Chin. Ann. Math. Ser. B
39(4), 2018, 695–704
DOI: 10.1007/s11401-018-0090-8

Chinese Annals of Mathematics, Series B

A Schwarz Lemma at the Boundary of Hilbert Balls*

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Abstract In this paper, the authors prove a general Schwarz lemma at the boundary for the holomorphic mapping f between unit balls $\mathbb{B}$ and $\mathbb{B}'$ in separable complex Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, respectively. It is found that if the mapping $f \in C^{1+\alpha}$ at $z_0 \in \partial\mathbb{B}$ with $f(z_0) = w_0 \in \partial\mathbb{B}'$, then the Fréchet derivative operator $\mathrm{D}f(z_0)$ maps the tangent space $T_{z_0}(\partial\mathbb{B}^n)$ to $T_{w_0}(\partial\mathbb{B}')$, the holomorphic tangent space $T_{z_0}^{(1,0)}(\partial\mathbb{B}^n)$ to $T_{w_0}^{(1,0)}(\partial\mathbb{B}')$, respectively.

Keywords Boundary Schwarz lemma, Separable Hilbert space, Holomorphic mapping, Unit ball
2000 MR Subject Classification 32H02, 46E20, 30C80

1 Introduction

We begin with some notations. Let $\mathcal{H}$ be a separable complex Hilbert space. For any $z, w \in \mathcal{H}$, the inner product and the corresponding norm are given by $\langle z,\ w\rangle,\ \|z\| = \langle z, z\rangle^{\frac{1}{2}}$. Given $c \in \mathbb{C}$, we have

$$\langle cz, w\rangle = c\langle z, w\rangle, \quad \overline{\langle z, w\rangle} = \langle w, z\rangle, \quad |\langle z, w\rangle| \le \|z\| \cdot \|w\|.$$

Two vectors $a,\ b \in \mathcal{H}$ are called orthogonal and we write $a \perp b$ provided by $\langle a, b\rangle = 0$. For a subset $E \subset \mathcal{H}$, the set $E^\perp$ is defined by

$$E^\perp = \{a \in \mathcal{H} \mid \langle a, b\rangle = 0 \quad \text{for all } b \in E\}.$$

We choose a normal orthonormal basis $e_1, e_2, \cdots$ for $\mathcal{H}$ such that $\langle e_i, e_j\rangle = \delta_{ij}$ which equals 0 for $i \ne j$ and equals 1 for $i = j$. Then for each $z \in \mathcal{H}$, we write

$$z = \sum_{j=1}^{\infty} z_j e_j \quad \text{and} \quad z' = \sum_{j=2}^{\infty} z_j e_j,$$

or denote $z = (z_1, z_2, \cdots)$ and $z' = (z_2, z_3, \cdots)$. For the other separable complex Hilbert space $\mathcal{H}'$, its normal orthonormal basis is denoted by $e_1', e_2', \cdots$.

Manuscript received April 3, 2015. Revised August 30, 2016.
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*This work was supported by the National Natural Science Foundation of China (Nos. 11671361, 11571256) and the Zhejiang Provincial Natural Science Foundation of China (No. LY14A010008).

Let V_0 be an open subset of $\mathcal{H}$, and f be a continuous mapping of V_0 to $\mathcal{H}'$. We say that f is a holomorphic mapping if for any $z \in V_0$, there is a bounded linear operator $\mathrm{D}f(z)$ from $\mathcal{H}$ into $\mathcal{H}'$ such that

$$\lim_{\beta \in \mathcal{H}, \|\beta\| \to 0} \frac{\|f(z+\beta) - f(z) - \mathrm{D}f(z)\beta\|}{\|\beta\|} = 0,$$

where $\|\cdot\|$ is the norm of appropriate space. $\mathrm{D}f(z)$ denotes the Fréchet derivative of f at z, and $\mathrm{D}f(z)\beta$ is called the Fréchet derivative of f at z in the direction β. The chain rule of the Fréchet derivative is given as follows.

Lemma 1.1 (see [1]) *Let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}''$ be Hilbert spaces. Suppose $f : V_0 \subset \mathcal{H} \to V_0' \subset \mathcal{H}'$ and $g : V_0' \subset \mathcal{H}' \to V_0'' \subset \mathcal{H}''$ are differentiable maps. Then the composite $g \circ f$ is also differentiable and*

$$\mathrm{D}(g \circ f)(u) = \mathrm{D}(g)(f(u)) \circ \mathrm{D}f(u), \quad u \in V_0.$$

Denote by $B(\mathcal{H}, \mathcal{H}')$ the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}'$. Then the adjoint of $L \in B(\mathcal{H}, \mathcal{H}')$ is the unique operator $L^* : \mathcal{H}' \to \mathcal{H}$ that satisfies

$$\langle a, Lb \rangle = \langle L^* a, b \rangle, \quad b \in \mathcal{H}, \ a \in \mathcal{H}'.$$

From the definition of Fréchet derivative, $\mathrm{D}f(z)$ is a bounded linear operator, and its adjoint is denoted by $\mathrm{D}^* f(z)$.

Let $\mathbb{B} = \{z \in \mathcal{H} \mid \|z\| < 1\}$ be the unit ball of $\mathcal{H}$, $\partial\mathbb{B} = \{z \in \mathcal{H} \mid \|z\| = 1\}$ be its boundary, and $H(\mathbb{B}, \mathbb{B}')$ be the set of all holomorphic mappings from $\mathbb{B}$ to $\mathbb{B}'$. For an open subset $V_0 \subset \mathcal{H}$ and $0 < \alpha < 1$, $C^\alpha(V_0)$ is the set of all functions f on V_0 for which

$$\sup\Big\{ \frac{\|f(z) - f(z')\|}{\|z - z'\|^\alpha} \,\Big|\, z, z' \in V_0 \Big\}$$

is finite. $C^{k+\alpha}(V_0)$ is the set of all functions f on V_0 whose kth order partial derivatives exist and belong to $C^\alpha(V_0)$ for an integer $k \geq 0$.

The Schwarz lemma is one of the most important results in complex analysis. A variant of the Schwarz lemma is known as the Schwarz-Pick lemma, which tells that a holomorphic self-mapping of the unit disk decreases the distance of points in the Poincaré metric. It has been generalized to the derivatives of arbitrary order in [2–3]. When it comes to several complex variables, Rudin [4] gave a first derivative estimate for the bounded holomorphic functions on the polydisc, which is really a precursor to Schwarz-Pick lemma in high dimensions. On the other hand, the unit ball is a distinguished bounded domain, in which many interesting results are obtained (see [5]). For the Schwarz-Pick lemma of arbitrary order, [6–7] generalized Schwarz-Pick lemma to the holomorphic mappings on the unit ball in $\mathbb{C}^n$.

On the other hand, the Schwarz lemma at the boundary is one of the popular topics in complex analysis (see [8]), which has been applied to geometric function theory of one complex variable and several complex variables (see [9]). The following result is the classical boundary version of the Schwarz lemma in one complex variable.

Theorem 1.1 (see [8]) *Let D be the unit disk in $\mathbb{C}$, and f be the self-holomorphic mapping of D. If f is holomorphic at $z = 1$ with $f(0) = 0$ and $f(1) = 1$, then $f'(1) \geq 1$.*

Some multidimensional generalizations of the Schwarz lemma at the boundary in several complex variables were given by [9–12] recently. In [13], the high order Schwarz-Pick lemma for holomorphic mappings between Hilbert balls was studied. However, the boundary Schwarz lemma for infinite dimensions Hilbert spaces is open and challenging, and in this paper, we study it for separable complex Hilbert spaces.

For $z_0 \in \partial\mathbb{B}$, the tangent space $T_{z_0}(\partial\mathbb{B})$ and holomorphic tangent space $T^{1,0}_{z_0}(\partial\mathbb{B})$ at z_0 are defined by

$$T_{z_0}(\partial\mathbb{B}) = \{\beta \in \mathcal{H} \mid \mathrm{Re}\langle z_0, \beta\rangle = 0\}, \quad T^{(1,0)}_{z_0}(\partial\mathbb{B}) = \{\beta \in \mathcal{H} \mid \langle z_0, \beta\rangle = 0\}, \tag{1.1}$$

respectively. In this paper, we study the mapping $f \in H(\mathbb{B}, \mathbb{B}')$. Our main results are listed as follows.

Theorem 1.2 *Let $f \in H(\mathbb{B}, \mathbb{B}')$. If f is $C^{1+\alpha}$ at $z_0 \in \partial\mathbb{B}$ and $f(z_0) = w_0 \in \partial\mathbb{B}'$, then it shows*

(I) $\mathrm{D}f(z_0)\beta \in T_{w_0}(\partial\mathbb{B}')$ *for any* $\beta \in T_{z_0}(\partial\mathbb{B})$, *and* $\mathrm{D}f(z_0)\beta \in T^{(1,0)}_{w_0}(\partial\mathbb{B}')$ *for any* $\beta \in T^{(1,0)}_{z_0}(\partial\mathbb{B})$.

(II) *There exists $\lambda \in \mathbb{R}$ such that $\mathrm{D}^* f(z_0)w_0 = \lambda z_0$ with $\lambda \geq \frac{|1-\langle w_0,a\rangle|^2}{1-\|a\|^2} > 0$ where $a = f(0)$ and $\mathrm{D}^* f(z_0)$ is the adjoint operator of $\mathrm{D}f(z_0)$.*

We notice that for $\mathcal{H} = \mathcal{H}' = \mathbb{C}$, the theorem tells $f'(z_0) > 0$, so the image $f(\partial\mathbb{B})$ at w_0 is always smooth. However, it is not necessarily true for $\mathbb{C}^n$ with $n \geq 2$. This theorem can be regarded as a general Schwarz lemma at the boundary for holomorphic mappings between unit balls in separable complex Hilbert spaces. It shows that the Fréchet derivative operator preserves tangent space and holomorphic tangent space at the boundary of the unit balls. When $\mathcal{H} = \mathcal{H}' = \mathbb{C}^n$, as a special case considered in this paper, Theorem 1.2 reduces (1) and (2) in [9, Theorem 3.1]. For $\mathcal{H} = \mathcal{H}' = \mathbb{C}$, part (II) of the theorem gives Theorem 1.1 from [8].

2 Preliminaries

Before proving the main results, we give some preparation. Lemma 2.1 was given in [13] for $p \in \mathbb{B}$.

Lemma 2.1 *For given $p \in \mathbb{B} \cup \partial\mathbb{B}$ and $q \in \mathcal{H}$ with $q \neq 0$, let $L(\xi) = p + \xi q$ for $\xi \in \mathbb{C}$. Then*

$$L(D_{p,q}) \subset \mathbb{B}, \quad L(\partial D_{p,q}) \subset \partial\mathbb{B},$$

where $D_{p,q} = \{\xi \in \mathbb{C} \mid |\xi - c_{p,q}| < r_{p,q}\}$ with $c_{p,q} = -\frac{\langle p,q\rangle}{\|q\|^2}$, $r_{p,q} = \sqrt{\frac{1-\|p\|^2}{\|q\|^2} + \left|\frac{\langle p,q\rangle}{\|q\|^2}\right|^2}$.

Proof Assume $\|L(D_{p,q})\|^2 < 1$, which means

$$\|p\|^2 + 2\mathrm{Re}\langle p, \xi q\rangle + \|\xi q\|^2 < 1$$

and

$$\frac{\|p\|^2}{\|q\|^2} + 2\frac{\mathrm{Re}\langle p, \xi q\rangle}{\|q\|^2} + |\xi|^2 < \frac{1}{\|q\|^2},$$

i.e.,

$$\left|\xi + \frac{\langle p, q\rangle}{\|q\|^2}\right|^2 < \frac{1 - \|p\|^2}{\|q\|^2} + \left|\frac{\langle p, q\rangle}{\|q\|^2}\right|^2.$$

The proof is finished.

Let $\phi_z(w)$ be the automorphism of $\mathbb{B}$, and

$$\phi_z(w) = \frac{z - P_z(w) - sQ_z(w)}{1 - \langle w, z\rangle}, \quad z, w \in \mathbb{B}$$

with P_z being the orthogonal projection of $\mathcal{H}$ by

$$P_z(w) = \frac{\langle w, z\rangle}{\|z\|^2} z, \quad \text{if } z \neq 0,$$

$Q_z = \mathrm{I} - P_z$ with I being the identity mapping, and $s = \sqrt{1 - \|z\|^2}$ (see [5]). It is found that $\phi_z(0) = z$, $\phi_z(z) = 0$ and $\phi_z = \phi_z^{-1}$. For simplicity, motivated by [14], we can rewrite $\phi_z(w)$ by

$$\phi_z(w) = \Gamma \frac{z - w}{1 - \langle w, z\rangle}, \tag{2.1}$$

where $\Gamma \in B(\mathcal{H}, \mathcal{H})$ is expressed by $\Gamma = s\mathrm{I} + \frac{\langle \cdot, z\rangle z}{1+s}$. Then it is easy to obtained that

$$\Gamma^2 = s^2 \mathrm{I} + \langle \cdot, z\rangle z, \quad \Gamma(z) = z.$$

From (2.1), for a fixed $z \in \mathbb{B}$ and any $w \in \mathbb{B}$,

$$D\phi_z(w) = \frac{1}{1 - \langle w, z\rangle} \Gamma\Big(-\mathrm{I} + \frac{\langle \cdot, z\rangle (z - w)}{1 - \langle w, z\rangle}\Big). \tag{2.2}$$

Lemma 2.2 *If $f \in H(\mathbb{B}, \mathbb{B}')$ and $f(0) = 0$, then $\|f(w)\| \le \|w\|$, $w \in \mathbb{B}$.*

It is a well-known Schwarz type result (see [5]), and we give a simple proof here.

Proof The Kobayashi distances for the unit ball in $\mathcal{H}$ could be expressed by

$$K_{\mathbb{B}}(z, w) = \frac{1}{2} \log \frac{1 + \|\phi_z(w)\|}{1 - \|\phi_z(w)\|}, \quad z, w \in \mathbb{B}. \tag{2.3}$$

It is the fact that the Kobayashi distance decreases under holomorphic mappings (see [15]). By (2.3), we have $K_{\mathbb{B}'}(0, f(w)) \le K_{\mathbb{B}}(0, w)$, i.e.,

$$\frac{1}{2} \log \frac{1 + \|f(w)\|}{1 - \|f(w)\|} \le \frac{1}{2} \log \frac{1 + \|w\|}{1 - \|w\|}.$$

Since $t \to \frac{1}{2} \log \frac{1+t}{1-t}$ is an increasing function for $t \in [0, 1)$, so that $\|f(w)\| \le \|w\|$.

Lemma 2.3 *Given any $b \in \mathcal{H}$ and $a \in \mathcal{H}'$, we have the operator $A = \langle \cdot, b\rangle a \in B(\mathcal{H}, \mathcal{H}')$ and*

$$A^* = \langle \cdot, a\rangle b.$$

Proof From the definition of the inner product, it is easy to see $A \in B(\mathcal{H}, \mathcal{H}')$. For any $y \in \mathcal{H}$ and $x \in \mathcal{H}'$,

$$\langle x, Ay\rangle = \langle x, a\langle y, b\rangle\rangle = \overline{\langle y, b\rangle}\langle x, a\rangle = \langle b, y\rangle \langle x, a\rangle = \langle \langle x, a\rangle b, y\rangle \triangleq \langle A^* x, y\rangle, \tag{2.4}$$

which gives $A^* = \langle \cdot, a\rangle b$ from the uniqueness of A^*.

If $A = \langle \cdot, b\rangle ac$ with $c \in \mathbb{C}$, Lemma 2.3 also gives $A^* = \langle \cdot, a\rangle b\overline{c}$.

3 Proof of Theorem 1.2

In the following, we will prove Theorem 1.2 in five steps.

Step 1 Assume $z_0 = e_1 \in \partial\mathbb{B}$, and f is $C^{1+\alpha}$ in a neighborhood V of z_0. Moreover, we assume $f(0) = 0$ and $f(z_0) = w_0 = e_1'$.

Let $p = z_0$, $q = (-1 + \mathrm{i}k)z_0$ for any given $k \in \mathbb{R}$. Then $p + tq = (1 - t + \mathrm{i}kt)z_0$ for $t \in \mathbb{R}$. From Lemma 2.1, $\|p+tq\| < 1 \Leftrightarrow |1-t+\mathrm{i}kt| < 1 \Leftrightarrow 0 < t < \frac{2}{1+k^2}$, which means that for a given $k \in \mathbb{R}$ when $t \to 0^+$, $p+tq \in \mathbb{B}\cap V$. For such t, taking the Taylor expansion of $f((1-t+\mathrm{i}kt)z_0)$ at $t = 0$, we have

$$f((1 - t + \mathrm{i}kt)z_0) = w_0 + \mathrm{D}f(z_0)(-1 + \mathrm{i}k)z_0 t + O(t^{1+\alpha}).$$

By Lemma 2.2,

$$\|f((1 - t + \mathrm{i}kt)z_0)\|^2 = \|w_0 + \mathrm{D}f(z_0)(-1 + \mathrm{i}k)z_0 t + O(t^{1+\alpha})\|^2 \le \|(1 - t + \mathrm{i}kt)z_0\|^2,$$

i.e.,

$$1 + 2\mathrm{Re}\langle w_0, \mathrm{D}f(z_0)(-1 + \mathrm{i}k)z_0 t\rangle + O(t^{1+\alpha}) \le 1 - 2t + O(t^2).$$

Substituting $w_0 = e_1'$, $z_0 = e_1$ and letting $t \to 0^+$, we have

$$\mathrm{Re}\langle e_1', \mathrm{D}f(z_0)(-1 + \mathrm{i}k)e_1\rangle \le -1,$$

i.e.,

$$-\mathrm{Re}\langle e_1', \mathrm{D}f(z_0)e_1\rangle + k\mathrm{Im}\langle e_1', \mathrm{D}f(z_0)e_1\rangle \le -1,$$

which gives

$$k\mathrm{Im}\langle e_1', \mathrm{D}f(z_0)e_1\rangle \le \mathrm{Re}\langle e_1', \mathrm{D}f(z_0)e_1\rangle - 1. \tag{3.1}$$

Since (3.1) is valid for any $k \in \mathbb{R}$, we have

$$\mathrm{Im}\langle e_1', \mathrm{D}f(z_0)e_1\rangle = 0,$$

which implies

$$0 \le \mathrm{Re}\langle e_1', \mathrm{D}f(z_0)e_1\rangle - 1$$

and

$$\langle e_1', \mathrm{D}f(z_0)e_1\rangle = \mathrm{Re}\langle e_1', \mathrm{D}f(z_0)e_1\rangle \ge 1. \tag{3.2}$$

Step 2 Let $p = z_0$, $q = -z_0 + \mathrm{i}ke_j$ for $j \ge 2$ and $k \in \mathbb{R}$. Then $p + tq = (1 - t)z_0 + \mathrm{i}kte_j$ for $t \in \mathbb{R}$. By Lemma 2.1, $\|p + tq\| < 1 \Leftrightarrow |1 - t|^2 + |\mathrm{i}kt|^2 < 1 \Leftrightarrow 0 < t < \frac{2}{1+k^2}$. Therefore, given a $k \in \mathbb{R}$, when $t \to 0^+$, $p+tq \in \mathbb{B}\cap V$. Similarly, taking the Taylor expansion of $f((1-t)z_0+\mathrm{i}kte_j)$ at $t = 0$, we have

$$f((1 - t)z_0 + \mathrm{i}kte_j) = w_0 + \mathrm{D}f(z_0)(-z_0 + \mathrm{i}ke_j)t + O(t^{1+\alpha}).$$

By Lemma 2.2,

$$\begin{aligned}\|f((1 - t)z_0 + \mathrm{i}kte_j)\|^2 &= \|w_0 + \mathrm{D}f(z_0)(-z_0 + \mathrm{i}ke_j)t + O(t^{1+\alpha})\|^2 \\ &\le \|(1 - t)z_0 + \mathrm{i}kte_j\|^2,\end{aligned}$$

i.e.,

$$1+2\mathrm{Re}\langle w_0,\mathrm{D}f(z_0)(-z_0+\mathrm{i}ke_j)t\rangle+O(t^{1+\alpha})\le 1-2t+O(t^2).$$

Using $w_0=e_1',\ z_0=e_1$ and letting $t\to 0^+$, it follows that

$$\mathrm{Re}\langle e_1',\mathrm{D}f(z_0)(-e_1+\mathrm{i}ke_j)\rangle\le -1,\quad j\ge 2,$$

i.e.,

$$-\mathrm{Re}\langle e_1',\mathrm{D}f(z_0)e_1\rangle+k\mathrm{Im}\langle e_1',\mathrm{D}f(z_0)e_j\rangle\le -1.$$

With a similar argument to Step 1, we have

$$\mathrm{Im}\langle e_1',\mathrm{D}f(z_0)e_j\rangle=0,\quad j\ge 2.$$

Meanwhile, if we assume $p=z_0$, $q=-z_0+ke_j$ for $j\ge 2$ and any $k\in\mathbb{R}$. It is easy to find

$$\mathrm{Re}\langle e_1',\mathrm{D}f(z_0)e_j\rangle=0,\quad j\ge 2.$$

Therefore

$$\langle e_1',\mathrm{D}f(z_0)e_j\rangle=0,\quad j\ge 2. \tag{3.3}$$

Combining (3.2) and (3.3), and using the adjoint operator $\mathrm{D}^*f(z_0)$ of $\mathrm{D}f(z_0)$, one gets

$$\begin{aligned}&\langle \mathrm{D}^*f(z_0)e_1',e_1\rangle\ge 1,\\&\langle \mathrm{D}^*f(z_0)e_1',e_j\rangle=0,\quad j\ge 2.\end{aligned}$$

Assume $\mathrm{D}^*f(z_0)e_1'=\sum\limits_{l=1}^{\infty}\lambda_l e_l$ for $\lambda_l\in\mathbb{C}$. The above equations give

$$\begin{aligned}&\Big\langle\sum_{l=1}^{\infty}\lambda_l e_l,e_1\Big\rangle=\lambda_1\ge 1,\\&\Big\langle\sum_{l=1}^{\infty}\lambda_l e_l,e_j\Big\rangle=\lambda_j=0,\quad j\ge 2.\end{aligned}$$

Therefore

$$\mathrm{D}^*f(z_0)w_0=\lambda_1 z_0, \tag{3.4}$$

where $\lambda_1=\langle \mathrm{D}^*f(z_0)w_0,z_0\rangle=\langle w_0,\mathrm{D}f(z_0)z_0\rangle$ for $w_0=e_1',\ z_0=e_1$.

Step 3 Now let z_0 be any given point at $\partial\mathbb{B}$ which is not necessary e_1. Then there exists a unitary matrix U_{z_0} such that $U_{z_0}(z_0)=e_1$. Assume $f(0)=0,\ f(z_0)=w_0\in\partial\mathbb{B}'$, which is not necessary e_1'. Similarly, there is a U_{w_0} such that $U_{w_0}(w_0)=e_1'$. Denote

$$g(z)=U_{w_0}\circ f\circ U_{z_0}^{-1}(z),\quad z\in\mathbb{B}\cup\{e_1\}.$$

Then $g(0)=0,\ g(e_1)=e_1'$. Moreover

$$\mathrm{D}g(z)=U_{w_0}\circ \mathrm{D}f(U_{z_0}^{-1}z)\circ U_{z_0}^{-1}(z),\quad z\in\mathbb{B}\cup\{e_1\}.$$

From Steps 1 and 2, we have

$$\mathrm{D}^*g(e_1)e_1'=\lambda_g e_1$$

with $\lambda_g = \langle e_1', \mathrm{D}g(e_1)e_1\rangle \geq 1$, which equals

$$(U_{w_0} \circ \mathrm{D}f(U_{z_0}^{-1}z) \circ U_{z_0}^{-1})^* e_1' = \lambda_g e_1,$$

i.e.,

$$U_{z_0} \circ \mathrm{D}^* f(z_0) \circ U_{w_0}^* e_1' = \lambda_g e_1.$$

Composing operator $U_{z_0}^{-1}$ at both sides of the above equation gives

$$U_{z_0}^{-1} \circ U_{z_0} \circ \mathrm{D}^* f(z_0) \circ U_{w_0}^{-1} e_1' = \lambda_g U_{z_0}^{-1} e_1,$$

i.e.,

$$\mathrm{D}^* f(z_0) w_0 = \lambda_g z_0,$$

where $\lambda_g = \langle e_1', \mathrm{D}g(e_1)e_1\rangle \geq 1$.

Step 4 Let $f(z_0) = w_0$ with $z_0 \in \partial\mathbb{B},\ w_0 \in \partial\mathbb{B}'$. If $f(0) = a \neq 0$, then we use the automorphism of $\mathbb{B}'$ to get the result. Assume that $\phi_a(w)$ is an automorphism of $\mathbb{B}'$ such that $\phi_a(a) = 0$. Then $\phi_a(w_0) \in \partial\mathbb{B}'$ and there exists a U_{ϕ_a} such that $U_{\phi_a}(\phi_a(w_0)) = w_0$. Let

$$h = U_{\phi_a} \circ \phi_a \circ f,$$

then $h(0) = 0,\ h(z_0) = w_0$. As a result of Step 3, there is a real number $\gamma \geq 1$ such that

$$\mathrm{D}^* h(z_0) w_0 = \gamma z_0.$$

According to the expression of h, it is obtained that

$$\begin{aligned} \mathrm{D}^* h(z_0) w_0 &= (U_{\phi_a} \circ \mathrm{D}\phi_a(w_0) \circ \mathrm{D}f(z_0))^* w_0 \\ &= \mathrm{D}^* f(z_0) \circ \mathrm{D}^* \phi_a(w_0) \circ U_{\phi_a}^{-1} w_0. \end{aligned} \tag{3.5}$$

Since $U_{\phi_a}(\phi_a(w_0)) = w_0$, we have $U_{\phi_a}^{-1} w_0 = \phi_a(w_0)$. Therefore,

$$\mathrm{D}^* \phi_a(w_0) \circ U_{\phi_a}^{-1} w_0 = \mathrm{D}^* \phi_a(w_0) \phi_a(w_0).$$

From (2.2) and Lemma 2.3,

$$\begin{aligned} \mathrm{D}^* \phi_z(w) &= \frac{1}{\overline{1 - \langle w, z\rangle}} \Big(-\mathrm{I} + \frac{\langle \cdot, z\rangle (z - w)}{1 - \langle w, z\rangle} \Big)^* \Gamma^* \\ &= \frac{1}{\overline{1 - \langle w, z\rangle}} \Big(-\mathrm{I} + \frac{\langle \cdot, z - w\rangle z}{\overline{1 - \langle w, z\rangle}} \Big) \Gamma. \end{aligned}$$

It comes from (2.1) that

$$\begin{aligned} &\mathrm{D}^* \phi_a(w_0) \phi_a(w_0) \\ &= \frac{1}{\overline{1 - \langle w_0, a\rangle}} \Big(-\mathrm{I} + \frac{\langle \cdot, a - w_0\rangle a}{\overline{1 - \langle w_0, a\rangle}} \Big) \Gamma^2 \frac{a - w_0}{1 - \langle w_0, a\rangle} \\ &= \frac{1}{|1 - \langle w_0, a\rangle|^2} \Big(-\mathrm{I} + \frac{\langle \cdot, a - w_0\rangle a}{\overline{1 - \langle w_0, a\rangle}} \Big) (a - s^2 w_0 - \langle w_0, a\rangle a) \\ &= \frac{1}{|1 - \langle w_0, a\rangle|^2} \Big[s^2 w_0 - (1 - \langle w_0, a\rangle) a + \frac{\langle a - s^2 w_0 - \langle w_0, a\rangle a, a - w_0\rangle a}{\overline{1 - \langle w_0, a\rangle}} \Big] \\ &= \frac{1}{|1 - \langle w_0, a\rangle|^2} \Big[s^2 w_0 - \Big(1 - \frac{\|a\|^2 - \langle a, w_0\rangle}{\overline{1 - \langle w_0, a\rangle}} - \frac{1 - \|a\|^2}{\overline{1 - \langle w_0, a\rangle}} \Big) (1 - \langle w_0, a\rangle) a \Big] \\ &= \frac{1 - \|a\|^2}{|1 - \langle w_0, a\rangle|^2} w_0. \end{aligned}$$

Combining with (3.5) we get

$$\mathrm{D}^* f(z_0)\frac{1-\|a\|^2}{|1-\langle w_0, a\rangle|^2} w_0 = \gamma z_0$$

for some $\gamma \geq 1$. As a result,

$$\mathrm{D}^* f(z_0) w_0 = \lambda z_0, \tag{3.6}$$

where $\lambda = \frac{|1-\langle w_0,a\rangle|^2}{1-\|a\|^2}\gamma \geq \frac{|1-\langle w_0,a\rangle|^2}{1-\|a\|^2} > 0$ and $a = f(0)$. The proof of (II) is completed.

Step 5 For any $\beta \in T_{z_0}(\partial\mathbb{B})$, from the definition of tangent space given by (1.1), we have

$$\mathrm{Re}\langle z_0, \beta\rangle = 0.$$

To prove $\mathrm{D}f(z_0)\beta \in T_{w_0}(\partial\mathbb{B}')$, it is sufficient to verify

$$\mathrm{Re}\langle w_0, \mathrm{D}f(z_0)\beta\rangle = 0.$$

From (3.6), $\mathrm{D}^* f(z_0) w_0 = \lambda z_0$ with $\lambda > 0$, which gives

$$\mathrm{Re}\langle w_0, \mathrm{D}f(z_0)\beta\rangle = \mathrm{Re}\langle \mathrm{D}^* f(z_0) w_0, \beta\rangle = \mathrm{Re}\langle \lambda z_0, \beta\rangle = \lambda \mathrm{Re}\langle z_0, \beta\rangle = 0.$$

On the other hand, for any $\beta \in T_{z_0}^{(1,0)}(\partial\mathbb{B})$, from (1.1), we have

$$\langle z_0, \beta\rangle = 0.$$

It comes from the above equation that

$$\langle w_0, \mathrm{D}f(z_0)\beta\rangle = \langle \mathrm{D}^* f(z_0) w_0, \beta\rangle = \langle \lambda z_0, \beta\rangle = 0.$$

Therefore $\mathrm{D}f(z_0)\beta \in T_{w_0}^{(1,0)}(\partial\mathbb{B}')$. The proof of (I) is finished.

4 The Boundary Version of Schwarz Lemma on the Upper Half-Plane

We decompose $\mathcal{H} = e_1\mathbb{C} \oplus F$, where $F = e_1^{\perp}$. In this decomposition, $z = (z_1, z') \in \mathcal{H}$ with $z_1 \in \mathbb{C}$ and $z' \in F$. Then the upper half-plane $\mathbb{H}$ of $\mathcal{H}$ could be described by $\{z \in \mathcal{H} \mid \mathrm{Im} z_1 > \|z'\|^2\}$. Similar notations are given for $\mathcal{H}'$.

The tangent space $T_0(\partial\mathbb{H})$ and holomorphic tangent space $T_0^{1,0}(\partial\mathbb{H})$ at 0 are defined by

$$T_0(\partial\mathbb{H}) = \{\beta \in \mathcal{H} \mid \mathrm{Re}\langle -\mathrm{i}e_1, \beta\rangle = 0\}, \quad T_{z_0}^{(1,0)}(\partial\mathbb{H}) = \{\beta \in \mathcal{H} \mid \langle -\mathrm{i}e_1, \beta\rangle = 0\},$$

respectively. The boundary version of Schwarz lemma on the upper half-plane is given as follows.

Theorem 4.1 *Let $g \in H(\mathbb{H}, \mathbb{H}')$ for $\mathbb{H} \in \mathcal{H}$ and $\mathbb{H}' \in \mathcal{H}'$ respectively. If g is $C^{1+\alpha}$ at $0 \in \partial\mathbb{H}$ and $g(0) = 0 \in \partial\mathbb{H}'$, then it holds that*

(I) $\mathrm{D}g(0)\beta \in T_0(\partial\mathbb{H}')$ *for any* $\beta \in T_0(\partial\mathbb{H})$, *and* $\mathrm{D}g(0)\beta \in T_0^{(1,0)}(\partial\mathbb{H}')$ *for any* $\beta \in T_0^{(1,0)}(\partial\mathbb{H})$.

(II) *There exists $\lambda \in \mathbb{R}$ such that $\mathrm{D}^* g(0)(-\mathrm{i}e_1') = \lambda(-\mathrm{i}e_1)$ with $\lambda \geq \frac{|1-\langle e_1', a\rangle|^2}{1-\|a\|^2} > 0$, where $a = -\frac{2g(\mathrm{i}e_1)}{\mathrm{i}+g_1(\mathrm{i}e_1)} + e_1'$.*

Proof First, there exists a biholomorphic mapping (Cayley transform) $\Phi : \mathbb{B} \to \mathbb{H}$ given by

$$\Phi(z) = \mathrm{i}\frac{e_1 - z}{1 + z_1} = \Big(\mathrm{i}\frac{1 - z_1}{1 + z_1}, \mathrm{i}\frac{-z'}{1 + z_1}\Big) \triangleq (w_1, w') \in \mathbb{H}.$$

Then it is easy to see $\operatorname{Im} w_1 - \|w'\|^2 = \frac{1-\|z\|^2}{|1+z_1|^2} > 0$, and $\Phi(e_1) = 0$. Moreover,

$$\Phi^{-1}(w) = -\frac{2w}{\mathrm{i} + w_1} + e_1$$

and $\Phi^{-1}(0) = e_1$.

Step 1 Let $g(w) \in H(\mathbb{H}, \mathbb{H}')$ which is also $C^{1+\alpha}$ at 0. We assume $g(0) = 0$ and $g(\mathrm{i}e_1) = \mathrm{i}e_1'$. Construct a mapping

$$f = \Phi'^{-1} \circ g \circ \Phi, \tag{4.1}$$

then $f \in H(\mathbb{B}, \mathbb{B}')$ and $f(0) = 0,\ f(e_1) = e_1'$. From (3.4) we have

$$\mathrm{D}^* f(e_1)e_1' = \lambda_f e_1 \tag{4.2}$$

for $\lambda_f = \langle e_1', \mathrm{D}f(e_1)e_1\rangle \geq 1$. On the other hand,

$$\begin{aligned} \mathrm{D}f(e_1) &= \mathrm{D}(\Phi'^{-1} \circ g \circ \Phi)(e_1) \\ &= \mathrm{D}\Phi'^{-1}(0) \circ \mathrm{D}g(0) \circ D\Phi(e_1) \\ &= (2\mathrm{i})I \circ \mathrm{D}g(0) \circ \Big(\frac{-\mathrm{i}}{2}\Big)\mathrm{I} \\ &= \mathrm{D}g(0). \end{aligned} \tag{4.3}$$

Substituting (4.3) into (4.2) gives

$$\mathrm{D}^* g(0)e_1' = \lambda_f e_1,$$

i.e.,

$$\mathrm{D}^* g(0)(-\mathrm{i}e_1') = \lambda_f(-\mathrm{i}e_1),$$

where $\lambda_f \geq 1$ and $-\mathrm{i}e_1$ denotes the normal vector of $\mathbb{H}$ at 0.

Step 2 If $g(0) = 0$ and $g(\mathrm{i}e_1) \neq \mathrm{i}e_1'$, we assume $g(\mathrm{i}e_1) = b$. Consider the mapping given by (4.1), then $f(0) = \Phi'^{-1}(b) \triangleq a \in \mathbb{B}',\ f(e_1) = e_1'$. From (3.6) we have

$$\mathrm{D}^* f(e_1)e_1' = \lambda e_1,$$

where $\lambda \geq \frac{|1-\langle e_1', a\rangle|^2}{1-\|a\|^2} > 0$.

In addition, from (4.3), one gets

$$\mathrm{D}^* f(e_1)e_1' = \mathrm{D}^* g(0)e_1' = \lambda e_1,$$

i.e.,

$$\mathrm{D}^* g(0)(-\mathrm{i}e_1') = \lambda(-\mathrm{i}e_1),$$

where $\lambda \geq \frac{|1-\langle e_1', a\rangle|^2}{1-\|a\|^2} > 0$. Therefore, part (II) is proved.

The proof of part (I) is the same as that of Theorem 1.2, and it is omitted here.

Acknowledgements The work was finished while the second author visited the third author at the department of mathematical sciences, Indiana University–Purdue University Fort Wayne. The first author appreciates the comfortable research environment and all support provided by the institution in the 2014 academic year.

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