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On Affine Connections Induced on the $(1,1)$-Tensor Bundle

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Abstract Let M be an n-dimensional differentiable manifold with an affine connection without torsion and $T_1^1(M)$ its $(1,1)$-tensor bundle. In this paper, the authors define a new affine connection on $T_1^1(M)$ called the intermediate lift connection, which lies somewhere between the complete lift connection and horizontal lift connection. Properties of this intermediate lift connection are studied. Finally, they consider an affine connection induced from this intermediate lift connection on a cross-section $\sigma_\xi(M)$ of $T_1^1(M)$ defined by a $(1,1)$-tensor field ξ and present some of its properties.

Keywords Connections, Geodesic, Semi-symmetry type condition, Sasaki metric, Tensor bundle
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1 Introduction

Let M be a differentiable manifold and $T_1^1(M)$ be its $(1,1)$-tensor bundle. Given an affine connection ∇ on M, $T_1^1(M)$ can be viewed as an almost product manifold. Affine connections on almost product manifolds have been studied the various aspects by several authors. Walker [6] presented the conditions for the existence of a torsion-free affine connection with respect to which the complementary distributions H and V are relatively parallel and path parallel. Yano [7] reformulated these conditions in terms of local coordinates with respect to adapted frame. Davies [1] defined this connection on the tangent bundle of M and showed how certain special connections lead to some simple expressions for the curvature tensor of the tangent bundle. Later, Mok [3] considered this connection on the cotangent bundle of M and called it as intermediate lift of the connection ∇. In this paper, we construct the intermediate lift connection on $T_1^1(M)$, the construction being the analogue of the connection on the tangent bundle that was considered by Davies [1]. We have computed the components of the curvature tensors of the intermediate and horizontal lift connections on $T_1^1(M)$ with respect to adapted frame and investigate their curvature conditions of semi-symmetry type and Ricci semi-symmetry type. Finally, we present some properties concerning an affine connection induced from the intermediate lift connection on the cross-section $\sigma_\xi(M)$ of $T_1^1(M)$ defined by a $(1,1)$-tensor field ξ with respect to the adapted (B,C)-frame.

We assume in the sequel that the manifolds, functions, tensor fields and connections under consideration are all of differentiability of class C^∞.

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2 Preliminaries

Throughout, M denotes an n-dimensional C^∞-differentiable manifold. Its $(1,1)$-tensor bundle is denoted by $T_1^1(M)$ and $\pi : T_1^1(M) \to M$ is the projection mapping. Recall that $T_1^1(M)$ has a structure of $(n+n^2)$-dimensional differentiable manifold induced from the differentiable manifold structure of M. Let (U, x^j) be a coordinate neighborhood of M, where (x^j) is a system of local coordinates defined in the neighborhood U. Let (t_j^i) be the system of Cartesian coordinates in each $(1,1)$-tensor space $T_{1(P)}^1(M)$ of M at P with respect to the natural frame $\{\frac{\partial}{\partial x^i}\}$, where P is an arbitrary point belonging to U. Then, in $\pi^{-1}(U)$ of $T_1^1(M)$, we can introduce the local coordinates $(\pi^{-1}(U), x^j, t_j^i)$, which are called the induced coordinates. From now on, we denote the induced coordinates by $(x^J) = (x^j, x^{\overline{j}}) = (x^j, t_j^i)$, $j = 1, \cdots, n$, $\overline{j} = n+1, \cdots, n+n^2$. We also denote the natural frame in $\pi^{-1}(U)$ by $\big(\frac{\partial}{\partial x^J}\big) = \big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^{\overline{j}}}\big) = \big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial t_j^i}\big)$.

If $X = X^i \frac{\partial}{\partial x^i}$ and $A = A_j^i \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j$ are the local expressions in U of a vector field X and a $(1,1)$-tensor field A on M, then the vertical lift ${}^V A$ and the horizontal lift ${}^H X$ are given, with respect to the induced coordinates (x^j, t_j^i), by

$$ {}^V A = \begin{pmatrix} {}^V A^j \\ {}^V A^{\overline{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_j^i \end{pmatrix} \tag{2.1} $$

and

$$ {}^H X = \begin{pmatrix} {}^H X^j \\ {}^H X^{\overline{j}} \end{pmatrix} = \begin{pmatrix} X^j \\ X^s(\Gamma_{sj}^m t_m^i - \Gamma_{sm}^i t_j^m) \end{pmatrix}, \tag{2.2} $$

respectively, where Γ_{ij}^h are the coefficients of the torsion-free affine connection ∇.

Put $X_{(j)} = \partial_j = \delta_j^h \partial_h$ in (2.1) and $A^{(\overline{j})} = \partial_i \otimes \mathrm{d}x^j = \delta_i^k \delta_h^j \partial_k \otimes \mathrm{d}x^h$ in (2.2). Then we get in each induced coordinate neighborhood $\pi^{-1}(U)$ of $T_1^1(M)$ a frame field which consists of the following $n + n^2$ linearly independent vector fields:

$$ \begin{aligned} E_j &= {}^H X_{(j)} = \partial_j + (-t_h^s \Gamma_{js}^k + t_s^k \Gamma_{jh}^s)\partial_{\overline{h}}, \\ E_{\overline{j}} &= {}^V A^{(\overline{j})} = \delta_i^k \delta_h^j \partial_{\overline{h}}. \end{aligned} $$

We can write the adapted frame as $\{E_\alpha\} = \{E_j, E_{\overline{j}}\}$. The indices $\alpha, \beta, \gamma, \cdots = 1, \cdots, n+n^2$ indicate the indices with respect to the adapted frame.

Using (2.1)–(2.2), we obtain

$$ {}^V A = \begin{pmatrix} 0 \\ A_j^i \end{pmatrix} $$

and

$$ {}^H X = \begin{pmatrix} X^j \\ 0 \end{pmatrix} $$

with respect to the adapted frame $\{E_\alpha\}$ (for details, see [4]). By the routine calculations, we state the lemma below.

Lemma 2.1 *The Lie brackets of the adapted frame of $T_1^1(M)$ satisfy the following identities:*

$$ [E_l, E_j] = (t_s^v R_{ljr}{}^s - t_r^s R_{ljs}{}^v) E_{\overline{r}}, $$

$$[E_l, E_{\overline{j}}] = (\delta_r^j \Gamma_{li}^v - \delta_i^v \Gamma_{lr}^j) E_{\overline{r}},$$
$$[E_{\overline{i}}, E_{\overline{j}}] = 0,$$

where ${R_{ijl}}^s$ *denote the components of the curvature tensor of* (M, ∇) (see [4]).

3 $T_1^1(M)$ as an Almost Product Manifold

The vertical distribution V is given by the fibres and the horizontal distribution H is determined uniquely by ∇ as complementary distribution to V on the $(1,1)$-tensor bundle $T_1^1(M)$ of M. The pair (H, V) defines an almost product structure on $T_1^1(M)$, i.e., $T_1^1(M)$ becomes an almost product manifold.

The $(1,1)$-type projection tensors of $T_1^1(M)$ onto H and V will again be denoted by H and V. Also, they satisfy the following conditions:

$$H^2 = H, \quad V^2 = V, \quad HV = VH = 0, \quad H + V = I,$$

where I is the identity tensor.

When we have an affine connection $\widetilde{\nabla}$ on an almost product manifold, we can investigate some parallelism conditions of the distributions of the almost product manifolds. These conditions are locally presented in [7]. On $T_1^1(M)$, these conditions are in the following forms:

$$\begin{aligned} &H \text{ is path-parallel iff } \widetilde{\Gamma}_{ji}^{\overline{h}} + \widetilde{\Gamma}_{ij}^{\overline{h}} = 0, \\ &H \text{ is parallel along } V \text{ iff } \widetilde{\Gamma}_{\overline{j}i}^{\overline{h}} = 0, \\ &V \text{ is path-parallel iff } \widetilde{\Gamma}_{\overline{j}\overline{i}}^{h} + \widetilde{\Gamma}_{\overline{i}\overline{j}}^{h} = 0, \\ &V \text{ is parallel along } H \text{ iff } \widetilde{\Gamma}_{j\overline{i}}^{h} = 0. \end{aligned} \tag{3.1}$$

In [1], Davies considered two special tensor fields marked by A and B on a tangent bundle of a manifold M. Now, by following the same method employed by Davies, we shall define the two special tensor fields on $T_1^1(M)$. For all vector fields $\widetilde{X}$ and $\widetilde{Y}$ on $T_1^1(M)$, the A-tensor field is achieved from two configuration tensor fields such that

$$T_{\widetilde{X}}\widetilde{Y} = H\widetilde{\nabla}_{V\widetilde{X}}(V\widetilde{Y}) + V\widetilde{\nabla}_{V\widetilde{X}}(H\widetilde{Y}),$$
$$O_{\widetilde{X}}\widetilde{Y} = H\widetilde{\nabla}_{H\widetilde{X}}(V\widetilde{Y}) + V\widetilde{\nabla}_{H\widetilde{X}}(H\widetilde{Y}),$$

where $\widetilde{\nabla}$ is a torsion-free affine connection on $T_1^1(M)$. With adding these two equations together, the A-tensor field is defined by

$$A(\widetilde{X}, \widetilde{Y}) = T_{\widetilde{X}}\widetilde{Y} + O_{\widetilde{X}}\widetilde{Y}.$$

The B-tensor field is coming out with using the A-tensor field as $2B(\widetilde{X}, \widetilde{Y}) = -2[A(\widetilde{X}, \widetilde{Y}) + A(\widetilde{Y}, \widetilde{X})] + [A(H\widetilde{X}, H\widetilde{Y}) + A(H\widetilde{Y}, H\widetilde{X})] + [A(V\widetilde{X}, V\widetilde{Y}) + A(V\widetilde{Y}, V\widetilde{X})]$. The A-tensor and B-tensor fields are both the $(1,2)$- type tensor fields on $T_1^1(M)$ and locally expressed in the following forms:

$$A_{ji}^{h} = 0, \quad A_{ji}^{\overline{h}} = \widetilde{\Gamma}\,_{ji}^{\overline{h}}, \quad A_{j\overline{i}}^{h} = \widetilde{\Gamma}_{j\overline{i}}^{h}, \quad A_{j\overline{i}}^{\overline{h}} = 0,$$
$$A_{\overline{j}i}^{h} = 0, \quad A_{\overline{j}i}^{\overline{h}} = \widetilde{\Gamma}_{\overline{j}i}^{\overline{h}}, \quad A_{\overline{j}\overline{i}}^{h} = \widetilde{\Gamma}_{\overline{j}\overline{i}}^{h}, \quad A_{\overline{j}\overline{i}}^{\overline{h}} = 0,$$

$$B^{h}_{ji}=0,\quad B^{\overline{h}}_{ji}=-\frac{1}{2}(\widetilde{\Gamma}^{\overline{h}}_{ji}+\widetilde{\Gamma}^{\overline{h}}_{ij}),\quad B^{h}_{j\overline{i}}=-\widetilde{\Gamma}^{h}_{j\overline{i}},$$
$$B^{\overline{h}}_{j\overline{i}}=-\widetilde{\Gamma}^{\overline{h}}_{\overline{i}j},\quad B^{h}_{\overline{j}i}=-\widetilde{\Gamma}^{h}_{i\overline{j}},\quad B^{\overline{h}}_{\overline{j}i}=-\widetilde{\Gamma}^{\overline{h}}_{\overline{j}i},$$
$$B^{h}_{\overline{j}\overline{i}}=-\frac{1}{2}(\widetilde{\Gamma}^{h}_{\overline{j}\overline{i}}+\widetilde{\Gamma}^{h}_{\overline{i}\overline{j}}),\quad B^{\overline{h}}_{\overline{j}\overline{i}}=0$$

with respect to the adapted frame $\{E_\alpha\}$.

4 Lifts of a Torsion-Free Affine Connection to $T^1_1(M)$

The horizontal lift ${}^H\nabla$ of any torsion-free connection ∇ on M is defined by

$$\begin{aligned} &{}^H\nabla_{{}^VA}{}^VB=0, && {}^H\nabla_{{}^VA}{}^HY=0,\\ &{}^H\nabla_{{}^HX}{}^VB={}^V(\nabla_XB), && {}^H\nabla_{{}^HX}{}^HY={}^H(\nabla_XY)\end{aligned}$$

for any vector fields X and Y and $(1,1)$-tensor fields A and B on M. The non-zero components ${}^H\Gamma^{\gamma}_{\alpha\beta}$ of the horizontal lift connection ${}^H\nabla$ are as follows:

$${}^H\Gamma^{r}_{lj}=\Gamma^{r}_{lj},\quad {}^H\Gamma^{\overline{r}}_{l\overline{j}}=\Gamma^{v}_{li}\delta^{j}_{r}-\Gamma^{j}_{lr}\delta^{v}_{i} \tag{4.1}$$

with respect to the adapted frame $\{E_\alpha\}$ (see [2, 4], for (p,q)-tensor bundles, see [5]).

The complete lift ${}^C\nabla$ of any torsion-free connection ∇ on $T^1_1(M)$ is given by

$${}^C\nabla={}^H\nabla+\gamma R(\ ,X)Y+\widetilde{\gamma}R(\ ,Y)X$$

for vector fields X and Y on M, where R is the curvature tensor field of ∇ and

$$\gamma R(\ ,X)Y+\widetilde{\gamma}R(\ ,Y)X=\begin{pmatrix}0\\ R_{rjl}{}^{s}t^{v}_{s}+R_{slj}{}^{v}t^{s}_{r}\end{pmatrix}$$

with respect to the adapted frame (see [5]). With the help of (4.1), we find the non-zero coefficients of the complete lift connection ${}^C\nabla$ with respect to the adapted frame as follows:

$${}^C\Gamma^{r}_{lj}=\Gamma^{r}_{lj},\quad {}^C\Gamma^{\overline{r}}_{l\overline{j}}=\Gamma^{v}_{li}\delta^{j}_{r}-\Gamma^{j}_{lr}\delta^{v}_{i},\quad {}^C\Gamma^{\overline{r}}_{lj}=R_{rjl}{}^{s}t^{v}_{s}+R_{slj}{}^{v}t^{s}_{r}. \tag{4.2}$$

When these components are compared with the conditions in (3.1), it can be seen that V is path parallel and is parallel along H, and H is parallel along V but H is not path parallel.

We are going to define the intermediate lift ${}^I\nabla$ of any torsion-free connection ∇. To do this, we need the B-tensor field associated with the complete lift connection ${}^C\nabla$. The non-zero component of B is only

$$B^{\overline{r}}_{lj}=-\frac{1}{2}(R_{rjl}{}^{s}+R_{rlj}{}^{s})t^{v}_{s}-\frac{1}{2}(R_{slj}{}^{v}+R_{sjl}{}^{v})t^{s}_{r}. \tag{4.3}$$

The intermediate lift ${}^I\nabla$ of ∇ is defined by

$${}^I\nabla={}^C\nabla+B.$$

Using (4.2)–(4.3), we obtain

$${}^I\Gamma^{r}_{lj}=\Gamma^{r}_{lj},\quad {}^I\Gamma^{\overline{r}}_{l\overline{j}}=\Gamma^{v}_{li}\delta^{j}_{r}-\Gamma^{j}_{lr}\delta^{v}_{i},\quad {}^I\Gamma^{\overline{r}}_{lj}=\frac{1}{2}(R_{ljr}{}^{s}t^{v}_{s}+R_{jls}{}^{v}t^{s}_{r}), \tag{4.4}$$

and others being zero, where ${}^I\Gamma^{\gamma}_{\alpha\beta}$ are the components of the intermediate lift connection ${}^I\nabla$. From (3.1) and (4.4), we can immediately state the following proposition.

Proposition 4.1 *Let M be a differentiable manifold with torsion-free affine connection ∇ and $T_1^1(M)$ be its tensor bundle with the intermediate lift connection ${}^I\nabla$. Then, with respect to ${}^I\nabla$,*

(i) *H is parallel along V,*

(ii) *V is parallel along H,*

(iii) *H and V are path-parallel in $T_1^1(M)$.*

Since the skew-symmetric part of B of the complete lift connection ${}^C\nabla$ is zero, the intermediate lift connection ${}^I\nabla$ is a torsion-free connection. The horizontal lift connection ${}^H\nabla$ can be obtained from ${}^I\nabla$, by using the formula

$$ {}^H\nabla = {}^I\nabla - A. $$

In fact, the non-zero component of A-tensor field associated with the complete lift connection ${}^C\nabla$ is

$$ A_{lj}^{\overline{r}} = -\frac{1}{2}(R_{jlr}{}^s t_s^v + R_{ljs}{}^v t_r^s). $$

The affine connection ${}^I\nabla - A$ has components as non-zero: $\widetilde{\Gamma}_{lj}^r = \Gamma_{lj}^r$ and $\widetilde{\Gamma}_{l\overline{j}}^{\overline{r}} = \Gamma_{li}^v \delta_r^j - \Gamma_{lr}^j \delta_i^v$. But these are the same as (4.1), i.e., ${}^I\nabla - A = {}^H\nabla$. In view of the definition of the intermediate lift connection ${}^I\nabla$, we can say that ${}^I\nabla$ is somewhere between the horizontal lift connection ${}^H\nabla$ and the complete lift connection ${}^C\nabla$, so it can be named as "intermediate lift".

We would like to remind Sasaki metric ${}^Sg$ on $T_1^1(M)$ to show under which conditions the horizontal, complete lift and intermediate lift connections are metrical with respect to ${}^Sg$. For detailed interpretation of ${}^Sg$, see [4]. The Sasaki metric ${}^Sg$ on the $(1,1)$-tensor bundle $T_1^1(M)$ over a Riemannian manifold (M, g) has the components with respect to the adapted frame $\{E_\alpha\}$ (see [2, 4]):

$$ ({}^Sg)_{\beta\gamma} = \begin{pmatrix} ({}^Sg)_{jl} & ({}^Sg)_{j\overline{l}} \\ ({}^Sg)_{\overline{j}l} & ({}^Sg)_{\overline{j}\overline{l}} \end{pmatrix} = \begin{pmatrix} g_{jl} & 0 \\ 0 & g_{it}g^{jl} \end{pmatrix}, \quad x^{\overline{l}} = t_l^t. $$

Calculating the covariant derivatives of ${}^Sg$ with respect to the horizontal, complete and intermediate lift connections, we get respectively their non-zero components as follows:

$$ \text{(i)} \quad \begin{cases} {}^H\nabla_p \, {}^Sg_{jl} = \nabla_p g_{jl}, \\ {}^H\nabla_p \, {}^Sg_{\overline{j}\overline{l}} = (\nabla_p g_{it})g^{jl} + g_{it}(\nabla_p g^{jl}), \end{cases} $$

$$ \text{(ii)} \quad \begin{cases} {}^C\nabla_p \, {}^Sg_{jl} = \nabla_p g_{jl}, \\ {}^C\nabla_p \, {}^Sg_{\overline{j}l} = -g_{ib}R^{j\,a}_{.\,lp}t_a^b - g^{js}R_{apli}t_s^a, \\ {}^C\nabla_p \, {}^Sg_{j\overline{l}} = -g_{bk}R^{l\,a}_{.\,jp}t_a^b - g^{sl}R_{apjk}t_s^a, \\ {}^C\nabla_p \, {}^Sg_{\overline{j}\overline{l}} = (\nabla_p g_{it})g^{jl} + g_{it}(\nabla_p g^{jl}), \end{cases} $$

$$ \text{(iii)} \quad \begin{cases} {}^I\nabla_p \, {}^Sg_{jl} = \nabla_p g_{jl}, \\ {}^I\nabla_p \, {}^Sg_{\overline{j}l} = \frac{1}{2}[g_{ib}R_{lp.}{}^{ja}t_a^b + g^{js}R_{plai}t_s^a], \\ {}^I\nabla_p \, {}^Sg_{j\overline{l}} = \frac{1}{2}[g_{bt}R_{jp.}{}^{la}t_a^b + g^{sl}R_{pjat}t_s^a], \\ {}^I\nabla_p \, {}^Sg_{\overline{j}\overline{l}} = (\nabla_p g_{it})g^{jl} + g_{it}(\nabla_p g^{jl}). \end{cases} $$

From the equations above, we get the following proposition.

Proposition 4.2 *Let (M,g) be a Riemannian manifold with the torsion-free affine connection ∇ and $(T_1^1(M),{}^S g)$ be its tensor bundle with Sasaki metric. Then the following conditions are equivalent:*

(i) *∇ is the Levi-Civita connection of g and is locally flat.*

(ii) *The complete lift connection ${}^C\nabla$ is metrical with respect to ${}^S g$.*

(iii) *The intermediate lift connection ${}^I\nabla$ is metrical with respect to ${}^S g$.*

In this case, ${}^C\nabla = {}^I\nabla = {}^H\nabla$ and it is the Levi-Civita connection of ${}^S g$.

An important geometric problem is to find the geodesics on the smooth manifolds with respect to the affine connections. Let C be a curve in M expressed locally by $x^r = x^r(t)$. We define a curve $\widetilde{C}$ in $T_1^1(M)$ by

$$\begin{cases} x^r = x^r(t), \\ x^{\overline{r}} = t_r^v(t), \end{cases} \tag{4.5}$$

where $t_r^v(t)$ is a $(1,1)$-tensor field along C. The curve $C = \pi \circ \widetilde{C}$ is the projection of the curve $\widetilde{C}$ in $T_1^1(M)$.

The geodesics of any connection $\widetilde{\nabla}$ is given by the differential equations

$$\frac{\delta^2 x^A}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 x^A}{\mathrm{d}t^2} + \widetilde{\Gamma}_{CB}^A \frac{\mathrm{d}x^C}{\mathrm{d}t}\frac{\mathrm{d}x^B}{\mathrm{d}t} = 0 \tag{4.6}$$

with respect to the induced coordinates $(x^r, x^{\overline{r}})$, where t is the arc length of a curve in $T_1^1(M)$. We write down the form equivalent to (4.6), namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\theta^\alpha}{\mathrm{d}t}\Big) + \widetilde{\Gamma}_{\gamma\beta}^\alpha \frac{\theta^\gamma}{\mathrm{d}t}\frac{\theta^\beta}{\mathrm{d}t} = 0 \tag{4.7}$$

with respect to the adapted frame $\{E_\alpha\}$, where

$$\frac{\theta^h}{\mathrm{d}t} = \frac{\mathrm{d}x^h}{\mathrm{d}t}, \qquad \frac{\theta^{\overline{h}}}{\mathrm{d}t} = \frac{\delta t_h^k}{\mathrm{d}t}$$

along a curve $\widetilde{C}$ in $T_1^1(M)$ (see [8]). Taking account of (4.4), then (4.7) reduces to

$$\begin{cases} \text{(a)}\ \dfrac{\mathrm{d}^2 x^r}{\mathrm{d}t^2} + \Gamma_{lj}^r \dfrac{\mathrm{d}x^l}{\mathrm{d}t}\dfrac{\mathrm{d}x^j}{\mathrm{d}t} = 0, \\ \text{(b)}\ \dfrac{\delta^2 t_r^v}{\mathrm{d}t^2} = 0. \end{cases} \tag{4.8}$$

Thus we have the following result.

Theorem 4.1 *Let $\widetilde{C}$ be a curve in $T_1^1(M)$ locally expressed by $x^r = x^r(t)$, $x^{\overline{r}} = t_r^v(t)$ with respect to the induced coordinates $(x^r, x^{\overline{r}})$ in $\pi^{-1}(U) \subset T_1^1(M)$. The curve $\widetilde{C}$ is a geodesic with respect to the intermediate lift connection ${}^I\nabla$ if the projection C of $\widetilde{C}$ is a geodesic in M with the torsion-free connection ∇ and $t_r^v(t)$ satisfies the differential equation* (b) *in* (4.8).

The curvature tensor of the intermediate lift connection ${}^I\nabla$ is denoted by ${}^I R$:

$${}^I R(E_\alpha, E_\beta)E_\gamma = {}^I\nabla_{E_\alpha}\, {}^I\nabla_{E_\beta} E_\gamma - {}^I\nabla_{E_\beta}\, {}^I\nabla_{E_\alpha} E_\gamma - {}^I\nabla_{[E_\alpha, E_\beta]} E_\gamma$$

with respect to the adapted frame. Its non-zero components in the adapted frame are found to be

$$\begin{aligned}
&{}^{I}R_{mlj}{}^{r} = R_{mlj}{}^{r},\\
&{}^{I}R_{mlj}{}^{\overline{r}} = \frac{1}{2}[(\nabla_m R_{ljr}{}^{s} - \nabla_l R_{mjr}{}^{s})t_s^v + (\nabla_m R_{jls}{}^{v} - \nabla_l R_{jms}{}^{v})t_r^s],\\
&{}^{I}R_{\overline{m}lj}{}^{\overline{r}} = \frac{1}{2}(R_{ljr}{}^{m}\delta_n^v + R_{jln}{}^{v}\delta_r^m),\\
&{}^{I}R_{ml\overline{j}}{}^{\overline{r}} = R_{mli}{}^{v}\delta_r^j - R_{mlr}{}^{j}\delta_i^v.
\end{aligned} \tag{4.9}$$

Let $\widetilde{X}$ and $\widetilde{Y}$ be vector fields of $T_1^1(M)$. The curvature operator ${}^{I}R(\widetilde{X},\widetilde{Y})$ is a differential operator on $T_1^1(M)$. Similarly, for vector fields X and Y of M, $R(X,Y)$ is a differential operator on M. Now, we operate the curvature operator to the curvature tensor ${}^{I}R$. That is, for all $\widetilde{Z},\widetilde{W}$ and $\widetilde{U}$ on $T_1^1(M)$, we consider the condition $({}^{I}R(\widetilde{X},\widetilde{Y}){}^{I}R)(\widetilde{Z},\widetilde{W})\widetilde{U} = 0$.

The tensor $({}^{I}R(\widetilde{X},\widetilde{Y}){}^{I}R)(\widetilde{Z},\widetilde{W})\widetilde{U}$ has components

$$\begin{aligned}
&(({}^{I}R(\widetilde{X},\widetilde{Y}){}^{I}R)(\widetilde{Z},\widetilde{W})\widetilde{U})_{\alpha\beta\gamma\theta\sigma}{}^{\varepsilon}\\
&= {}^{I}R_{\alpha\beta\tau}{}^{\varepsilon}\,{}^{I}R_{\gamma\theta\sigma}{}^{\tau} - {}^{I}R_{\alpha\beta\gamma}{}^{\tau}\,{}^{I}R_{\tau\theta\sigma}{}^{\varepsilon} - {}^{I}R_{\alpha\beta\theta}{}^{\tau}\,{}^{I}R_{\gamma\tau\sigma}{}^{\varepsilon} - {}^{I}R_{\alpha\beta\sigma}{}^{\tau}\,{}^{I}R_{\gamma\theta\tau}{}^{\varepsilon}
\end{aligned} \tag{4.10}$$

with respect to the adapted frame $\{E_\alpha\}$. Similarly, for all Z,W and U on M,

$$\begin{aligned}
&((R(X,Y)R)(Z,W)U)_{ijklm}{}^{n}\\
&= R_{ijp}{}^{n}R_{klm}{}^{p} - R_{ijk}{}^{p}R_{plm}{}^{n} - R_{ijl}{}^{p}R_{kpm}{}^{n} - R_{ijm}{}^{p}R_{klp}{}^{n}.
\end{aligned} \tag{4.11}$$

As is known, if the Riemannian curvature tensor of a Riemannian manifold satisfies the condition (4.11), then the Riemannian manifold is called a semi-symmetric manifold.

Using (4.9)–(4.10), computing the coefficients of $(({}^{I}R(\widetilde{X},\widetilde{Y}){}^{I}R)(\widetilde{Z},\widetilde{W})\widetilde{U})_{\alpha\beta\gamma\theta\sigma}{}^{\varepsilon}$ for different indices, we get

$$\begin{aligned}
\text{(i)}\quad &(({}^{I}R(\widetilde{X},\widetilde{Y}){}^{I}R)(\widetilde{Z},\widetilde{W})\widetilde{U})_{pamlj}{}^{r}\\
&= {}^{I}R_{pak}{}^{r}\,{}^{I}R_{mlj}{}^{k} + {}^{I}R_{pa\overline{k}}{}^{r}\,{}^{I}R_{mlj}{}^{\overline{k}} - {}^{I}R_{pam}{}^{k}\,{}^{I}R_{klj}{}^{r} - {}^{I}R_{pa\overline{m}}{}^{\overline{k}}\,{}^{I}R_{\overline{k}lj}{}^{r}\\
&\quad - {}^{I}R_{pal}{}^{k}\,{}^{I}R_{mkj}{}^{r} - {}^{I}R_{pal}{}^{\overline{k}}\,{}^{I}R_{m\overline{k}j}{}^{r} - {}^{I}R_{paj}{}^{k}\,{}^{I}R_{mlk}{}^{r} - {}^{I}R_{paj}{}^{\overline{k}}\,{}^{I}R_{ml\overline{k}}{}^{r}\\
&= ((R(X,Y)R)(Z,W)U)_{pamlj}{}^{r},\\
\text{(ii)}\quad &(({}^{I}R(\widetilde{X},\widetilde{Y}){}^{I}R)(\widetilde{Z},\widetilde{W})\widetilde{U})_{pa\overline{m}lj}{}^{\overline{r}}\\
&= {}^{I}R_{pak}{}^{\overline{r}}\,{}^{I}R_{\overline{m}lj}{}^{k} + {}^{I}R_{pa\overline{k}}{}^{\overline{r}}\,{}^{I}R_{\overline{m}lj}{}^{\overline{k}} - {}^{I}R_{pa\overline{m}}{}^{k}\,{}^{I}R_{klj}{}^{\overline{r}} - {}^{I}R_{pa\overline{m}}{}^{\overline{k}}\,{}^{I}R_{\overline{k}lj}{}^{\overline{r}}\\
&\quad - {}^{I}R_{pal}{}^{k}\,{}^{I}R_{\overline{m}kj}{}^{\overline{r}} - {}^{I}R_{pal}{}^{\overline{k}}\,{}^{I}R_{\overline{m}\overline{k}j}{}^{\overline{r}} - {}^{I}R_{paj}{}^{k}\,{}^{I}R_{\overline{m}lk}{}^{\overline{r}} - {}^{I}R_{paj}{}^{\overline{k}}\,{}^{I}R_{\overline{m}l\overline{k}}{}^{\overline{r}}\\
&= \frac{1}{2}[((R(X,Y)R)(Z,W)U)_{pajln}{}^{v}\delta_r^m - ((R(X,Y)R)(Z,W)U)_{pajlr}{}^{m}\delta_n^v],\\
\text{(iii)}\quad &(({}^{I}R(\widetilde{X},\widetilde{Y}){}^{I}R)(\widetilde{Z},\widetilde{W})\widetilde{U})_{paml\overline{j}}{}^{\overline{r}}\\
&= {}^{I}R_{pak}{}^{\overline{r}}\,{}^{I}R_{ml\overline{j}}{}^{k} + {}^{I}R_{pa\overline{k}}{}^{\overline{r}}\,{}^{I}R_{ml\overline{j}}{}^{\overline{k}} - {}^{I}R_{pam}{}^{k}\,{}^{I}R_{kl\overline{j}}{}^{\overline{r}} - {}^{I}R_{pam}{}^{\overline{k}}\,{}^{I}R_{\overline{k}l\overline{j}}{}^{\overline{r}}\\
&\quad - {}^{I}R_{pal}{}^{k}\,{}^{I}R_{mk\overline{j}}{}^{\overline{r}} - {}^{I}R_{pal}{}^{\overline{k}}\,{}^{I}R_{m\overline{k}\overline{j}}{}^{\overline{r}} - {}^{I}R_{pa\overline{j}}{}^{k}\,{}^{I}R_{mlk}{}^{\overline{r}} - {}^{I}R_{pa\overline{j}}{}^{\overline{k}}\,{}^{I}R_{ml\overline{k}}{}^{\overline{r}}\\
&= ((R(X,Y)R)(Z,W)U)_{pamli}{}^{v}\delta_r^j - ((R(X,Y)R)(Z,W)U)_{pamlr}{}^{j}\delta_i^v,\\
\text{(iv)}\quad &(({}^{I}R(\widetilde{X},\widetilde{Y}){}^{I}R)(\widetilde{Z},\widetilde{W})\widetilde{U})_{pamlj}{}^{\overline{r}}
\end{aligned}$$

$$= {}^I R_{pak}{}^{\bar r}\, {}^I R_{mlj}{}^{k} + {}^I R_{pa\bar k}{}^{\bar r}\, {}^I R_{mlj}{}^{\bar k} - {}^I R_{pam}{}^{k}\, {}^I R_{klj}{}^{\bar r} - {}^I R_{pam}{}^{\bar k}\, {}^I R_{\bar k lj}{}^{\bar r}$$
$$- {}^I R_{pal}{}^{k}\, {}^I R_{mkj}{}^{\bar r} - {}^I R_{pal}{}^{\bar k}\, {}^I R_{m\bar k j}{}^{\bar r} - {}^I R_{paj}{}^{k}\, {}^I R_{mlk}{}^{\bar r} - {}^I R_{paj}{}^{\bar k}\, {}^I R_{ml\bar k}{}^{\bar r},$$

(v) all other $(({}^I R(\widetilde{X},\widetilde{Y})^I R)(\widetilde{Z},\widetilde{W})\widetilde{U})_{\alpha\beta\gamma\theta\sigma}{}^{\varepsilon}$ are zero.

The above conditions give the following theorem.

Theorem 4.2 *Let M be a differentiable manifold with torsion-free affine connection ∇ and $T_1^1(M)$ be its tensor bundle with the intermediate lift connection ${}^I\nabla$. Under the assumption that ${}^I R_{mlj}{}^{\bar r} = \frac{1}{2}[(\nabla_m R_{ljr}{}^{s} - \nabla_l R_{mjr}{}^{s})t_s^v + (\nabla_m R_{jls}{}^{v} - \nabla_l R_{jms}{}^{v})t_r^s] = 0$, where R and ${}^I R$ are the curvature tensors of the torsion-free affine connection ∇ and the intermediate lift connection ${}^I\nabla$, $({}^I R(\widetilde{X},\widetilde{Y})^I R)(\widetilde{Z},\widetilde{W})\widetilde{U} = 0$ for all $\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}$ and $\widetilde{U}$ on $T_1^1(M)$ if and only if $(R(X,Y)R)(Z,W)U = 0$ for all X,Y,Z,W and U on M.*

We obtain the Ricci tensor of ${}^I R$ by using the well known contraction ${}^I R_{\beta\gamma} = {}^I R_{\alpha\beta\gamma}{}^{\alpha}$. The non-zero component of ${}^I R_{\beta\gamma}$ is obtained as

$${}^I R_{lj} = R_{lj}.$$

Now, we operate the curvature operator ${}^I R(\widetilde{X},\widetilde{Y})$ to the Ricci tensor. The tensors $({}^I R(\widetilde{X},\widetilde{Y})\widetilde{\mathrm{Ric}})(\widetilde{Z},\widetilde{W})$ and $(R(X,Y)\mathrm{Ric})(Z,W)$ have coefficients

$$(({}^I R(\widetilde{X},\widetilde{Y})\widetilde{\mathrm{Ric}})(\widetilde{Z},\widetilde{W}))_{\alpha\beta\gamma\theta} = {}^I R_{\alpha\beta\gamma}{}^{\varepsilon}\, {}^I R_{\varepsilon\theta} + {}^I R_{\alpha\beta\theta}{}^{\varepsilon}\, {}^I R_{\gamma\varepsilon}$$

and

$$((R(X,Y)\mathrm{Ric})(Z,W))_{ijkl} = R_{ijk}{}^{p} R_{pl} + R_{ijl}{}^{p} R_{kp},$$

respectively. By putting $\alpha = m, \beta = l, \gamma = j, \theta = r$, it follows that

$$\begin{aligned}(({}^I R(\widetilde{X},\widetilde{Y})\widetilde{\mathrm{Ric}})(\widetilde{Z},\widetilde{W}))_{mljr} &= {}^I R_{mlj}{}^{s}\, {}^I R_{sr} + {}^I R_{mlr}^{s}\, {}^I R_{js} \\ &= R_{mlj}{}^{s}\, R_{sr} + R_{mlr}{}^{s}\, R_{js} \\ &= ((R(X,Y)\mathrm{Ric})(Z,W))_{mljr},\end{aligned}$$

all the others being zero. Therefore we get the following theorem.

Theorem 4.3 *Let M be a differentiable manifold with torsion-free affine connection ∇ and $T_1^1(M)$ be its tensor bundle with the intermediate lift connection ${}^I\nabla$. Then $({}^I R(\widetilde{X},\widetilde{Y})\widetilde{\mathrm{Ric}})(\widetilde{Z},\widetilde{W}) = 0$ for all $\widetilde{X},\widetilde{Y},\widetilde{Z}$ and $\widetilde{W}$ on $T_1^1(M)$ if and only if $(R(X,Y)\mathrm{Ric})(Z,W) = 0$ for all X,Y,Z and W on M.*

The curvature tensor of the horizontal lift connection ${}^H\nabla$ has the following non-zero components

$${}^H R_{mlj}^{r} = R_{mlj}^{r}, \quad {}^H R_{ml\bar j}^{\bar r} = R_{mli}^{v}\delta_r^j - R_{mlr}^{j}\delta_i^v.$$

Operating the curvature operator ${}^H R(\widetilde{X},\widetilde{Y})$ to the curvature tensor ${}^H R$, we obtain

(i) $(({}^H R(\widetilde{X},\widetilde{Y})^H R)(\widetilde{Z},\widetilde{W})\widetilde{U})_{pamlj}{}^{r}$

$$= {}^H R_{pak}{}^r\, {}^H R_{mlj}{}^k + {}^H R_{pa\bar{k}}{}^r\, {}^H R_{mlj}{}^{\bar{k}} - {}^H R_{pam}{}^k\, {}^H R_{klj}{}^r - {}^H R_{pam}{}^{\bar{k}}\, {}^H R_{\bar{k}lj}{}^r$$
$$- {}^H R_{pal}{}^k\, {}^H R_{mkj}{}^r - {}^H R_{pal}{}^{\bar{k}}\, {}^H R_{m\bar{k}j}{}^r - {}^H R_{paj}{}^k\, {}^H R_{mlk}{}^r - {}^H R_{paj}{}^{\bar{k}}\, {}^H R_{ml\bar{k}}{}^r$$
$$= ((R(X,Y)R)(Z,W)U)_{pamlj}{}^r,$$

(ii) $(({}^H R(\widetilde{X},\widetilde{Y})^H R)(\widetilde{Z},\widetilde{W})\widetilde{U})_{paml\bar{j}}{}^{\bar{r}}$

$$= {}^H R_{pak}{}^{\bar{r}}\, {}^H R_{ml\bar{j}}{}^k + {}^H R_{pa\bar{k}}{}^{\bar{r}}\, {}^H R_{ml\bar{j}}{}^{\bar{k}} - {}^H R_{pam}{}^k\, {}^H R_{kl\bar{j}}{}^{\bar{r}} - {}^H R_{pam}{}^{\bar{k}}\, {}^H R_{\bar{k}l\bar{j}}{}^{\bar{r}}$$
$$- {}^H R_{pal}{}^k\, {}^H R_{mk\bar{j}}{}^{\bar{r}} - {}^H R_{pal}{}^{\bar{k}}\, {}^H R_{m\bar{k}\bar{j}}{}^{\bar{r}} - {}^H R_{pa\bar{j}}{}^k\, {}^H R_{mlk}{}^{\bar{r}} - {}^H R_{pa\bar{j}}{}^{\bar{k}}\, {}^H R_{ml\bar{k}}{}^{\bar{r}}$$
$$= ((R(X,Y)R)(Z,W)U)_{pamli}{}^r \delta_r^j - ((R(X,Y)R)(Z,W)U)_{pamlr}{}^j \delta_i^v,$$

(iii) all other $(({}^H R(\widetilde{X},\widetilde{Y})^H R)(\widetilde{Z},\widetilde{W})\widetilde{U})_{\alpha\beta\gamma\theta\sigma}{}^{\varepsilon}$ are zero.

Hence, we have the following result.

Theorem 4.4 *Let M be a differentiable manifold with torsion-free affine connection ∇ and $T_1^1(M)$ be its tensor bundle with the horizontal lift connection ${}^H\nabla$. Then $({}^H R(\widetilde{X},\widetilde{Y})^H R)(\widetilde{Z},\widetilde{W})\widetilde{U} = 0$ for all $\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}$ and $\widetilde{U}$ on $T_1^1(M)$ if and only if $(R(X,Y)R)(Z,W)U = 0$ for all X,Y,Z,W and U on M.*

5 The Affine Connection Induced on a Cross-Section from the Intermediate Lift Connection

We shall first find the components of the intermediate lift connection ${}^I\nabla$ with respect to the natural frame in $T_1^1(M)$. Let ${}^I\widetilde{\Gamma}_{ij}^h$ be components of the intermediate lift connection ${}^I\nabla$ with respect to the natural frame. The law of transformation of the intermediate lift connection ${}^I\nabla$ is as follows:

$$ {}^I\widetilde{\Gamma}_{MS}^N = \widetilde{A}^L{}_M \widetilde{A}^J{}_S \widetilde{A}_R{}^N\, {}^I\Gamma_{LJ}^R + \widetilde{A}_J{}^N \frac{\partial \widetilde{A}^J{}_S}{\partial x^M}, \tag{5.1}$$

where

$$(\widetilde{A}_\beta{}^\alpha) = \begin{pmatrix} \widetilde{A}_j{}^h & \widetilde{A}_{\bar{j}}{}^h \\ \widetilde{A}_j{}^{\bar{h}} & \widetilde{A}_{\bar{j}}{}^{\bar{h}} \end{pmatrix} = \begin{pmatrix} \delta_j^h & 0 \\ -\Gamma_{js}^k t_h^s + \Gamma_{jh}^s t_s^k & \delta_i^k \delta_h^j \end{pmatrix}$$

is the inverse of the matrix

$$(\widetilde{A}^\alpha{}_\beta) = \begin{pmatrix} \widetilde{A}^h{}_j & \widetilde{A}^h{}_{\bar{j}} \\ \widetilde{A}^{\bar{h}}{}_j & \widetilde{A}^{\bar{h}}{}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta_j^h & 0 \\ \Gamma_{js}^k t_h^s - \Gamma_{jh}^s t_s^k & \delta_i^k \delta_h^j \end{pmatrix}$$

of frames changes $\partial_A = \widetilde{A}^\beta{}_A e_\beta$. Substituting (4.4) into (5.1), we get the components ${}^I\widetilde{\Gamma}_{MS}^N$ in the natural frame as follows:

$$\begin{aligned}
{}^I\widetilde{\Gamma}_{ms}^i &= \Gamma_{ms}^i, \\
{}^I\widetilde{\Gamma}_{\bar{m}s}^{\bar{i}} &= \Gamma_{sn}^j \delta_i^m - \Gamma_{si}^m \delta_n^j, \\
{}^I\widetilde{\Gamma}_{m\bar{s}}^{\bar{i}} &= \Gamma_{mb}^j \delta_i^s - \Gamma_{mi}^s \delta_b^j, \\
{}^I\widetilde{\Gamma}_{ms}^{\bar{i}} &= (\partial_m \Gamma_{sa}^j + \Gamma_{sa}^r \Gamma_{mr}^j - \Gamma_{ms}^r \Gamma_{ra}^j) t_i^a \\
&\quad + (-\partial_m \Gamma_{si}^a + \Gamma_{mi}^r \Gamma_{sr}^a - \Gamma_{ms}^r \Gamma_{ri}^a) t_a^j \\
&\quad - (\Gamma_{sa}^j \Gamma_{mi}^v t_v^a + \Gamma_{si}^a \Gamma_{mr}^j t_a^r) + \frac{1}{2}(R_{msi}{}^a t_a^j - R_{msa}{}^j t_i^a),
\end{aligned} \tag{5.2}$$

all the other ${}^I\widetilde{\Gamma}_{MS}^N$ are zero.

Given a $(1,1)$-tensor field ξ on M, the correspondence $x \mapsto \xi_x$, ξ_x being the value of ξ at $x \in M$, determines a mapping $\sigma_\xi : M \mapsto T_1^1(M)$, such that $\pi \circ \sigma_\xi = \mathrm{id}_M$, and the n-dimensional submanifold $\sigma_\xi(M)$ of $T_1^1(M)$ is called the cross-section determined by ξ. The cross-section $\sigma_\xi(M)$ is locally expressed by

$$\begin{cases} x^k = x^k, \\ x^{\overline{k}} = \xi_k^h(x^k), \end{cases} \tag{5.3}$$

where ξ_k^h are the components of ξ. Differentiating (5.3) by x^j, we get the components of n-tangent vector B_j to $\sigma_\xi(M)$ as follows:

$$(B_j^K) = \Big(\frac{\partial x^K}{\partial x^j}\Big) = \begin{pmatrix} \delta_j^k \\ \partial_j \xi_k^h \end{pmatrix}. \tag{5.4}$$

On the other hand, the fibre is locally expressed by

$$\begin{cases} x^k = \text{const.}, \\ t_k^h = t_k^h, \end{cases}$$

t_k^h being considered as parameters. Differentiating the above relations by $x^{\overline{j}} = t_j^i$, we obtain the components of n^2-tangent vectors $C_{\overline{j}}$ to the fibre as follows:

$$(C_{\overline{j}}^K) = \Big(\frac{\partial x^K}{\partial x^{\overline{j}}}\Big) = \begin{pmatrix} 0 \\ \delta_k^j \delta_i^h \end{pmatrix}. \tag{5.5}$$

The $n + n^2$ local vectors $\{B_j, C_{\overline{j}}\}$ define a local family of frames along $\sigma_\xi(M)$, which is called the adapted (B,C)-frame of $\sigma_\xi(M)$.

We now investigate an affine connection induced from the intermediate lift connection ${}^I\nabla$ on the cross-section $\sigma_\xi(M)$ with respect to the adapted (B,C)-frame. The vector fields $C_{\overline{j}}$ given by (5.5) are linearly independent and not tangent to $\sigma_\xi(M)$. Here, we take the vector fields $C_{\overline{j}}$ as normals to the cross-section $\sigma_\xi(M)$. The components $\widetilde{\Gamma}_{ji}^h$ of the affine connection $\widetilde{\nabla}$ induced on $\sigma_\xi(M)$ from the intermediate lift connection ${}^I\nabla$ are in the following form:

$$\widetilde{\Gamma}_{ji}^h = (\partial_j B_i{}^A + {}^I\widetilde{\Gamma}_{CB}^A B_j{}^C B_i{}^B) B^h{}_A, \tag{5.6}$$

where $B^h{}_A$ are defined by

$$(B^h{}_A, C^h{}_A) = (B_i{}^A, C_i{}^A)^{-1}$$

and hence

$$B^h{}_A = (\delta_i^h, 0), \quad C^h{}_A = (-\partial_j \xi_k^h, \delta_k^j \delta_i^h).$$

Substitution (5.2) into (5.6) gives

$$\widetilde{\Gamma}_{ji}^h = \Gamma_{ji}^h,$$

where Γ_{ji}^h are components of the torsion-free affine connection ∇ on M.

From (5.6) we have

$$\partial_j B_i{}^A + {}^I\Gamma_{CB}^A B_j{}^C B_i{}^B - \Gamma_{ji}^h B_h{}^A = H_{ji}^{\overline{k}} C_k{}^A. \tag{5.7}$$

Putting $A = \overline{h}$ in (5.7), we obtain

$$H_{ji}^{\overline{h}} = \nabla_j \nabla_i \xi_h^k + \frac{1}{2}(R_{jia}{}^k \xi_h^a - R_{jih}{}^a \xi_a^k).$$

Denoting by $\widetilde{\nabla}_j B_i{}^A$ the left hand side of (5.7), we have

$$\widetilde{\nabla}_j B_i{}^A = \Big(\nabla_j \nabla_i \xi_h^k + \frac{1}{2}(R_{jia}{}^k \xi_h^a - R_{jih}{}^a \xi_a^k)\Big) C_{\overline{h}}{}^A, \tag{5.8}$$

which is the Gauss equation for the cross section $\sigma_\xi(M)$. Hence, we have the following proposition.

Proposition 5.1 *Let $\sigma_\xi(M)$ be a cross-section in $T_1^1(M)$ determined by a $(1,1)$-tensor field ξ on M with torsion-free affine connection ∇. Then, the cross-section $\sigma_\xi(M)$ is totally geodesic if and only if the condition*

$$\nabla_j \nabla_i \xi_h^k + \frac{1}{2}(R_{jia}{}^k \xi_h^a - R_{jih}{}^a \xi_a^k) = 0$$

is fulfilled.

Operating $\widetilde{\nabla}_p$ to (5.8), we obtain

$$\widetilde{\nabla}_p \widetilde{\nabla}_j B_i{}^A = \nabla_p \Big(\nabla_j \nabla_i \xi_h^k + \frac{1}{2}(R_{jia}{}^k \xi_h^a - R_{jih}{}^a \xi_a^k)\Big) C_{\overline{h}}{}^A. \tag{5.9}$$

Using

$$\widetilde{\nabla}_p \widetilde{\nabla}_j B_i{}^A - \widetilde{\nabla}_j \widetilde{\nabla}_k B_i{}^A = \widetilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B - R_{kji}{}^h B_h{}^A$$

and the Ricci identity for $(1,1)$-type tensor, it follows from (5.9) that

$$\begin{aligned}
&\widetilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B - R_{kji}{}^h B_h{}^A \\
&= \frac{1}{2}\{[(\nabla_p R_{jia}{}^k - \nabla_j R_{pia}{}^k)\xi_h^a - (\nabla_p R_{jih}{}^a - \nabla_j R_{pih}{}^a)\xi_a^k \\
&\quad + (R_{jia}{}^k \nabla_p \xi_h^a - R_{jih}{}^a \nabla_p \xi_a^k - R_{pia}{}^k \nabla_j \xi_h^a + R_{pih}{}^a \nabla_j \xi_a^k)] \\
&\quad + R_{rsa}{}^k(\nabla_i \xi_h^k) - R_{rsi}{}^a(\nabla_a \xi_h^k) - R_{rsh}{}^k(\nabla_i \xi_a^k)\} C_{\overline{h}}{}^A
\end{aligned}$$

from which we have the following proposition.

Proposition 5.2 $\widetilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B$ *is tangent to the cross-section $\sigma_\xi(M)$ if and only if the condition*

$$\begin{aligned}
&\frac{1}{2}[(\nabla_p R_{jia}{}^k - \nabla_j R_{pia}{}^k)\xi_h^a - (\nabla_p R_{jih}{}^a - \nabla_j R_{pih}{}^a)\xi_a^k \\
&\quad + (R_{jia}{}^k \nabla_p \xi_h^a - R_{jih}{}^a \nabla_p \xi_a^k - R_{pia}{}^k \nabla_j \xi_h^a + R_{pih}{}^a \nabla_j \xi_a^k)] \\
&\quad + R_{rsa}{}^k(\nabla_i \xi_h^k) - R_{rsi}{}^a(\nabla_a \xi_h^k) - R_{rsh}{}^k(\nabla_i \xi_a^k) = 0
\end{aligned}$$

is fulfilled.

References

[1] Davies, E. T., On the curvature of the tangent bundle, *Annali di Mat.*, **181**, 1969, 193–204.

[2] Lai, K. F. and Mok, K. P., On the differential geometry of the (1,1) tensor bundle, *Tensor* (*N.S.*), **63**(1), 2002, 15–27.

[3] Mok, K. P., Metrics and connections on cotangent bundle, *Kodai Math. Sem. Rep.*, **28**, 1977, 226–238.

[4] Salimov, A. and Gezer, A., On the geometry of the (1,1)-tensor bundle with Sasaki type metric, *Chin. Ann. Math. Ser. B*, **32**(3), 2011, 369–386.

[5] Salimov, A. A., Gezer, A. and Akbulut, K., Geodesics of Sasakian metrics on tensor bundles, *Mediterr. J. Math.*, **6**(2), 2009, 135–147.

[6] Walker, A. G., Connections for parallel distributions in the large, *Quart. J. Math. Oxford*, **6**(2), 1958, 301–308.

[7] Yano, K., Affine connections in an almost product space, *Kodai Math. Sem. Rep.*, **11**, 1959, 1–24.

[8] Yano, K. and Ishihara, S., Tangent and Cotangent Bundles, Marcel Dekker, Inc., New York, 1973.