is

$$(u', v') = (u, v)U + \text{ terms of } (u, v) \text{ with higher degrees,}$$

where U is the traversal of all unitary matrices of degree 2, by Lemma 2.1, $Q^{-1} \left(\begin{smallmatrix} \mathrm{e}^{\mathrm{i}m\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}p\theta} \end{smallmatrix}\right) Q \in \mathrm{U}_2$, i.e.,

$$\begin{aligned} E &= \overline{\left(Q^{-1} \begin{pmatrix} \mathrm{e}^{im\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}p\theta} \end{pmatrix} Q\right)}^{\mathrm{T}} Q^{-1} \begin{pmatrix} \mathrm{e}^{im\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}p\theta} \end{pmatrix} Q \\ &= \overline{Q}^{\mathrm{T}} \begin{pmatrix} \mathrm{e}^{-im\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}p\theta} \end{pmatrix} \overline{Q^{-1}}^{\mathrm{T}} Q^{-1} \begin{pmatrix} \mathrm{e}^{im\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}p\theta} \end{pmatrix} Q. \end{aligned}$$

By a direct computation, we have

$$Q\overline{Q}^{\mathrm{T}} \begin{pmatrix} \mathrm{e}^{im\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}p\theta} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{im\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}p\theta} \end{pmatrix} Q\overline{Q}^{\mathrm{T}}.$$

Then $Q\overline{Q}^{\mathrm{T}} = \mathrm{diag}\{\alpha_1, \alpha_2\}$, where $\alpha_1, \alpha_2 \in \mathbb{C}$ do not equal 0. But $Q\overline{Q}^{\mathrm{T}} > 0$, it follows that $\alpha_1, \alpha_2 > 0$. Denote $\alpha_1 = \lambda_1^2$, $\alpha_2 = \lambda_2^2$, $\lambda_1 > 0$, $\lambda_2 > 0$. Hence, $Q = \mathrm{diag}\{\lambda_1, \lambda_2\} U_0$, where U_0 is a unitary matrix of degree 2. Note that D, D_1 are bounded domains, hence λ_1, λ_2 are finite.

Note that for a bounded positive (m, p)-circle domain D, if $\dim(\mathrm{Iso}(D)^0) = 2$, then $\mathrm{Iso}(D_1)^0$ has a 1-dimensional real Lie subgroup

$$u' = u\mathrm{e}^{\mathrm{i}m\theta}, \quad v' = v\mathrm{e}^{\mathrm{i}p\theta}, \quad 0 \leq \theta < 2\pi$$

or

$$u' = u\mathrm{e}^{\mathrm{i}p\theta}, \quad v' = v\mathrm{e}^{\mathrm{i}m\theta}, \quad 0 \leq \theta < 2\pi.$$

In the sense of holomorphic equivalence, we regard the two cases as the same class. If

$$\dim(\mathrm{Iso}(D)^0) = 4,$$

then $\mathrm{Iso}(D)^0$ is

$$
\begin{aligned}
(x',y') &= (x,y)QUQ^{-1} + \text{terms of } (x,y) \text{ with higher degrees}\\
&= (x,y)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} U_0 U \overline{U_0}^{\mathrm{T}} \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}^{-1} + \text{terms of } (x,y) \text{ with higher degrees}\\
&= (x,y)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} U \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}^{-1} + \text{terms of } (x,y) \text{ with higher degrees},
\end{aligned}
$$

where U is the traversal of all unitary matrices of degree 2. Set $\kappa:\ \widetilde{x} = \lambda_1 x,\ \widetilde{y} = \lambda_2 y$. Then $D_2 = \kappa(D)$ is a bounded positive (m,p)-circle domain, and $\mathrm{Iso}(D_2)^0$ is

$$(\widetilde{x}', \widetilde{y}') = (\widetilde{x}, \widetilde{y})U + \text{terms of } (\widetilde{x}, \widetilde{y}) \text{ with higher degrees},$$

where U is the traversal of all unitary matrices of degree 2. Without loss of generality, for a bounded positive (m,p)-circle domain D, suppose that the Taylor expansion at $z = 0$ of any $\sigma \in \mathrm{Iso}(D)^0$ is

$$\sigma:\quad (x',y') = (x,y)U + \text{terms of } (x,y) \text{ with higher degrees},$$

where U is a unitary matrices of degree 2.

Now we can get the following lemmas.

Lemma 2.5 *Let D be a bounded positive (m,p)-circle domain with $\mathrm{dim}(\mathrm{Iso}(D)^0) = 2$. If $p > 1$, then D is a Reinhardt domain; if $p = 1$, then D is holomorphically equivalent to a Reinhardt domain.*

Proof Suppose that the Lie algebra $\mathrm{iso}(D)$ has a group of bases $A = \mathrm{i}mx\frac{\partial}{\partial x} + \mathrm{i}py\frac{\partial}{\partial y}$ and

$$B = \Big((-c+\mathrm{i}d)y + \sum_{k=2}^{\infty}\sum_{j=0}^{k} a_{j,k-j}x^j y^{k-j}\Big)\frac{\partial}{\partial x} + \Big((c+\mathrm{i}d)x + \mathrm{i}by + \sum_{k=2}^{\infty}\sum_{j=0}^{k} b_{j,k-j}x^j y^{k-j}\Big)\frac{\partial}{\partial y},$$

where $0 \neq (b,c,d) \in \mathbb{R}^3$, $a_{j,k-j}, b_{j,k-j} \in \mathbb{C}$, $k \geq 2$, $0 \leq j \leq k$. Since $[A,B] \in \mathrm{iso}(D)$, there are $\lambda, \mu \in \mathbb{R}$, such that $[A,B] = \lambda A + \mu B$.

Firstly, we want to prove that $\lambda = \mu = 0$. In fact,

$$
\begin{aligned}
[A,B] &= \mathrm{i}\Big[(-m+p)(-c+\mathrm{i}d)y + \sum_{k=2}^{\infty}\sum_{j=0}^{k}(m(j-1)+p(k-j))a_{j,k-j}x^j y^{k-j}\Big]\frac{\partial}{\partial x}\\
&\quad + \mathrm{i}\Big[(m-p)(c+\mathrm{i}d)x + \sum_{k=2}^{\infty}\sum_{j=0}^{k}(mj+p(k-j-1))b_{j,k-j}x^j y^{k-j}\Big]\frac{\partial}{\partial y}\\
&= \lambda A + \mu B\\
&= \Big[\mathrm{i}\lambda mx + \mu(-c+\mathrm{i}d)y + \mu\sum_{k=2}^{\infty}\sum_{j=0}^{k} a_{j,k-j}x^j y^{k-j}\Big]\frac{\partial}{\partial x}\\
&\quad + \Big[\mu(c+\mathrm{i}d)x + \mathrm{i}(\lambda p + \mu b)y + \mu\sum_{k=2}^{\infty}\sum_{j=0}^{k} b_{j,k-j}x^j y^{k-j}\Big]\frac{\partial}{\partial y}.
\end{aligned}
$$

It follows that

$$\mathrm{i}\sum_{j=0}^{k}(m(j-1)+p(k-j))a_{j,k-j}x^jy^{k-j}=\mu\sum_{j=0}^{k}a_{j,k-j}x^jy^{k-j},\quad k\geq 2, \tag{2.1}$$

$$\mathrm{i}\sum_{j=0}^{k}(mj+p(k-j-1))b_{j,k-j}x^jy^{k-j}=\mu\sum_{j=0}^{k}b_{j,k-j}x^jy^{k-j},\quad k\geq 2 \tag{2.2}$$

and

$$\begin{aligned}&\mathrm{i}\lambda mx+[\mu(-c+\mathrm{i}d)-(m-p)(d+\mathrm{i}c)]y=0,\\&[(m-p)(\mathrm{i}c-d)-\mu(c+\mathrm{i}d)]x-\mathrm{i}(\lambda p+\mu b)y=0,\end{aligned}$$

hence $\lambda=0$, $\mu b=0$, $(m-p)c=\mu d$, $(m-p)d=-\mu c$.

Supposing that $\mu\neq 0$, we have $d(m-p)=-c\mu$ and $c(m-p)=d\mu$. Hence $-cd\mu=d^2(m-p)=-c^2(m-p)$. When $m\neq p$, then $c^2+d^2=0$, hence $c=d=0$. By $\mu b=0$, we have $b=0$. But $(b,c,d)\neq 0$, it gives a contradiction.

Suppose that $\lambda=\mu=0$, hence $[A,B]=0$, $c=d=0$, $b\neq 0$.

When $p>1$, from (2.1)–(2.2), we have $a_{jk}=b_{jk}=0$, $j+k\geq 2$, $B=\mathrm{i}by\frac{\partial}{\partial y}$. So the Lie algebra $\mathrm{iso}(D)$ has a group of bases $\mathrm{i}x\frac{\partial}{\partial x}$, $\mathrm{i}y\frac{\partial}{\partial y}$, and D is a Reinhardt domain.

When $p=1$, from (2.1)–(2.2), we have $a_{jk}=b_{jk}=0$, $j+k\geq 2$, but we do not know a_{0m}, $B=a_{0m}y^m\frac{\partial}{\partial x}+\mathrm{i}by\frac{\partial}{\partial y}$. So the Lie algebra $\mathrm{iso}(D)$ has a group of bases A, B, where

$$A=\mathrm{i}mx\frac{\partial}{\partial x}+\mathrm{i}y\frac{\partial}{\partial y},\quad B=ay^m\frac{\partial}{\partial x}+\mathrm{i}y\frac{\partial}{\partial y},\quad a\in\mathbb{C}.$$

The one-parameter subgroup $\exp(tB)$ $(t\in\mathbb{R})$ is a unique holomorphic solution of the complex ordinary differential equation systems

$$\frac{\mathrm{d}x'}{\mathrm{d}t}=a(y')^m,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{i}y',\quad t\in\mathbb{R}$$

with the initial values $x'(0)=x$, $y'(0)=y$. So $x'=x+\frac{ay^m}{\mathrm{i}m}(\mathrm{e}^{\mathrm{i}mt}-1)$, $y'=y\mathrm{e}^{\mathrm{i}t}$ $(t\in\mathbb{R})$. Therefore, $\mathrm{Iso}(D)^0$ is

$$x'=x\mathrm{e}^{\mathrm{i}m\theta}+\frac{a(y\mathrm{e}^{\mathrm{i}\theta})^m}{\mathrm{i}m}(\mathrm{e}^{\mathrm{i}m\varphi}-1),\quad y'=y\mathrm{e}^{\mathrm{i}(\theta+\varphi)},\quad \theta,\varphi\in[0,2\pi).$$

Given a holomorphic isomorphism $\sigma: u=x-\frac{a}{\mathrm{i}m}y^m$, $v=y$, then

$$A^*=\sigma_*(A)=\mathrm{i}mu\frac{\partial}{\partial u}+\mathrm{i}v\frac{\partial}{\partial v},\quad B^*=\sigma_*(B)=\mathrm{i}v\frac{\partial}{\partial v}.$$

Therefore, $\mathrm{iso}(\sigma(D))$ has a group of bases $\mathrm{i}u\frac{\partial}{\partial u}$, $\mathrm{i}v\frac{\partial}{\partial v}$, and $\sigma(D)$ is a Renhardt domain.

Lemma 2.6 *Let D be a bounded positive (m,p)-circle domain with* $\dim(\mathrm{Iso}(D)^0)=4$. *Then D is holomorphically equivalent to the unit ball in $\mathbb{C}^2$.*

Proof Since $\mathrm{Iso}(D)^0$ is

$$(x',y')=(x,y)U+\text{ terms of }(u,v)\text{ with higher degrees},$$

where U is the traversal of all unitary matrices of degree 2, $\mathrm{Iso}(D)^0$ is isomorphic to U_2. By the knowledge of Lie group, $\mathrm{Iso}(D)^0$ has a 2-dimensional Lie subgroup H which consists of the elements

$$x' = x\mathrm{e}^{\mathrm{i}m\theta}, \quad y' = y\mathrm{e}^{\mathrm{i}p\theta}, \quad 0 \le \theta < 2\pi.$$

By Lemma 2.5, the Lie algebra of H has a group of bases $\mathrm{i}x\frac{\partial}{\partial x}$, $\mathrm{i}y\frac{\partial}{\partial y}$ (in the sense of holomorphic isomorphism). Cartan [3] proved that for any bounded circle domain, its isotropic subgroup is constructed by the linear transformation, so $\mathrm{Iso}(D)^0$ is

$$(x', y') = (x, y)U, \quad U \in \mathrm{U}_2.$$

Hence D is holomorphically equivalent to the unit ball in $\mathbb{C}^2$.

It is known that $\dim(\mathrm{Iso}(\Delta^2)^0) = \dim(\mathrm{Iso}(D_\lambda)^0) = 2$, $\dim(\mathrm{Iso}(B^2)^0) = 4$. By the Thullen's classification of Reinhardt domains in $\mathbb{C}^2$ and Lemmas 2.5–2.6, we immediately obtain the following result.

Theorem 2.2 *Let D be a bounded positive (m, p)-circle domain. If $\dim(\mathrm{Iso}(D)^0) = 2$, then D is holomorphically equivalent to a Reinhardt domain which is one of the two classes:*

(1) *Polydisc* $\Delta^2 = \{(x, y) \in \mathbb{C}^2 : \ |x| < 1, \ |y| < 1\}$;

(2) *Thullen domain* $D_\lambda = \{(x, y) \in \mathbb{C}^2 : \ |x|^2 + |y|^\lambda < 1\} (\lambda > 0, \ \lambda \ne 2)$.

If $\dim(\mathrm{Iso}(D)^0) = 4$, *then D is holomorphically equivalent to the unit ball*

$$B^2 = \{(x, y) \in \mathbb{C}^2 : \ |x|^2 + |y|^2 < 1\}.$$

In the next section, we use a new approach to prove Theorem 2.2. By the results of Xu [5], the classification of bounded positive (m, p)-circle domains in the sense of holomorphic isomorphism is given.

3 The Classification of Reinhardt Domains

In the above section, we prove that for a bounded positive (m, p)-circle domain D, if $\dim(\mathrm{Iso}(D)^0) = 2$, then D is holomorphically equivalent to a Reinhardt domain; if $\dim(\mathrm{Iso}(D)^0) = 4$, then D is holomorphically equivalent to the unit ball in $\mathbb{C}^2$. Thus in order to classify bounded positive (m, p)-circle domains with the isotropic subgroup of dimensions 2 or 4, we just do the classification of bounded Reinhardt domains in $\mathbb{C}^2$. Thullen [3] gave the classification of bounded Reinhardt domains in $\mathbb{C}^2$ by the geometric property of domains. In this section, firstly we give the Lie algebra of holomorphic automorphism group and holomorphic automorphism group, then prove the Thullen's classification result again by another method.

In a way similar to [5], we assert that bounded positive (m, p)-circle domains allow holomorphic automorphism families

$$x' = \frac{x + \alpha}{1 + x\overline{\alpha}}, \quad y' = g(x, y, \alpha, \overline{\alpha}), \quad |\alpha| < 1,$$

where $\alpha \in \mathbb{C}$. The function $g(x, y, \alpha, \overline{\alpha})$ is continuous on $(x, y) \in D$, $|\alpha| < 1$. Then the point set

$$\mathfrak{T} = \Big\{(0, y) \in \mathbb{C}^2 \ \Big| \ |y| < \tau = \sup_{(x,y)\in D} |g(x, y, -x, -\overline{x})|\Big\}$$

is in D. In fact, the point set

$$\Delta_0 = \{y \in \mathbb{C} \mid (0, y) \in D\}$$

is a bounded open subset in the y-plane. Since $\mathrm{e}^{\mathrm{i}\theta}\Delta_0 \subset \Delta_0$, $\forall\theta \in [0, 2\pi)$, the point set Δ_0 is constructed by some concentric real circle disks. We consider the sectional set of Δ_0 by the positive axis, the section set is a sequence of open intervals

$$[a_0, a_1),\ (a_2, a_3),\ \cdots,\ (a_{2\ell}, a_{2\ell+1}),\ \cdots,$$

where

$$0 = a_0 < a_1 < a_2 < \cdots < a_{2\ell} < a_{2\ell+1} < \cdots \le M \text{ (a fixed postive constant).}$$

If $(x_1, y_1) \in D$, then $\phi(x_1, y_1) = (0, g(x_1, y_1, -x_1, -\overline{x}_1)) \in D$, where

$$\phi:\ x' = \frac{x - x_1}{1 - x\overline{x}_1}, \quad y' = g(x, y, -x_1, -\overline{x}_1).$$

It yields $g(x_1, y_1, -x_1, -\overline{x}_1) \in \Delta_0$. If $|g(x_1, y_1, -x_1, -\overline{x}_1)| \ne 0$, then there is an index k, such that

$$a_{2k} < |g(x_1, y_1, -x_1, -\overline{x}_1)| < a_{2k+1}.$$

Let

$$D_\ell = \{(x, y) \in D \mid a_{2\ell} < |g(x, y, -x, -\overline{x})| < a_{2\ell+1}\},$$

$\ell = 0, 1, 2, \cdots$. Then

$$D = D_0 \cup D_1 \cup D_2 \cup \cdots \cup D_\ell \cup \cdots.$$

When $j < k$, we have $D_j \cap D_k = \emptyset$. Since D is a connected open subset in $\mathbb{C}^2$,

$$D = D_0 = \{(x, y) \in D \mid |g(x, y, -x, -\overline{x})| < a_1\}.$$

Thus

$$\Delta_0 = \{y \in \mathbb{C} \mid (0, y) \in D\} = \{y \in \mathbb{C} \mid |y| < M\}.$$

In other words, $\{(0, y) \in \mathbb{C}^2 \mid |y| < M\} \subset D$, and $\sup\limits_{(0,y)\in D} |y| = M$. $\forall(x_1, y_1) \in D$, we have $|g(x_1, y_1, -x_1, -\overline{x}_1)| < M$. So $\tau = \sup\limits_{(x,y)\in D} |g(x, y, -x, -\overline{x})| \le M$, the assertion holds.

In order to express bounded positive (m, p)-circle domains by the function g, we introduce a point set

$$\mathfrak{D} = \{(x, y) \in \mathbb{C}^2 \mid \text{for all } \alpha \in \mathbb{C}\ (|\alpha| < 1), \text{ the function } g(x, y, \alpha, \overline{\alpha}) \text{ is analysis on } (x, y)\}.$$

Clearly, $\mathfrak{D}$ consists of bounded positive (m, p)-circle domains. Whereas, we have the following lemma.

Lemma 3.1 *Suppose that D is a bounded positive (m, p)-circle domain, for any $(x_1, y_1) \in D$, satisfying*

$$g(0, g(x_1, y_1, -x_1, -\overline{x}_1), x_1, \overline{x}_1) \equiv y_1.$$

Then D is the point set

$$\widehat{D} = \mathfrak{D} \cap \{(x, y) \in \mathbb{C}^2 \mid |g(x, y, -x, -\overline{x})| < \tau,\ |x| < 1\}.$$

Proof Clearly, $\widehat{D} \supset D$. Now $\forall(x_0, y_0) \in \widehat{D}$, $(0, g(x_0, y_0, -x_0, -\overline{x}_0)) \in D$. Now $|x_0| < 1$. Hence there exists a holomorphic automorphism map $x' = \frac{x+x_0}{1+x\overline{x}_0}$, $y' = g(x, y, x_0, \overline{x}_0)$ on D. By the supposition, it maps $(0, g(x_0, y_0, -x_0, -\overline{x}_0))$ to (x_0, y_0). That is, $(x_0, y_0) \in D$, thus $\widehat{D} = D$.

Lemma 3.2 *If D is a bounded Reinhardt domain in $\mathbb{C}^2$ with $\dim(\mathrm{Iso}(D)^0) = 4$, then* $\mathrm{iso}(D)$ *has a group of bases A, B, C, D, where*

$$A = \mathrm{i}x\frac{\partial}{\partial x}, \quad B = \mathrm{i}y\frac{\partial}{\partial y}, \quad C = x\frac{\partial}{\partial y} - ay\frac{\partial}{\partial x}, \quad D = \mathrm{i}x\frac{\partial}{\partial y} + \mathrm{i}ay\frac{\partial}{\partial x}, \quad a > 0.$$

Proof If D is a bounded Reinhardt domain in $\mathbb{C}^2$, then $A = \mathrm{i}x\frac{\partial}{\partial x}$, $B = \mathrm{i}y\frac{\partial}{\partial y} \in \mathrm{iso}(D)$. Since $\mathrm{iso}(D)$ is a real linear space, we have

$$A + B = \mathrm{i}x\frac{\partial}{\partial x} + \mathrm{i}y\frac{\partial}{\partial y} \in \mathrm{iso}(D).$$

Suppose

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \in \mathrm{iso}(D),$$

where $\xi(x,y) = \sum\limits_{j,k=0}^{\infty} a_{jk}x^j y^k$ and $\eta(x,y) = \sum\limits_{j,k=0}^{\infty} b_{jk}x^j y^k$ are two power series expansion at the origin, respectively. By Lemma 2.3, we have $a_{00} = b_{00} = 0$. By a direct calculation,

$$\begin{aligned}
&[A + B, X] \\
&= \Big[\mathrm{i}x\frac{\partial}{\partial x} + \mathrm{i}y\frac{\partial}{\partial y}, \sum_{j,k=0}^{\infty} a_{jk}x^j y^k\frac{\partial}{\partial x} + \sum_{j,k=0}^{\infty} b_{jk}x^j y^k\frac{\partial}{\partial y}\Big] \\
&= \mathrm{i}\sum_{j,k=0}^{\infty}(j+k-1)a_{jk}x^j y^k\frac{\partial}{\partial x} + \mathrm{i}\sum_{j,k=0}^{\infty}(j+k-1)b_{jk}x^j y^k\frac{\partial}{\partial y} \\
&= \mathrm{i}\sum_{j+k\geq 2, j,k\geq 0}(j+k-1)a_{jk}x^j y^k\frac{\partial}{\partial x} + \mathrm{i}\sum_{j+k\geq 2, j,k\geq 0}(j+k-1)b_{jk}x^j y^k\frac{\partial}{\partial y} \in \mathrm{iso}(D).
\end{aligned}$$

By Lemma 2.2, we obtain that $[A + B, X] = 0$, so

$$X = (a_{10}x + a_{01}y)\frac{\partial}{\partial x} + (b_{10}x + b_{01}y)\frac{\partial}{\partial y}.$$

Hence

$$\begin{aligned}
[B, X] &= \mathrm{i}a_{01}y\frac{\partial}{\partial x} - \mathrm{i}b_{10}x\frac{\partial}{\partial y} \in \mathrm{iso}(D), \\
[A, [B, X]] &= a_{01}y\frac{\partial}{\partial x} + b_{10}x\frac{\partial}{\partial y} \in \mathrm{iso}(D), \\
X - [A, [B, X]] &= a_{10}x\frac{\partial}{\partial x} + b_{01}y\frac{\partial}{\partial y} \in \mathrm{iso}(D),
\end{aligned}$$

thus $\mathrm{Re}(a_{10}) = \mathrm{Re}(b_{01}) = 0$. We denote

$$\begin{aligned}
C &= [A, [B, X]] = a_{01}y\frac{\partial}{\partial x} + b_{10}x\frac{\partial}{\partial y}, \\
D &= [A, C] = -\mathrm{i}a_{01}y\frac{\partial}{\partial x} + \mathrm{i}b_{10}x\frac{\partial}{\partial y}.
\end{aligned}$$

If $a_{01} = 0$, then $C = b_{10}x\frac{\partial}{\partial y} \in \mathrm{iso}(D)$, $D = \mathrm{i}b_{10}x\frac{\partial}{\partial y} \in \mathrm{iso}(D)$. By Lemma 2.3, we get $b_{10} = 0$. For the same reason, if $b_{10} = 0$, then $a_{01} = 0$. Therefore if $a_{01} \neq 0$, then $b_{10} \neq 0$; if $b_{10} \neq 0$, then

$a_{01}\neq 0$. Since $\forall\zeta\in\mathbb{C}$, $\zeta b_{10}x\frac{\partial}{\partial y}+\overline{\zeta}a_{01}y\frac{\partial}{\partial x}\in \mathrm{iso}(D)$, without loss of generality, when $b_{10}\neq 0$, we suppose $b_{10}=1$, $a_{01}\neq 0$. By $[C,D]=-2a_{01}\big(\mathrm{i}x\frac{\partial}{\partial x}-\mathrm{i}y\frac{\partial}{\partial y}\big)\in\mathrm{iso}(D)$, we have $a_{01}\in\mathbb{R}$.

The one-parameter subgroup $(x'(t),y'(t))=\exp(tC)$ $(t\in\mathbb{R})$ is a unique holomorphic solution of the complex ordinary differential equation systems

$$\frac{\mathrm{d}x'}{\mathrm{d}t}=a_{01}y',\quad \frac{\mathrm{d}y'}{\mathrm{d}t}=x',\quad t\in\mathbb{R}$$

with the initial values $x'(0)=x$, $y'(0)=y$. If $a_{01}>0$, then the solution is

$$\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}\mathrm{ch}(\sqrt{a_{01}}t) & \sqrt{a_{01}}\mathrm{sh}(\sqrt{a_{01}}t)\\ \frac{1}{\sqrt{a_{01}}}\mathrm{sh}(\sqrt{a_{01}}t) & \mathrm{ch}(\sqrt{a_{01}}t)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix},$$

which transforms $(x_0,0)$ to $(\mathrm{ch}(\sqrt{a_{01}}t)x_0,\frac{1}{\sqrt{a_{01}}}\mathrm{sh}(\sqrt{a_{01}}t)x_0)$. But D is a bounded domain, it is a contraction. If $a_{01}<0$, then the solution is

$$\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}\cos(\sqrt{-a_{01}}t) & -\sqrt{-a_{01}}\sin(\sqrt{-a_{01}}t)\\ \frac{1}{\sqrt{-a_{01}}}\sin(\sqrt{-a_{01}}t) & \cos(\sqrt{-a_{01}}t)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

Therefore $a_{01}<0$, denoting $a=-a_{01}>0$, we get $C=x\frac{\partial}{\partial y}-ay\frac{\partial}{\partial x}$, $D=\mathrm{i}x\frac{\partial}{\partial y}+\mathrm{i}ay\frac{\partial}{\partial x}$, $a>0$.

Lemma 3.3 *Suppose that D is a bounded Reinhardt domain in $\mathbb{C}^2$, $\mathrm{aut}(D)$ is a direct sum of two subspaces: one is $\mathrm{iso}(D)$, the other is linear combination of some vector fields as follows*

$$X_1=(1-\rho_1x^2)\frac{\partial}{\partial x}-\rho_2xy\frac{\partial}{\partial y},\qquad Y_1=\mathrm{i}(1+\rho_1x^2)\frac{\partial}{\partial x}+\mathrm{i}\rho_2xy\frac{\partial}{\partial y};$$
$$X_2=-\lambda_2xy\frac{\partial}{\partial x}+(1-\lambda_1y^2)\frac{\partial}{\partial y},\quad Y_2=\mathrm{i}\lambda_2xy\frac{\partial}{\partial x}+\mathrm{i}(1+\lambda_1y^2)\frac{\partial}{\partial y},$$

where $\rho_1,\ \lambda_1\geq 0,\ \rho_2,\ \lambda_2\geq 0$.

Proof Since D satisfies the Thullen condition, there exists $(a_{00},b_{00})\neq 0$, such that

$$X=\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y}\in\mathrm{aut}(D),\quad X\notin\mathrm{iso}(D),$$

where $\xi(x,y)=\sum\limits_{j,k=0}^{\infty}a_{jk}x^jy^k$ and $\eta(x,y)=\sum\limits_{j,k=0}^{\infty}b_{jk}x^jy^k$ are two power series expansion at the origin, respectively. By a direct calculation, we have

$$[A+B,X]=\mathrm{i}\sum_{j,k=0}^{\infty}(j+k-1)a_{jk}x^jy^k\frac{\partial}{\partial x}+\mathrm{i}\sum_{j,k=0}^{\infty}(j+k-1)b_{jk}x^jy^k\frac{\partial}{\partial y},$$

$$[A+B,[A+B,X]]=-\sum_{j,k=0}^{\infty}(j+k-1)^2a_{jk}x^jy^k\frac{\partial}{\partial x}-\sum_{j,k=0}^{\infty}(j+k-1)^2b_{jk}x^jy^k\frac{\partial}{\partial y},$$

$$\begin{aligned}&[A+B,[A+B,[A+B,X]]]\\ =&-\mathrm{i}\sum_{j,k=0}^{\infty}(j+k-1)^3a_{jk}x^jy^k\frac{\partial}{\partial x}-\mathrm{i}\sum_{j,k=0}^{\infty}(j+k-1)^3b_{jk}x^jy^k\frac{\partial}{\partial y},\\ &[A+B,[A+B,[A+B,[A+B,X]]]]\\ =&\sum_{j,k=0}^{\infty}(j+k-1)^4a_{jk}x^jy^k\frac{\partial}{\partial x}+\sum_{j,k=0}^{\infty}(j+k-1)^4b_{jk}x^jy^k\frac{\partial}{\partial y}.\end{aligned}$$

By

$$\begin{aligned}
&[A+B,[A+B,[A+B,[A+B,X]]]]+[A+B,[A+B,X]]\\
=&\sum_{j,k=0}^{\infty}(j+k-1)^2((j+k-1)^2-1)a_{jk}x^jy^k\frac{\partial}{\partial x}\\
&+\sum_{j,k=0}^{\infty}(j+k-1)^2((j+k-1)^2-1)b_{jk}x^jy^k\frac{\partial}{\partial y}
\end{aligned}$$

and Lemma 2.2, we know

$$[A+B,[A+B,[A+B,[A+B,X]]]]+[A+B,[A+B,X]]=0,$$

therefore

$$\begin{aligned}
X&=\sum_{j+k\le 2,j,k\ge 0}a_{jk}x^jy^k\frac{\partial}{\partial x}+\sum_{j+k\le 2,j,k\ge 0}b_{jk}x^jy^k\frac{\partial}{\partial y},\\
X+[A+B,[A+B,X]]&=(a_{10}x+a_{01}y)\frac{\partial}{\partial x}+(b_{10}x+b_{01}y)\frac{\partial}{\partial y}\in \mathrm{iso}(D).
\end{aligned}$$

Without loss of generality, we suppose

$$X=(a_{00}+a_{20}x^2+a_{11}xy+a_{02}y^2)\frac{\partial}{\partial x}+(b_{00}+b_{02}y^2+b_{11}xy+b_{20}x^2)\frac{\partial}{\partial y},$$

then

$$\begin{aligned}
[A,X]&=\mathrm{i}(-a_{00}+a_{20}x^2-a_{02}y^2)\frac{\partial}{\partial x}+\mathrm{i}(b_{11}xy+2b_{20}x^2)\frac{\partial}{\partial y},\\
[B,[A,X]]&=-2a_{02}y^2\frac{\partial}{\partial x}+2b_{20}x^2\frac{\partial}{\partial y}\in \mathrm{iso}(D).
\end{aligned}$$

By Lemma 2.2, we get $a_{02}=b_{20}=0$. So we have

$$X=(a_{00}+a_{20}x^2+a_{11}xy)\frac{\partial}{\partial x}+(b_{00}+b_{02}y^2+b_{11}xy)\frac{\partial}{\partial y}.$$

Since

$$\begin{aligned}
[A,X]&=\mathrm{i}(-a_{00}+a_{20}x^2)\frac{\partial}{\partial x}+\mathrm{i}b_{11}xy\frac{\partial}{\partial y}\in \mathrm{aut}(D),\\
[A,[A,X]]&=-(a_{00}+a_{20}x^2)\frac{\partial}{\partial x}-b_{11}xy\frac{\partial}{\partial y}\in \mathrm{aut}(D),
\end{aligned}$$

we know that $\mathrm{aut}(D)$ has elements

$$\begin{aligned}
X_1&=-[A,[A,X]]=(a_{00}+a_{20}x^2)\frac{\partial}{\partial x}+b_{11}xy\frac{\partial}{\partial y},\\
Y_1&=-[A,X_1]=\mathrm{i}(a_{00}-a_{20}x^2)\frac{\partial}{\partial x}-\mathrm{i}b_{11}xy\frac{\partial}{\partial y}
\end{aligned}$$

and

$$\begin{aligned}
X_2&=X-X_1=a_{11}xy\frac{\partial}{\partial x}+(b_{00}+b_{02}y^2)\frac{\partial}{\partial y},\\
Y_2&=-[A,X_1]=-\mathrm{i}a_{11}xy\frac{\partial}{\partial x}+\mathrm{i}(b_{00}-b_{02}y^2)\frac{\partial}{\partial y}.
\end{aligned}$$

Note that if $a_{00}=0$, then $X_1=Y_1=0$; if $b_{00}=0$, then $X_2=Y_2=0$. When $a_{00}\neq 0$, denote $a_{00}^{-1}=a_1+\mathrm{i}a_2$, then $a_1X_1+a_2Y_1=(1+(a_1-\mathrm{i}a_2)a_{20}x^2)\frac{\partial}{\partial x}+(a_1-\mathrm{i}a_2)b_{11}xy\frac{\partial}{\partial y}\in\mathrm{aut}(D)$, thus we can suppose $a_{00}=1$. For the same reason, when $b_{00}\neq 0$, we also can suppose $b_{00}=1$.

If $a_{00}=1$, then we have

$$[X_1,Y_1]=-4\mathrm{i}a_{20}x\frac{\partial}{\partial x}-2\mathrm{i}b_{11}y\frac{\partial}{\partial y}\in\mathrm{iso}(D),$$

thus there exist $\rho_1,\ \rho_2\in\mathbb{R}$, such that $a_{20}=-\rho_1$, $b_{11}=-\rho_2$. For the same reason when $b_{00}=1$, there exist $\lambda_1,\lambda_2\in\mathbb{R}$, such that $b_{02}=-\lambda_1$, $a_{11}=-\lambda_2$.

The one-parameter subgroup $(x'(t),y'(t))=\exp(tX_1)$ $(t\in\mathbb{R})$ is a unique holomorphic solution of the complex ordinary differential equation systems

$$\frac{\mathrm{d}x'}{\mathrm{d}t}=1-\rho_1(x')^2,\quad \frac{\mathrm{d}y'}{\mathrm{d}t}=-\rho_2x'y',\quad t\in\mathbb{R}$$

with the initial values $x'(0)=x$, $y'(0)=y$. If $\rho_1=0$, then $x'(t)=x+t,\ t\in\mathbb{R}$, which contracts that D is a bounded domain. If $\rho_1<0$, then

$$x'(t)=\frac{x\cos(\sqrt{-\rho_1}t)+\frac{1}{\sqrt{-\rho_1}}\sin(\sqrt{-\rho_1}t)}{\cos(\sqrt{-\rho_1}t)+\sqrt{-\rho_1}x\sin(\sqrt{-\rho_1}t)}=\frac{x+\frac{1}{\sqrt{-\rho_1}}\tan(\sqrt{-\rho_1}t)}{1+\sqrt{-\rho_1}x\tan(\sqrt{-\rho_1}t)},\quad t\in\mathbb{R}.$$

It is a contraction that D is a bounded domain. If $\rho_1>0$, then

$$x'(t)=\frac{x\mathrm{ch}(\sqrt{\rho_1}t)+\frac{1}{\sqrt{\rho_1}}\mathrm{sh}(\sqrt{\rho_1}t)}{\mathrm{ch}(\sqrt{\rho_1}t)+\sqrt{\rho_1}x\mathrm{sh}(\sqrt{\rho_1}t)},\quad y'(t)=y(\mathrm{ch}(\sqrt{\rho_1}t)+\sqrt{\rho_1}x\mathrm{sh}(\sqrt{\rho_1}t))^{-\rho_2\rho_1^{-1}},\quad t\in\mathbb{R}.$$

Since D is a bounded domain, it yields $\rho_2\geq 0$. Therefore, we have proved $\rho_1>0,\ \rho_2\geq 0$. For the same reason, we can prove $\lambda_1>0,\ \lambda_2\geq 0$.

Now one can prove the following theorems.

Theorem 3.1 *Let D be a bounded Reinhardt domain in $\mathbb{C}^2$. If $\dim(\mathrm{Iso}(D)^0)=2$, then* $\mathrm{aut}(D)$ *has a group of bases $A=\mathrm{i}x\frac{\partial}{\partial x}$, $B=\mathrm{i}y\frac{\partial}{\partial y}$ and*

Case (I)

$$X_1=(1-\rho_1x^2)\frac{\partial}{\partial x},\quad Y_1=\mathrm{i}(1+\rho_1x^2)\frac{\partial}{\partial x},\quad \rho_1>0,\ \rho_2=0,$$
$$X_2=(1-\lambda_1y^2)\frac{\partial}{\partial y},\quad Y_2=\mathrm{i}(1+\lambda_1y^2)\frac{\partial}{\partial y},\quad \lambda_1>0,\ \lambda_2=0;$$

Case (II)

$$X_1=(1-\rho_1x^2)\frac{\partial}{\partial x}-\rho_2xy\frac{\partial}{\partial y},\quad Y_1=\mathrm{i}(1+\rho_1x^2)\frac{\partial}{\partial x}+\mathrm{i}\rho_2xy\frac{\partial}{\partial y},\quad \rho_1,\rho_2>0,\ \rho_1\neq\rho_2;$$

Case (III)

$$X_2=-\lambda_2xy\frac{\partial}{\partial x}+(1-\lambda_1y^2)\frac{\partial}{\partial y},\quad Y_2=\mathrm{i}\lambda_2xy\frac{\partial}{\partial x}+\mathrm{i}(1+\lambda_1y^2)\frac{\partial}{\partial y},\quad \lambda_1,\lambda_2>0,\ \lambda_1\neq\lambda_2.$$

If $\dim(\mathrm{Iso}(D)^0)=4$, then $\mathrm{aut}(D)$ *has a group of bases*

$$A=\mathrm{i}x\frac{\partial}{\partial x},\quad B=\mathrm{i}y\frac{\partial}{\partial y},\quad C=x\frac{\partial}{\partial y}-ay\frac{\partial}{\partial x},\quad D=\mathrm{i}x\frac{\partial}{\partial y}+\mathrm{i}ay\frac{\partial}{\partial x},$$
$$X_1=(1-cx^2)\frac{\partial}{\partial x}-cxy\frac{\partial}{\partial y},\quad Y_1=\mathrm{i}(1+cx^2)\frac{\partial}{\partial x}+\mathrm{i}cxy\frac{\partial}{\partial y},$$
$$X_2=-acxy\frac{\partial}{\partial x}+(1-acy^2)\frac{\partial}{\partial y},\quad Y_2=\mathrm{i}acxy\frac{\partial}{\partial x}+\mathrm{i}(1+acy^2)\frac{\partial}{\partial y},$$

where $a>0$, $c>0$.

Proof If $X_1=(1-\rho_1x^2)\frac{\partial}{\partial x}-\rho_2xy\frac{\partial}{\partial y}$, $X_1'=(1-\rho_1'x^2)\frac{\partial}{\partial x}-\rho_2'xy\frac{\partial}{\partial y}\in \mathrm{aut}(D)$, where $\rho_1,\rho_1'>0$, $\rho_2,\rho_2'\geq 0$, then

$$X_1-X_1'=(\rho_1'-\rho_1)x^2\frac{\partial}{\partial x}+(\rho_2'-\rho_2)xy\frac{\partial}{\partial y}\in \mathrm{iso}(D).$$

By Lemma 2.2, we have $\rho_1'=\rho_1$, $\rho_2'=\rho_2$.

(1) If $\dim(\mathrm{Iso}(D)^0)=2$, then $\mathrm{Iso}(D)^0$ is a real Lie group

$$x'=\mathrm{e}^{\mathrm{i}\theta}x,\quad y'=\mathrm{e}^{\mathrm{i}\varphi}y,\quad \theta,\varphi\in[0,2\pi),$$

and $\mathrm{iso}(D)$ has a group of bases $A=\mathrm{i}x\frac{\partial}{\partial x}$, $B=\mathrm{i}y\frac{\partial}{\partial y}$.

If $a_{00}b_{00}\neq 0$, then

$$[X_1,X_2]=-(\lambda_2y+\lambda_2(\rho_1-\rho_2)x^2y)\frac{\partial}{\partial x}+(\rho_2x+\rho_2(\lambda_1-\lambda_2)xy^2)\frac{\partial}{\partial y}\in \mathrm{iso}(D).$$

It yields $\rho_2=\lambda_2=0$. Case (I) holds.

If $(a_{00},b_{00})=(1,0)$, then $X_1\neq 0$, $X_2=0$. The one-parameter subgroup $(x'(t),y'(t))=\exp(t((\cos\varsigma)X_1+(\sin\varsigma)Y_1))$ $(t\in\mathbb{R})$ is a unique holomorphic solution of the complex ordinary differential equation systems

$$\frac{\mathrm{d}x'}{\mathrm{d}t}=\mathrm{e}^{\mathrm{i}\varsigma}-\mathrm{e}^{-\mathrm{i}\varsigma}\rho_1(x')^2,\quad \frac{\mathrm{d}y'}{\mathrm{d}t}=-\rho_2x'y',\quad t\in\mathbb{R}$$

with the initial values $x'(0)=x$, $x'(0)=y$, where $\varsigma\in[0,2\pi)$. We get the solution

$$x'=\frac{\mathrm{ch}(\sqrt{\rho_1}t)x+(\sqrt{\rho_1})^{-1}\mathrm{e}^{\mathrm{i}\varsigma}\mathrm{sh}(\sqrt{\rho_1}t)}{\mathrm{ch}(\sqrt{\rho_1}t)+x\mathrm{e}^{-\mathrm{i}\varsigma}\sqrt{\rho_1}\mathrm{sh}(\sqrt{\rho_1}t)},\quad y'=y(\mathrm{ch}(\sqrt{\rho_1}t)+x\mathrm{e}^{-\mathrm{i}\varsigma}\sqrt{\rho_1}\mathrm{sh}(\sqrt{\rho_1}t))^{-\frac{\rho_2}{\rho_1}},\quad t\in\mathbb{R}.$$

If $\rho_2=0$, then

$$\sqrt{\rho_1}x'=\frac{\sqrt{\rho_1}x+\mathrm{e}^{\mathrm{i}\varsigma}\mathrm{tgh}(\sqrt{\rho_1}t)}{1+\sqrt{\rho_1}x\mathrm{e}^{-\mathrm{i}\varsigma}\mathrm{tgh}(\sqrt{\rho_1}t)},\quad y'=y,\quad t\in\mathbb{R}.$$

By Lemma 3.1, it yields $D=\{(x,y)\in\mathbb{C}^2\mid|\sqrt{\rho_1}x|<1,\ |y|<M\}$. But we can see $\mathrm{Aut}(D)^0$ allows

$$x'=x,\quad y'=M\frac{y+M\beta}{M+y\overline{\beta}},\quad |\beta|<1,$$

so $\rho_2=0$ does not hold.

Next we prove $\rho_1\neq\rho_2$. If $\rho_1=\rho_2$, then

$$\sqrt{\rho_1}x'=\frac{\sqrt{\rho_1}x+\mathrm{e}^{\mathrm{i}\varsigma}\mathrm{tgh}(\sqrt{\rho_1}t)}{1+\sqrt{\rho_1}x\mathrm{e}^{-\mathrm{i}\varsigma}\mathrm{tgh}(\sqrt{\rho_1}t)},\quad y'=y\frac{\sqrt{1-|\mathrm{tgh}(\sqrt{\rho_1}t)|^2}}{1+\sqrt{\rho_1}x\mathrm{e}^{-\mathrm{i}\varsigma}\mathrm{tgh}(\sqrt{\rho_1}t)},\quad t\in\mathbb{R}.$$

By Lemma 3.1, it yields $D=\{(x,y)\in\mathbb{C}^2\mid|x|^2<\rho_1^{-1}(1-|y|^2)\}$. But we can see $\mathrm{Iso}(D)^0$ allows

$$(x',y')=(x,y)\begin{pmatrix}\sqrt{\rho_1}&0\\0&1\end{pmatrix}U\begin{pmatrix}\sqrt{\rho_1}&0\\0&1\end{pmatrix}^{-1},\quad U\in\mathrm{U}_2,$$

so $\rho_1=\rho_2$ does not hold. Therefore, Case (II) holds.

If $(a_{00}, b_{00}) = (0, 1)$, then $X_1 = 0$, $X_2 \neq 0$. As the same reason as the case $(a_{00}, b_{00}) = (1, 0)$, we get $\lambda_2 \neq 0$, and $\lambda_1 \neq \lambda_2$. Therefore, Case (III) holds.

(2) If $\dim(\mathrm{Iso}(D)^0) = 4$, then $\mathrm{iso}(D)$ has a group of bases

$$A = \mathrm{i}x\frac{\partial}{\partial x}, \quad B = \mathrm{i}y\frac{\partial}{\partial y}, \quad C = x\frac{\partial}{\partial y} - ay\frac{\partial}{\partial x}, \quad D = \mathrm{i}x\frac{\partial}{\partial y} + \mathrm{i}ay\frac{\partial}{\partial x}, \quad a > 0.$$

If $a_{00} \neq 0$, then $X_1 \neq 0$. By a direct computation

$$\begin{aligned}
&[C, X_1] = (2\rho_1 - \rho_2)axy\frac{\partial}{\partial x} - (1 - (\rho_1 - \rho_2)x^2 - \rho_2 ay^2)\frac{\partial}{\partial y},\\
&[A, [C, X_1]] = \mathrm{i}2(\rho_1 - \rho_2)x^2\frac{\partial}{\partial y} \in \mathrm{iso}(D),
\end{aligned}$$

we get $\rho_1 = \rho_2$. Then

$$\begin{aligned}
-[C, X_1] &= -a\rho_1 xy\frac{\partial}{\partial x} + (1 - a\rho_1 y^2)\frac{\partial}{\partial y} \in \mathrm{aut}(D),\\
[X_1, -[C, X_1]] &= \rho_1\Big(x\frac{\partial}{\partial y} - ay\frac{\partial}{\partial x}\Big) = \rho_1 C \in \mathrm{iso}(D).
\end{aligned}$$

Thus $b_{00} \neq 0$.

If $b_{00} \neq 0$, then $X_2 \neq 0$. By a direct calculation

$$\begin{aligned}
&[C, X_2] = (a - \lambda_2 x^2 - a(\lambda_1 - \lambda_2)y^2)\frac{\partial}{\partial x} - (2\lambda_1 - \lambda_2)xy\frac{\partial}{\partial y} \in \mathrm{aut}(D),\\
&[B, [C, X_2]] = -\mathrm{i}2a(\lambda_1 - \lambda_2)y^2\frac{\partial}{\partial x} \in \mathrm{iso}(D),
\end{aligned}$$

we get $\lambda_1 = \lambda_2$. Then

$$\begin{aligned}
&a^{-1}[C, X_2] = (1 - a^{-1}\lambda_1 x^2)\frac{\partial}{\partial x} - a^{-1}\lambda_1 xy\frac{\partial}{\partial y} \in \mathrm{aut}(D),\\
&[a^{-1}[C, X_2], X_2] = a^{-1}\lambda_1\Big(x\frac{\partial}{\partial y} - ay\frac{\partial}{\partial x}\Big) = a^{-1}\lambda_1 C \in \mathrm{iso}(D).
\end{aligned}$$

Thus $a_{00} \neq 0$.

When $a_{00}b_{00} \neq 0$, then

$$[X_1, X_2] = -\lambda_2 y\frac{\partial}{\partial x} + \rho_2 x\frac{\partial}{\partial y} \in \mathrm{iso}(D),$$

there exists $c \in \mathbb{R}$, such that $\lambda_2 = a\rho_2 = ac$. So we have $\rho_1 = \rho_2 = c > 0$, $\lambda_1 = \lambda_2 = ac$.

Therefore, we have proved that $\mathrm{aut}(D)$ has a group of bases

$$\begin{aligned}
&A = \mathrm{i}x\frac{\partial}{\partial x}, \quad B = \mathrm{i}y\frac{\partial}{\partial y}, \quad C = x\frac{\partial}{\partial y} - ay\frac{\partial}{\partial x}, \quad D = \mathrm{i}x\frac{\partial}{\partial y} + \mathrm{i}ay\frac{\partial}{\partial x},\\
&X_1 = (1 - cx^2)\frac{\partial}{\partial x} - cxy\frac{\partial}{\partial y}, \quad Y_1 = \mathrm{i}(1 + cx^2)\frac{\partial}{\partial x} + \mathrm{i}cxy\frac{\partial}{\partial y},\\
&X_2 = -acxy\frac{\partial}{\partial x} + (1 - acy^2)\frac{\partial}{\partial y}, \quad Y_2 = \mathrm{i}acxy\frac{\partial}{\partial x} + \mathrm{i}(1 + acy^2)\frac{\partial}{\partial y},
\end{aligned}$$

where $a > 0$, $c > 0$.

Theorem 3.2 *Suppose that D is a bounded Reinhardt domain in $\mathbb{C}^2$. Then there exists a linear isomorphism σ, such that $\sigma(D)$ is one of the three classes:*

(1) *Polydisc* $\Delta^2=\{(x,y)\in\mathbb{C}^2:\ |x|<1,\ |y|<1\}$;

(2) *Thullen domain* $D_\lambda=\{(x,y)\in\mathbb{C}^2:\ |x|^2+|y|^\lambda<1\}$ $(\lambda>0,\ \lambda\neq 2)$;

(3) *Unit ball* $B^2=\{(x,y)\in\mathbb{C}^2:\ |x|^2+|y|^2<1\}$.

Proof If $\dim(\mathrm{Iso}(D)^0)=2$, then $\mathrm{iso}(D)$ has a group of bases: $A=\mathrm{i}x\frac{\partial}{\partial x}$, $B=\mathrm{i}y\frac{\partial}{\partial y}$.

For Case (I), given a linear isomorphism $\sigma:(u,v)=(\sqrt{\rho_1}x,\sqrt{\lambda_1}y)$, then $\mathrm{aut}(\sigma(D))$ has a group of bases

$$\begin{aligned}&A^*=\sigma_*(A)=\mathrm{i}u\frac{\partial}{\partial u},\quad B^*=\sigma_*(B)=\mathrm{i}v\frac{\partial}{\partial v},\\ &X_1^*=\frac{1}{\sqrt{\rho_1}}\sigma_*(X_1)=(1-u^2)\frac{\partial}{\partial u},\quad Y_1^*=\frac{1}{\sqrt{\rho_1}}\sigma_*(Y_1)=\mathrm{i}(1+u^2)\frac{\partial}{\partial u},\\ &X_2^*=\frac{1}{\sqrt{\lambda_1}}\sigma_*(X_2)=(1-v^2)\frac{\partial}{\partial v},\quad Y_2^*=\frac{1}{\sqrt{\lambda_1}}\sigma_*(Y_2)=\mathrm{i}(1+v^2)\frac{\partial}{\partial v}.\end{aligned}$$

For fixed $t_1,\ t_2\in\mathbb{R}$, the one-parameter subgroup

$$(u(t),v(t))=\exp(t(t_1(\cos\varsigma_1X_1^*+\sin\varsigma_1Y_1^*)+t_2(\cos\varsigma_2X_2^*+\sin\varsigma_2Y_2^*))),\quad t\in\mathbb{R}$$

is a unique holomorphic solution of the complex ordinary differential equation systems

$$\frac{\mathrm{d}u'}{\mathrm{d}t}=t_1(\mathrm{e}^{\mathrm{i}\varsigma_1}-\mathrm{e}^{-\mathrm{i}\varsigma_1}(u')^2),\quad \frac{\mathrm{d}v'}{\mathrm{d}t_2}=t_2(\mathrm{e}^{\mathrm{i}\varsigma_2}-\mathrm{e}^{-\mathrm{i}\varsigma_2}(v')^2),\quad t\in\mathbb{R}$$

with the initial values $u'(0)=u$, $v'(0)=v$, where $\varsigma_1,\ \varsigma_2\in[0,2\pi)$. We get the one-parameter analytic transformation:

$$u'=\frac{u+\mathrm{e}^{\mathrm{i}\varsigma_1}\mathrm{tgh}(tt_1)}{1+u\mathrm{e}^{-\mathrm{i}\varsigma_1}\mathrm{tgh}(tt_1)},\quad v'=\frac{v+\mathrm{e}^{\mathrm{i}\varsigma_2}\mathrm{tgh}(tt_2)}{1+v\mathrm{e}^{-\mathrm{i}\varsigma_2}\mathrm{tgh}(tt_2)}.$$

Thus D is linearly equivalent to $D_1^*=\sigma(D)$, $\mathrm{Aut}(D_1^*)^0$ is a real Lie group with $\dim(\mathrm{Aut}(D_1^*)^0)=6$:

$$u'=\mathrm{e}^{\mathrm{i}\theta}\frac{u+\alpha}{1+u\overline{\alpha}},\quad v'=\mathrm{e}^{\mathrm{i}\varphi}\frac{v+\beta}{1+v\overline{\beta}},$$

where $\theta,\varphi\in[0,2\pi)$, $|\alpha|=|\mathrm{e}^{\mathrm{i}\varsigma_1}\mathrm{tgh}t_1|<1$, $|\beta|=|\mathrm{e}^{\mathrm{i}\varsigma_2}\mathrm{tgh}\,t_2|<1$. By Lemma 3.1, one can obtain

$$D_1^*=\Delta^2=\{(u,v)\in\mathbb{C}^2\mid |u|<1,\ |v|<1\}.$$

For Case (II), given a linear isomorphism $\sigma:(u,v)=(\sqrt{\rho_1}x,y)$, then $\mathrm{aut}(\sigma(D))$ has a group of bases

$$\begin{aligned}&A^*=\sigma_*(A)=\mathrm{i}u\frac{\partial}{\partial u},\quad B^*=\sigma_*(B)=\mathrm{i}v\frac{\partial}{\partial v},\\ &X_1^*=\frac{1}{\sqrt{\rho_1}}\sigma_*(X_1)=(1-u^2)\frac{\partial}{\partial u}-\frac{\rho_2}{\rho_1}uv\frac{\partial}{\partial v},\\ &Y_1^*=\frac{1}{\sqrt{\rho_1}}\sigma_*(Y_1)=\mathrm{i}(1+u^2)\frac{\partial}{\partial u}+\mathrm{i}\frac{\rho_2}{\rho_1}uv\frac{\partial}{\partial v}.\end{aligned}$$

The one-parameter subgroup $(u'(t),v'(t))=\exp(t(\cos\varsigma X_1^*+\sin\varsigma Y_1^*))$ $(t\in\mathbb{R})$ is a unique holomorphic solution of the complex ordinary differential equation systems

$$\frac{\mathrm{d}u'}{\mathrm{d}t}=\mathrm{e}^{\mathrm{i}\varsigma}-\mathrm{e}^{-\mathrm{i}\varsigma}(u')^2,\quad \frac{\mathrm{d}v'}{\mathrm{d}t}=-\mathrm{e}^{-\mathrm{i}\varsigma}\frac{\rho_2}{\rho_1}u'v',\quad t\in\mathbb{R}$$

with the initial values $u'(0)=u$, $v'(0)=v$, where $\varsigma\in[0,2\pi)$. We get the solution

$$u'=\frac{u+\alpha}{1+u\overline{\alpha}},\quad v'=v\Big(\frac{\sqrt{1-|\alpha|^2}}{1+u\overline{\alpha}}\Big)^{\frac{\rho_2}{\rho_1}},$$

where $\alpha=\mathrm{e}^{\mathrm{i}\varsigma}\mathrm{tgh}\,t$. Therefore D is linearly isomorphic to $D_2^*=\sigma(D)$, and $\mathrm{Aut}(D_2^*)^0$ is a real Lie group with $\dim(\mathrm{Aut}(D_2^*)^0)=4$:

$$u'=\mathrm{e}^{\mathrm{i}\theta}\frac{u+\alpha}{1+u\overline{\alpha}},\quad v'=\mathrm{e}^{\mathrm{i}\varphi}v\Big(\frac{\sqrt{1-|\alpha|^2}}{1+u\overline{\alpha}}\Big)^{\frac{\rho_2}{\rho_1}},$$

where $\theta,\varphi\in[0,2\pi)$, $\alpha\in\mathbb{C}$, $|\alpha|<1$. By Lemma 3.1, we can obtain

$$D_2^*=\{(u,v)\in\mathbb{C}^2\mid |u|<1,\ |v|^{\frac{2\rho_1}{\rho_2}}<\tau(1-|u|^2)\},\quad \tau>0.$$

For Case (III), given a linear isomorphism $\sigma:(u,v)=(x,\sqrt{\lambda_1}y)$, then $\mathrm{aut}(\sigma(D))$ has a group of bases

$$\begin{aligned}
A^*&=\sigma_*(A)=\mathrm{i}u\frac{\partial}{\partial u},\quad B^*=\sigma_*(B)=\mathrm{i}v\frac{\partial}{\partial v},\\
X_2^*&=\frac{1}{\sqrt{\lambda_1}}\sigma_*(X_2)=-\frac{\lambda_2}{\lambda_1}uv\frac{\partial}{\partial u}+(1-u^2)\frac{\partial}{\partial v},\\
Y_2^*&=\frac{1}{\sqrt{\lambda_1}}\sigma_*(Y_2)=\mathrm{i}\frac{\lambda_2}{\lambda_1}uv\frac{\partial}{\partial u}+\mathrm{i}(1+u^2)\frac{\partial}{\partial v}.
\end{aligned}$$

The one-parameter subgroup $(u'(t),v'(t))=\exp(t(\cos\varsigma X_2^*+\sin\varsigma Y_2^*))$ $(t\in\mathbb{R})$ is a unique holomorphic solution of the complex ordinary differential equation systems

$$\frac{\mathrm{d}u'}{\mathrm{d}t}=-\mathrm{e}^{-\mathrm{i}\varsigma}\frac{\lambda_2}{\lambda_1}u'v',\quad \frac{\mathrm{d}v'}{\mathrm{d}t}=\mathrm{e}^{\mathrm{i}\varsigma}-\mathrm{e}^{-\mathrm{i}\varsigma}(v')^2,\quad t\in\mathbb{R}$$

with the initial values $u'(0)=u$, $v'(0)=v$, where $\varsigma\in[0,2\pi)$. We get the solution

$$u'=u\Big(\frac{\sqrt{1-|\alpha|^2}}{1+v\overline{\alpha}}\Big)^{\frac{\lambda_2}{\lambda_1}},\quad v'=\frac{v+\alpha}{1+v\overline{\alpha}},$$

where $\alpha=\mathrm{e}^{\mathrm{i}\varsigma}\mathrm{tgh}\,t$. Therefore D is linearly isomorphic to $D_3^*=\sigma(D)$, and $\mathrm{Aut}(D_3^*)^0$ is a real Lie group with $\dim(\mathrm{Aut}(D_3^*)^0)=4$:

$$u'=\mathrm{e}^{\mathrm{i}\theta}u\Big(\frac{\sqrt{1-|\alpha|^2}}{1+u\overline{\alpha}}\Big)^{\frac{\lambda_2}{\lambda_1}},\quad v'=\mathrm{e}^{\mathrm{i}\varphi}\frac{u+\alpha}{1+u\overline{\alpha}},\quad \theta,\varphi\in[0,2\pi),\ |\alpha|<1.$$

By Lemma 3.1, we can get

$$D_3^*=\{(u,v)\in\mathbb{C}^2\mid |u|^{\frac{2\lambda_1}{\lambda_2}}<\tau(1-|v|^2),\ |v|<1\},\quad \tau>0.$$

We can see for the domain

$$D_1=\{(x,y)\in\mathbb{C}^2\mid |x|^2+|y|^\lambda<1\},\quad \lambda>0,\ \lambda\neq 2,$$

given a linear isomorphism $\sigma:\ (x,y)\longrightarrow(y,x)$, then D_1 is linearly isomorphic to

$$D_2=\{(x,y)\in\mathbb{C}^2\mid |x|^\lambda+|y|^2<1\},\quad \lambda>0,\ \lambda\neq 2.$$

Therefore we regard Case (II) and Case (III) as the same class, denote the canonical domain

$$D_\lambda = \{(x,y) \in \mathbb{C}^2 \mid |x|^2 + |y|^\lambda < 1\}, \quad \lambda > 0,\ \lambda \neq 2.$$

If $\dim(\mathrm{iso}(D)) = 4$, given a linear isomorphism map $\sigma : (u,v) = (\sqrt{c}x, \sqrt{ac}y)$, then $\mathrm{aut}(\sigma(D))$ has a group of bases

$$\begin{aligned}
A^* &= \sigma_*(A) = \mathrm{i}u\frac{\partial}{\partial u}, \quad B^* = \sigma_*(B) = \mathrm{i}v\frac{\partial}{\partial v},\\
C^* &= \frac{1}{\sqrt{a}}\sigma_*(C) = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u},\\
D^* &= \frac{1}{\sqrt{a}}\sigma_*(D) = \mathrm{i}u\frac{\partial}{\partial v} + \mathrm{i}v\frac{\partial}{\partial u},\\
X_1^* &= \frac{1}{\sqrt{c}}\sigma_*(X_1) = (1-u^2)\frac{\partial}{\partial u} - uv\frac{\partial}{\partial v},\\
Y_1^* &= \frac{1}{\sqrt{c}}\sigma_*(Y_1) = \mathrm{i}(1+u^2)\frac{\partial}{\partial u} + \mathrm{i}uv\frac{\partial}{\partial v},\\
X_2^* &= \frac{1}{\sqrt{ac}}\sigma_*(X_2) = -uv\frac{\partial}{\partial u} + (1-v^2)\frac{\partial}{\partial v},\\
Y_2^* &= \frac{1}{\sqrt{ac}}\sigma_*(Y_2) = \mathrm{i}uv\frac{\partial}{\partial u} + \mathrm{i}(1+v^2)\frac{\partial}{\partial v}.
\end{aligned}$$

Denote $z = (u,v)$. In fact

$$\mathrm{iso}(\sigma(D)) = \Big\{ zK\frac{\partial^{\mathrm{T}}}{\partial z} \;\Big|\; K \in \mathrm{gl}(2,\mathbb{C}),\ K + \overline{K}^{\mathrm{T}} = 0 \Big\}.$$

The one-parameter subgroup $w(t) = \exp\big(t\big(zK\frac{\partial^{\mathrm{T}}}{\partial z}\big)\big)$ $(t \in \mathbb{R})$ is a unique holomorphic solution of the complex ordinary differential equation systems

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = w(t)K, \quad t \in \mathbb{R}$$

with the initial values $w(0) = z$. We get the one-parameter analytic transformation: $w(t) = z\mathrm{e}^{tK}$. Therefore $\mathrm{Iso}(\sigma(D))^0$ is a real Lie group with $\dim(\mathrm{Iso}(\sigma(D))^0) = 4$:

$$(u',v') = (u,v)U, \tag{3.1}$$

where $U = \mathrm{e}^K \in \mathrm{U}_2$, $K = \begin{pmatrix} it_1 & \zeta \\ -\overline{\zeta} & it_2 \end{pmatrix}$, $t_1, t_2 \in \mathbb{R}$, $\zeta \in \mathbb{C}$.

The one-parameter subgroup $(u'(t), v'(t)) = \exp(t(\cos\theta_1 X_1^* + \sin\theta_1 Y_1^*))$ $(t \in \mathbb{R})$ is a unique holomorphic solution of the complex ordinary differential equation systems

$$\frac{\mathrm{d}u'}{\mathrm{d}t} = \mathrm{e}^{\mathrm{i}\theta_1} - \mathrm{e}^{-\mathrm{i}\theta_1}(u')^2, \quad \frac{\mathrm{d}v'}{\mathrm{d}t} = -\mathrm{e}^{-\mathrm{i}\theta_1}u'v', \quad t \in \mathbb{R}$$

with the initial values $(u'(0), v'(0)) = (u,v)$, where $0 \le \theta_1 < 2\pi$. Denote $\alpha_1 = \mathrm{e}^{\mathrm{i}\theta_1}\mathrm{tgh}\, t$, $t \in \mathbb{R}$. The solution is

$$\begin{aligned}
(u',v') &= (1 + u\overline{\alpha}_1)^{-1}(u + \alpha, \sqrt{1-|\alpha_1|^2}v)\\
&= (\mathrm{ch}\, t \cdot z + \mathrm{e}^{\mathrm{i}\theta_1}\mathrm{sh}\, t \cdot e_1)[\mathrm{e}^{-\mathrm{i}\theta_1}\mathrm{sh}\, t \cdot e_1^{\mathrm{T}} z + E + (\mathrm{ch}\, t - 1)E_{11}]^{-1},
\end{aligned}$$

where $e_1 = (1, 0)$, $E_{11} = \mathrm{diag}\{1, 0\}$.

The one-parameter subgroup $(u'(t), v'(t)) = \exp(t(\cos\theta_2 X_2^* + \sin\theta_2 Y_2^*))$ $(t \in \mathbb{R})$ is a unique holomorphic solution of the complex ordinary differential equation systems

$$\frac{\mathrm{d}u'}{\mathrm{d}t} = -\mathrm{e}^{-\mathrm{i}\theta_2} u'v', \quad \frac{\mathrm{d}v'}{\mathrm{d}t} = \mathrm{e}^{\mathrm{i}\theta_2} - \mathrm{e}^{-\mathrm{i}\theta_2}(v')^2, \quad t \in \mathbb{R}$$

with the initial values $(u'(0), v'(0)) = (u, v)$, where $0 \le \theta_2 < 2\pi$. Denote $\alpha_2 = \mathrm{e}^{\mathrm{i}\theta_2}\mathrm{tgh}\, t$, $t \in \mathbb{R}$. The solution is

$$\begin{aligned}(u', v') &= (1 + v\overline{\alpha}_2)^{-1}(\sqrt{1 - |\alpha_2|^2}u, v + \alpha_2) \\ &= (\mathrm{ch}\, t \cdot z + \mathrm{e}^{\mathrm{i}\theta_2}\mathrm{sh}\, t \cdot e_2)[\mathrm{e}^{-\mathrm{i}\theta_2}\mathrm{sh}\, t \cdot e_2^{\mathrm{T}} z + E + (\mathrm{ch}\, t - 1)E_{22}]^{-1},\end{aligned}$$

where $e_2 = (0, 1)$, $E_{22} = \mathrm{diag}\{0, 1\}$.

Set $\widetilde{A} = \mathrm{ch}\, t$, $\widetilde{B} = \mathrm{e}^{\mathrm{i}\theta_k}\mathrm{sh}\, t \cdot e_k$, $\widetilde{C} = \mathrm{e}^{-\mathrm{i}\theta_k}\mathrm{sh}\, t \cdot e_k^{\mathrm{T}}$, $\widetilde{D} = E + (\mathrm{ch}\, t - 1)E_{kk}$, $k = 1, 2$. By [10], $\begin{pmatrix}\widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D}\end{pmatrix}$ is composed of the one-parameter subgroup of Lorentz group of type $(1, 2)$. It determines the Lie algebra $\begin{pmatrix}0 & \mathrm{e}^{\mathrm{i}\theta_k}e_k \\ \mathrm{e}^{-\mathrm{i}\theta_k}e_k^{\mathrm{T}} & 0\end{pmatrix}$. Take over $k = 1, 2$, one can get four base elements $\begin{pmatrix}0 & e_k \\ e_k^{\mathrm{T}} & 0\end{pmatrix}$, $\begin{pmatrix}0 & \mathrm{i}e_k \\ -\mathrm{i}e_k^{\mathrm{T}} & 0\end{pmatrix}$, $k = 1, 2$. By [10], the unit connected component of the holomorphic automorphism group of the domain $\sigma(D)$, construed by the four linearly independent one-parameter subgroup and the isotropic subgroup with the origin which is defined by (3.1), is

$$w = (\widetilde{A}z + \widetilde{B})(\widetilde{C}z + \widetilde{D})^{-1},$$

where $\begin{pmatrix}\widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D}\end{pmatrix}$ is the traversal of Lorentz group of type $(1, 2)$. By [10], one can get

$$\sigma(D) = \{z \in \mathbb{C}^2 \mid E - \overline{z}^{\mathrm{T}} z > 0\} = \{z \in \mathbb{C}^2 \mid \|z\|^2 = z\overline{z}^{\mathrm{T}} < 1\}.$$

By Lemmas 2.5–2.6 and Theorem 3.2, we immediately get Theorem 2.2 again.

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