Chin. Ann. Math. Ser. B
39(4), 2018, 721–738
DOI: 10.1007/s11401-018-0092-6

Chinese Annals of Mathematics, Series B

Endpoint Estimates for Generalized Multilinear Fractional Integrals on the Non-homogeneous Metric Spaces*

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Abstract In this paper, some endpoint estimates for the generalized multilinear fractional integrals $I_{\alpha,m}$ on the non-homogeneous metric spaces are established.

Keywords Generalized multilinear fractional integrals, Lipschitz space, RBMO space, Morrey space, Non-homogeneous metric space
2000 MR Subject Classification 42B25, 42B30

1 Introduction and Notation

Spaces of homogeneous type — (quasi-)metric spaces equipped with a so-called doubling measure—were introduced by Cofiman and Weiss [7] as a general framework in which many results from real and harmonic analysis on Euclidean spaces have their natural extensions (see for example [6, 11–12]). It is now well known that a metric space $(\mathcal{X}, d)$ equipped with a nonnegative Borel measure μ is called a space of homogeneous type — if $(\mathcal{X}, d, \mu)$ satisfies the following measure doubling condition that there exists a positive constant C_μ, depending on μ, such that for any ball $B(x,r) = \{y \in \mathcal{X} : d(x,y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0,\infty)$,

$$0 < \mu(B(x,r)) \le C_\mu \mu\Big(B\Big(x, \frac{r}{2}\Big)\Big). \tag{1.1}$$

The doubling condition (1.1) plays a key role in the classical theory of Calderón-Zygmund operators.

Meanwhile, recent developments in the Calderón-Zygmund theory (which one might think of it as "zeroth order calculus", as only integrability of the functions on which one operator is considered) have shown that a number of interesting problems cannot be, and need not be, embedded into the homogeneous framework. The measure can be replaced by a less demanding condition such as the polynomial growth condition.

Let μ be a non-negative Radon measure on $\mathbb{R}^n$ which only satisfies the polynomial growth condition, namely, there exist positive constants C and $\kappa \in (0,n]$ such that for all $x \in \mathbb{R}^n$ and

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*This work was supported by the National Natural Science Foundation of China (Nos. 11271330, 11261023, 11461033, 11401269) and the Jiangxi Provincial Natural Science Foundation of China (No. 20142BAB201003).

$r \in (0, \infty)$,

$$\mu(\{y \in \mathbb{R}^n : |x - y| < r\}) \le C_0 r^\kappa. \tag{1.2}$$

The analysis associated with such nondoubling measures μ is proved to play a striking role in solving the long-standing open painlevé's problem by Tolsa [20]. Obviously, the measure μ satisfies the polynomial growth condition may not satisfy the doubling condition. To unify both the doubling condition and polynomial growth condition, Hytönen [14] introduced a new class of metric measure spaces satisfying so-called geometrically doubling and the upper doubling conditions (see Definitions 1.1–1.2 respectively), which are called non-homogeneous spaces. We refer the reader to the survey (see [21]) and the monograph (see [22]) for more progress on the theory of Hardy spaces and singular integrals over nonhomogeneous metric measure spaces.

Definition 1.1 *A metric measure space $(\mathcal{X}, d, \mu)$ is called upper doubling if μ is a Borel measure on $\mathcal{X}$ and there exists a dominating function*

$$\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$$

and a positive constant $C_\lambda > 1$ such that for each $x \in \mathcal{X}$, $r \to \lambda(x, r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \le \lambda(x, r) \le C_\lambda \lambda\Big(x, \frac{r}{2}\Big). \tag{1.3}$$

Remark 1.1 (i) A space of homogeneous type is a special case of upper doubling spaces, where one can take the dominating function $\lambda(x, r) \equiv \mu(B(x, r))$. On the other hand, a metric space $(\mathcal{X}, d, \mu)$ satisfying the polynomial growth condition is also an upper doubling measure space if we take $\lambda(x, r) \equiv Cr^k$.

(ii) Let $(\mathcal{X}, d, \mu)$ be an upper doubling space and λ be a dominating function on $\mathcal{X} \times (0, +\infty)$ as in Definition 1.1. In [15], it was showed that there exists another dominating function $\widetilde{\lambda}$ such that for all $x, y \in \mathcal{X}$ with $d(x, y) \le r$,

$$\widetilde{\lambda}(x, r) \le \widetilde{C}\widetilde{\lambda}(y, r). \tag{1.4}$$

Based on this, in this paper, we always assume that the dominating function λ also satisfies (1.4).

Definition 1.2 *A metric measure space $(\mathcal{X}, d)$ is called geometrically doubling if there exists a positive integer N_0 such that for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite covering $\big\{B\big(x_i, \frac{r}{2}\big)\big\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.*

In this paper, we will consider the boundedness of generalized multilinear fractional integrals on nonhomogeneous spaces. First, let us give some symbols and notation. We start with the notion of multilinear fractional kernel of order α and regularity δ.

Definition 1.3 *Let $0 < \alpha < mn$ and $0 < \delta \le 1$. A function $K_\alpha \in L^1_{\rm loc}(\mathcal{X} \times \cdots \times \mathcal{X} \setminus \{(x, y_1, \cdots, y_m) : x = y_1 = \cdots = y_m\})$ is said to be a multilinear fractional kernel of order α and regularity δ if it satisfies the following two conditions:*

(i) *There exists s positive constant C such that for all $x, y_1, \cdots, y_m \in \mathcal{X}$ with $x \neq y_j$ for some j,*

$$|K_\alpha(x, y_1, \cdots, y_m)| \leq \frac{C}{\Big[\sum_{i=1}^{m} \lambda(x, d(x, y_i))\Big]^{m-\frac{\alpha}{n}}}. \tag{1.5}$$

(ii) *For all $x, x', y_1, \cdots, y_m \in \mathcal{X}$ with* $\max\{d(x, y_1), \cdots, d(x, y_m)\} \geq 2d(x, x')$,

$$\begin{aligned}&|K_\alpha(x, y_1, \cdots, y_m) - K_\alpha(x', y_1, \cdots, y_m)|\\ &\leq \frac{C[\lambda(x, d(x, x'))]^{\frac{\delta}{n}}}{\Big[\sum_{i=1}^{m} \lambda(x, d(x, y_i))\Big]^{m-\frac{\alpha}{n}+\frac{\delta}{n}}}.\end{aligned} \tag{1.6}$$

Now, we will give the definition of the generalized multilinear fractional integral operator $I_{\alpha,m}$ associated with K_α.

Definition 1.4 *For $0 < \alpha < mn$, the generalized multilinear fractional integral operator $I_{\alpha,m}$ associated with K_α is assumed to be bounded from $L^1(\mu) \times \cdots \times L^1(\mu)$ into $L^{\frac{n}{mn-\alpha},\infty}(\mu)$ and satisfies that for all bounded functions $f_1, f_2, \cdots, f_m$ with bounded support and μ-almost every $x \in \mathcal{X} \backslash (\bigcap_{j=1}^{m} \mathrm{supp} f_j)$,*

$$\begin{aligned}&I_{\alpha,m}(\vec{f})(x) = I_{\alpha,m}(f_1, \cdots, f_m)(x)\\ &= \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} K_\alpha(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) \mathrm{d}\mu(y_1) \cdots \mathrm{d}\mu(y_m).\end{aligned} \tag{1.7}$$

When $m = 1$, the operator $I_{\alpha,1}$ defined by (1.7) is adapted from the generalized fractional integral operator in nonhomogeneous metric spaces that appeared in [9] with $\big[\frac{d(x,x')}{d(x,y_i)}\big]^\delta$ replaced by $\big[\frac{\lambda(x,d(x,x'))}{\lambda(x,d(x,y_i))}\big]^{\frac{\delta}{n}}$. See also [13] for the case of Euclidean spaces associated with nondoubling measures.

As is well-known, when $(\mathcal{X}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mathrm{d}x)$, the classical multilinear fractional integrals operator $I_{\alpha,m}$ is bounded from $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n)$ into $L^{\frac{n}{mn-\alpha},\infty}(\mathbb{R}^n)$. However, in the case of non-homogeneous metric spaces, it is still unknown whether $I_{\alpha,m}$ has the $(L^1(\mu) \times \cdots \times L^1(\mu), L^{\frac{n}{mn-\alpha},\infty}(\mu))$-boundedness. It is known that many authors have been interested in studying the boundedness of this operator on various function spaces, see [10], [16], [18] and [5] etc. Recently, Tang [19] studied the classical multilinear fractional integral and obtained some endpoint estimates. He proved that $I_{\alpha,m}$ is bounded from $M_{p_1}^{q_1}(\mathbb{R}^n) \times \cdots \times M_{q_m}^{p_m}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ and $1 \leq q_i < p_i < \infty$, $p = \frac{n}{\alpha}$. In addition, he also obtained the $\big(M_{p_1}^{q_1}(\mathbb{R}^n) \times \cdots \times M_{q_m}^{p_m}(\mathbb{R}^n), \mathrm{Lip}\big(\alpha - \frac{n}{p}\big)\big)$-boundedness for $p > \frac{n}{\alpha}$ and $(M_{p_1}^{q_1}(\mathbb{R}^n) \times \cdots \times M_{q_m}^{p_m}(\mathbb{R}^n), M_s^{\frac{pn}{n-p\alpha}}(\mathbb{R}^n))$-boudedness and $(M_{p_1}^{q_1}(\mathbb{R}^n) \times \cdots \times M_{q_m}^{p_m}(\mathbb{R}^n), WM_s^{\frac{pn}{n-p\alpha}}(\mathbb{R}^n))$-boudedness with $\frac{1}{s} = \frac{1}{q} - \frac{\alpha}{n}$. See [8] for more information on the theory of generalized fractional integrals and H^p spaces over non-homogeneous metric measure spaces.

Inspired by [9] and [19], we will investigate the same endpoint estimates in [19] for generalized multilinear fractional integral on non-homogeneous metric spaces. We can formulate our main results as follows.

Theorem 1.1 *Let $m \in \mathbb{N}$, $(m-1)n < \alpha < mn$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ with $1 \le q_i < p_i < \infty$ for $i = 1, \cdots, m$. If $p = \frac{n}{\alpha}$ and λ satisfies the ε-weak reverse doubling condition with $\varepsilon \in \big(0, \min\limits_{i=1,\cdots,m} \{1 + \frac{\delta}{n} - \frac{1}{p_i}, \frac{\delta_i}{n}\}\big)$, where $0 < \delta_i$ and $\delta = \sum\limits_{i=1}^{m} \delta_i$, then*

$$\|I_{\alpha,m}(f_1, \cdots, f_m)\|_{\mathrm{RBMO}(\mu)} \le C \prod_{j=1}^{m} \|f_j\|_{M^{p_j}_{q_j}(3,\mu)}. \tag{1.8}$$

Theorem 1.2 *Let $m \in \mathbb{N}$, $0 < \alpha < mn$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ with $1 < q_i < p_i < \infty$ for $i = 1, \cdots, m$. If λ satisfies the ε-weak reverse doubling condition with $\varepsilon \in \big(0, \min\limits_{i=1,\cdots,m} \{(\frac{\alpha_i}{n} - \frac{1}{q_i})q_i', 1 + \frac{\delta}{n} + \frac{1}{p_i} - \frac{\alpha}{n}, \frac{\delta_i}{n} + \frac{1}{p_i} - \frac{\alpha_i}{n}\}\big)$, where $0 < \alpha_i, \delta_i$, $0 < \alpha_i - \frac{n}{q_i} < \delta_i$ and $\delta = \sum\limits_{i=1}^{m} \delta_i$, $\alpha = \sum\limits_{i=1}^{m} \alpha_i$, then*

$$\|I_{\alpha,m}(f_1, \cdots, f_m)\|_{\mathrm{Lip}(\alpha - \frac{n}{p})} \le C \prod_{j=1}^{m} \|f_j\|_{M^{p_j}_{q_j}(3,\mu)}. \tag{1.9}$$

Theorem 1.3 *Let $m \in \mathbb{N}$, $0 < \alpha < mn$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ with $1 < q_i \le p_i \le \infty$ for $i = 1, \cdots, m$. Assume that $\frac{1}{s} = \frac{1}{q} - \frac{\alpha}{n}$, $p < \frac{n}{\alpha}$ and λ satisfies the ε-weak reverse doubling condition with $\varepsilon \in \big(0, \min\limits_{i=1,\cdots,m} \{1 + \frac{1}{p_i} - \frac{\alpha}{n}, \frac{1}{p_i} - \frac{\alpha_i}{n}\}\big)$, where $0 < \alpha_i$ and $\alpha = \sum\limits_{i=1}^{m} \alpha_i$.*

(a) *If each $q_i > 1$, then*

$$\|I_{\alpha,m}(f_1, \cdots, f_m)\|_{M_s^{\frac{pn}{n-p\alpha}}(3,\mu)} \le C \prod_{j=1}^{m} \|f_j\|_{M^{p_j}_{q_j}(3,\mu)}. \tag{1.10}$$

(b) *If each $1 \le q_i < p_i$ and $s(\frac{1}{q} - \frac{1}{p}) > q_i(\frac{1}{q_i} - \frac{1}{p_i})$ for $i = 1, \cdots, m$, then*

$$\|I_{\alpha,m}(f_1, \cdots, f_m)\|_{WM_s^{\frac{pn}{n-p\alpha}}(3,\mu)} \le C \prod_{j=1}^{m} \|f_j\|_{M^{p_j}_{q_j}(3,\mu)}. \tag{1.11}$$

Without loss of generality, in this paper, we only consider the case of $m = 2$, and C always means a positive constant independent of the main parameters involved, but it may be different from line to line. The p' is the conjugate index of p, that is to say, $\frac{1}{p} + \frac{1}{p'} = 1$.

The paper is organized as follows. In Section 2, we collect some useful definitions and lemmas. Theorems will be proved in the last section.

2 Preliminaries

In this section, we will recall some necessary notions and notation and the boundedness of $I_{\alpha,2}$ in $L^s(\mu)$ which was established in [3]. We begin with the definition of (α, β)-doubling ball, which can be found in [14].

Definition 2.1 *Let $\alpha, \beta \in (1, \infty)$. A ball $B \subset \mathcal{X}$ is called (α, β)-doubling if $\mu(\alpha B) \le \beta \mu(B)$.*

It was proved in [14] that if a metric measure space $(\mathcal{X}, d, \mu)$ is upper doubling and $\beta > C_\lambda^{\log_2 \alpha} \equiv \alpha^\nu$, then for every ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$ such that $\alpha^j B$

is (α, β)-doubling. Moreover, let $(\mathcal{X}, d)$ be geometrically doubling, $\beta > \alpha^n$ with $n \equiv \log_2 N_0$ and μ a Borel measure on $\mathcal{X}$ which is finite on bounded sets. Hytönen [14] also showed that for μ-almost every $x \in \mathcal{X}$, there exist arbitrarily small (α, β)-doubling balls centered at x. Furthermore, the radius of these balls may be chosen to be of the form $\alpha^{-j} r$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0, \infty)$. For any $\alpha \in (1, \infty)$ and ball B, $\widetilde{B}^{\alpha}$ denotes the smallest (α, β_{α})-doubling ball of the form $\alpha^j B$ with $j \in \mathbb{N}$, where

$$\beta_{\alpha} := \max\{\alpha^{3n}, \alpha^{3\nu}\} + 30^n + 30^{\nu} = \alpha^{3(\max\{n,\nu\})} + 30^n + 30^{\nu}.$$

In this paper we choose $\alpha = 6$ and denote the ball $\widetilde{B}^{\alpha}$ simply by $\widetilde{B}$.

Next, we give the definitions of constant $K_{B,S}$ and regular BMO space RBMO introduced by Bui and Duong [1].

Definition 2.2 *For any two balls $B \subset S$, define*

$$K_{B,S} = 1 + \int_{2S \setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} \mathrm{d}\mu(x), \tag{2.1}$$

where c_B is the center of the ball B.

Remark 2.1 The following discrete version $\widetilde{K}_{B,S}$ of $K_{B,S}$ defined in Definition 2.2 was first introduced by Bui and Duong [1] in non-homogeneous metric measure spaces, which is more close to the quantity $K_{Q,R}$ introduced by Tolsa [20] in the setting of non-doubling measures. For any two balls $B \subset S$, let $\widetilde{K}_{B,S}$ be defined by

$$\widetilde{K}_{B,S} = 1 + \sum_{k=1}^{N_{B,S}} \frac{\mu(6^k B)}{\lambda(c_B, 6^k r_B)},$$

where r_B and r_S respectively denote the radii of the balls B and S, and $N_{B,S}$ the smallest integer satisfying $6^{N_{B,S}} r_B \geq r_S$. Then $K_{B,S} \leq C\widetilde{K}_{B,S}$, but, in general, it is not true that $K_{B,S} \sim \widetilde{K}_{B,S}$.

Now we introduce the fractional coefficient $\widetilde{K}^{\gamma}_{B,S}$ from [9]; see also [4] for the case of Euclidean spaces associated with non-doubling measures.

Definition 2.3 *For any two balls $B \subset S$, $\widetilde{K}^{\gamma}_{B,S}$ is defined by*

$$\widetilde{K}^{\gamma}_{B,S} = 1 + \sum_{k=1}^{N_{B,S}} \Big(\frac{\mu(6^k B)}{\lambda(c_B, 6^k r_B)}\Big)^{1-\gamma},$$

where $\gamma \in (0,1)$ and $N_{B,S}$ is defined as in Remark 2.1.

Next we give out some properties of $\widetilde{K}^{\gamma}_{B,S}$ appeared in [9, Lemma 3.4], which are completely analogous to [4, Lemma 3].

Lemma 2.1 *Let $\gamma \in (0,1)$.*

(i) *For all balls $B \subset R \subset S$, $\widetilde{K}^{\gamma}_{B,R} \leq 2\widetilde{K}^{\gamma}_{B,S}$.*

(ii) *For any $\rho \in [1, \infty)$, there exists a positive constant $C_{(\rho)}$, depending only on ρ, such that for all balls $B \subset S$ with $r_S \leq \rho r_B$, $\widetilde{K}^{\gamma}_{B,S} \leq C_{(\rho)}$.*

(iii) *There exists a positive constant $C_{(\gamma)}$, depending on γ, such that for all balls B, $\widetilde{K}^{\gamma}_{B,\widetilde{B}} \leq C_{(\gamma)}$.*

(iv) *There exists a positive constant c, depending on C_λ and γ, such that for all balls $B \subset R \subset S$, $\widetilde{K}^{\gamma}_{B,S} \leq \widetilde{K}^{\gamma}_{B,R} + c\widetilde{K}^{\gamma}_{R,S}$.*

(v) *There exists a positive constant $\widetilde{c}$, depending on C_λ and γ, such that for all balls $B \subset R \subset S$, $\widetilde{K}^{\gamma}_{R,S} \leq \widetilde{c}\widetilde{K}^{\gamma}_{B,S}$.*

Now we give the definition of regular BMO space RBMO introduced by Bui and Duong in [1].

Definition 2.4 *Let $1 < \rho < \infty$ be some fixed constant. A function $b \in L^1_{\mathrm{loc}}(\mu)$ is said to belong to* RBMO(μ) *if there exists a positive constant $C > 0$, such that for any ball B,*

$$\frac{1}{\mu(\rho B)} \int_B |b(x) - m_{\widetilde{B}}(b)| \mathrm{d}\mu(x) \leq C, \tag{2.2}$$

and for any two doubling balls B, S, such that $B \subset S$,

$$|m_B(b) - m_S(b)| \leq CK_{B,S}, \tag{2.3}$$

where

$$m_B(b) = \frac{1}{\mu(B)} \int_B b(x) \mathrm{d}\mu(x). \tag{2.4}$$

The minimal constant C appearing in (2.2), (2.3) *is defined as the* RBMO(μ) *norm of f and denoted by $\|b\|_*$ or $\|b\|_{\mathrm{RBMO}(\mu)}$.*

Now, we recall the definition of function space Lip(β) introduced by Zhou and Wang [23].

Definition 2.5 *Suppose that $\beta \in (0, 1]$, we say that the function $f : \mathcal{X} \to \mathbb{C}$ satisfies a Lipschitz condition of order β provided that*

$$|f(x) - f(y)| \leq C\lambda(x, d(x, y))^{\frac{\beta}{n}} \tag{2.5}$$

for every $x, y \in \mathcal{X}$ and the smallest constant in this inequality will be denoted by $\|f\|_{\mathrm{Lip}(\beta)}$.

It is easy to see that the linear space with the norm $\|.\|_{\mathrm{Lip}(\beta)}$ is a Banach space, and we call it Lip(β). The following Morrey $M^p_q(k, \mu)$ and weak Morrey space $WM^p_q(k, \mu)$ appear in [2]; see also [17].

Definition 2.6 *Let $k > 1$ and $1 \leq q \leq p < \infty$. Define*

$$M^p_q(k, \mu) = \{f \in L^q_{\mathrm{Loc}} : \|f\|_{M^p_q(k,\mu)} < \infty\}, \tag{2.6}$$

where

$$\|f\|_{M^p_q(k,\mu)} = \sup_B \mu(kB)^{\frac{1}{p}-\frac{1}{q}} \Big(\int_B |f|^q \mathrm{d}\mu \Big)^{\frac{1}{q}}. \tag{2.7}$$

Definition 2.7 *Let $k > 1$ and $1 \leq q \leq p < \infty$. We say that f belongs to weak Morrey space $WM^p_q(k, \mu)$ if*

$$\|f\|_{WM^p_q(k,\mu)} = \sup_B \mu(kB)^{\frac{1}{p}-\frac{1}{q}} \sup_{\lambda>0} (\lambda^q \mu(\{x \in B : |f(x)| > \lambda\}))^{\frac{1}{q}} < \infty. \tag{2.8}$$

Cao and Zhou showed in their paper [2] that for different k the Morrey spaces $M^p_q(k,\mu)$ are equivalent with each other.

Lemma 2.2 *Let $k, r > 1$. Then*

$$M^p_q(k,\mu) \approx M^p_q(r,\mu). \tag{2.9}$$

The following ε-weak reverse doubling condition was introduced by Fu, Yang and Yuan in [9].

Definition 2.8 *Let $\varepsilon \in (0,\infty)$. A dominating function λ is satisfying the ε-weak reverse doubling condition if, for all $r \in (0, 2\mathrm{diam}(\mathcal{X}))$ and $a \in \big(1, 2\mathrm{diam}\frac{(\mathcal{X})}{r}\big)$, there exists a number $C(a) \in [1,\infty)$, depending only on a, r and $\mathcal{X}$, such that for all $x \in \mathcal{X}$,*

$$\lambda(x, ar) \geq C(a)\lambda(x,r), \tag{2.10}$$

and moreover,

$$\sum_{i=1}^{\infty} \frac{1}{[C(a^i)]^{\varepsilon}} < \infty. \tag{2.11}$$

Remark 2.2 (i) It is easy to see that if $\varepsilon_1 < \varepsilon_2$, then λ also satisfies the ε_2-weak reverse doubling condition.

(ii) Assume that $\mathrm{diam}(\mathcal{X}) = \infty$. For any fixed $x \in \mathcal{X}$, we know that

$$\lim_{r\to 0} \lambda(x,r) = 0, \quad \lim_{r\to\infty} \lambda(x,r) = \infty. \tag{2.12}$$

(iii) It is easy to see that the ε-weak reverse doubling condition is much weaker than the assumption introduced by Bui and Duong in [1]: There exists $m \in (0,\infty)$ such that for all $x \in \mathcal{X}$ and $a, r \in (0,\infty)$, $\lambda(x, ar) = a^m \lambda(x,r)$.

Finally, we give the $(L^{q_1}(\mu) \times L^{q_2}(\mu), L^s(\mu))$ boundedness of general integral operator $I_{\alpha,2}$, which can be found in [3].

Lemma 2.3 *Suppose $1 < q_i < \infty$, $\alpha_i > 0$, $\frac{1}{s_i} = \frac{1}{q_i} - \frac{\alpha_i}{n}$ for $i = 1, 2$. Let $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}$ and $\alpha = \alpha_1 + \alpha_2$. If $I_{\alpha,2}$ is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{n}{2n-\alpha},\infty}(\mu)$, then there exits a positive constant C such that*

$$\|I_{\alpha,2}(f_1,f_2)\|_{L^s(\mu)} \leq C\|f_1\|_{L^{q_1}(\mu)}\|f_2\|_{L^{q_2}(\mu)}.$$

3 The Proofs of Theorems 1.1–1.3

In this section, we always suppose that the point $x \neq y$, B is the ball with center x and radius $r = d(x,y)$, obviously $2B \subset B(y, 3r) = 3B(y,r)$. First, we prove Theorem 1.1.

Proof of Theorem 1.1 For any ball $B \subset \mathcal{X}$, let $f_j^0 = f_j\chi_{2B}$, $f_j^\infty = f_j - f_j\chi_{2B}$, $j = 1, 2$, and set

$$h_B = m_B[I_{\alpha,2}(f_1^\infty, f_2^0) + I_{\alpha,2}(f_1^0, f_2^\infty) + I_{\alpha,2}(f_1^\infty, f_2^\infty)].$$

Then

$$\frac{1}{\mu(6B)}\int_B |I_{\alpha,2}(f_1,f_2)(x) - m_{\widetilde{B}}(I_{\alpha,2}(f_1,f_2))|\mathrm{d}\mu(x)$$

$$
\begin{aligned}
&\le \frac{1}{\mu(6B)}\int_B |I_{\alpha,2}(f_1,f_2)(x)-h_B|\,\mathrm{d}u(x)+|h_B-h_{\widetilde{B}}| \\
&\quad +|h_{\widetilde{B}}-m_{\widetilde{B}}(I_{\alpha,2}(f_1,f_2))| \\
&\le \frac{1}{\mu(6B)}\int_B |I_{\alpha,2}(f_1,f_2)(x)-h_B|\,\mathrm{d}u(x)+|h_B-h_{\widetilde{B}}| \\
&\quad +\frac{1}{\mu(\widetilde{B})}\int_{\widetilde{B}} |I_{\alpha,2}(f_1,f_2)(x)-h_{\widetilde{B}}|\,\mathrm{d}u(x)
\end{aligned}
\tag{3.1}
$$

and for any two $(6,\beta_6)$ doubling balls $B\subset S$,

$$
\begin{aligned}
&|m_B(I_{\alpha,2}(f_1,f_2))-m_S(I_{\alpha,2}(f_1,f_2))| \\
&\le |m_B(I_{\alpha,2}(f_1,f_2))-h_B|+|h_B-h_S|+|h_S-m_S(I_{\alpha,2}(f_1,f_2))| \\
&\le \frac{C}{\mu(6B)}\int_B |I_{\alpha,2}(f_1,f_2)(x)-h_B|\mathrm{d}\mu(x)+|h_B-h_S| \\
&\quad +\frac{C}{\mu(6S)}\int_S |I_{\alpha,2}(f_1,f_2)(x)-h_S|\mathrm{d}\mu(x).
\end{aligned}
\tag{3.2}
$$

Therefore, to prove Theorem 1.1, it suffices to show that for any ball B,

$$
\frac{1}{\mu(6B)}\int_B |I_{\alpha,2}(f_1,f_2)(x)-h_B|\mathrm{d}\mu(x)\le C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}
\tag{3.3}
$$

and that for all balls $B\subset S$ with S being $(6,\beta_6)$-doubling,

$$
|h_B-h_S|\le CK_{B,S}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.
\tag{3.4}
$$

Let us prove (3.3) firstly. Write

$$
\begin{aligned}
&\frac{1}{\mu(6B)}\int_B |I_{\alpha,2}(f_1,f_2)(x)-h_B|\mathrm{d}\mu(x) \\
&\le \frac{1}{\mu(6B)}\int_B |I_{\alpha,2}(f_1^0,f_2^0)(x)|\mathrm{d}\mu(x)+\frac{1}{\mu(6B)}\int_B (|I_{\alpha,2}(f_1^\infty,f_2^0)(x) \\
&\quad -m_B(I_{\alpha,2}(f_1^\infty,f_2^0))|+|I_{\alpha,2}(f_1^0,f_2^\infty)(x)-m_B(I_{\alpha,2}(f_1^0,f_2^\infty))| \\
&\quad +|I_{\alpha,2}(f_1^\infty,f_2^\infty)(x)-m_B(I_{\alpha,2}(f_1^\infty,f_2^\infty))|)\,\mathrm{d}\mu(x) \\
&\le \frac{1}{\mu(6B)}\int_B |I_{\alpha,2}(f_1^0,f_2^0)(x)|\mathrm{d}\mu(x)+\frac{1}{\mu(B)}\frac{1}{\mu(6B)}\int_B\int_B \\
&\quad \Big(\int_{2B}\int_{\mathcal{X}\setminus 2B}|K_\alpha(x,z_1,z_2)-K_\alpha(y,z_1,z_2)|\prod_{i=1}^{2}|f_i(z_i)|\mathrm{d}\mu(z_i) \\
&\quad +\int_{\mathcal{X}\setminus 2B}\int_{2B}|K_\alpha(x,z_1,z_2)-K_\alpha(y,z_1,z_2)|\prod_{i=1}^{2}|f_i(z_i)|\mathrm{d}\mu(z_i) \\
&\quad +\int_{\mathcal{X}\setminus 2B}\int_{\mathcal{X}\setminus 2B}|K_\alpha(x,z_1,z_2)-K_\alpha(y,z_1,z_2)|\prod_{i=1}^{2}|f_i(z_i)|\mathrm{d}\mu(z_i)\Big)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&:= E_1+\frac{1}{\mu(B)}\frac{1}{\mu(6B)}\int_B\int_B (E_2+E_3+E_4)\mathrm{d}\mu(x)\mathrm{d}\mu(y).
\end{aligned}
\tag{3.5}
$$

Therefore, to prove (3.3), we need only to prove that $E_i \le C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}$ for $i=1,\cdots,4$. Suppose that $\frac{\alpha_j}{n}=\frac{1}{p_j}$ for $j=1,2$, $\alpha=\alpha_1+\alpha_2$ and $\delta=\delta_1+\delta_2$, where $\delta_1,\delta_2>0$. Let us estimate E_1 first. For $\frac{1}{s}=\frac{1}{q}-\frac{\alpha}{n}$, by Hölder's inequality and Lemma 2.3, we get

$$
\begin{aligned}
E_1 &\le \frac{1}{\mu(6B)}\Big(\int_B |I_{\alpha,2}(f_1^0,f_2^0)(x)|^s \mathrm{d}\mu(x)\Big)^{\frac{1}{s}}\mu(B)^{\frac{1}{s}'}\\
&\le C\mu(6B)^{-\frac{1}{s}}\Big(\int_{2B}|f_1(z_1)|^{q_1}\mathrm{d}\mu(z_1)\Big)^{\frac{1}{q_1}}\Big(\int_{2B}|f_2(z_2)|^{q_2}\mathrm{d}\mu(z_2)\Big)^{\frac{1}{q_2}}\\
&= C\mu(6B)^{\frac{1}{p_1}-\frac{1}{q_1}}\Big(\int_{2B}|f_1(z_1)|^{q_1}\mathrm{d}\mu(z_1)\Big)^{\frac{1}{q_1}}\\
&\quad\times \mu(6B)^{\frac{1}{p_2}-\frac{1}{q_2}}\Big(\int_{2B}|f_2(z_2)|^{q_2}\mathrm{d}\mu(z_2)\Big)^{\frac{1}{q_2}}\\
&\le C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.
\end{aligned}
\tag{3.6}
$$

Now we estimate E_2. E_3 can be done in the same way by notice the fact that $\lambda(x,r)\le C\lambda(y,r)$ if $d(x,y)<r$. Using (1.6), Hölder's inequality, (2.9) and (2.11), we have

$$
\begin{aligned}
E_2 &\le C\int_{2B}\int_{\mathcal{X}\setminus 2B}\frac{\lambda(x,d(x,y))^{\frac{\delta}{n}}|f_1(z_1)f_2(z_2)|}{\Big[\sum_{i=1}^{2}\lambda(x,d(x,z_i))\Big]^{2-\frac{\alpha}{n}+\frac{\delta}{n}}}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
&\le C\lambda(x,d(x,y))^{\frac{\delta}{n}}\mu(2B)^{1-\frac{1}{q_1}}\Big(\int_{2B}|f_1(z_1)|^{q_1}\mathrm{d}\mu(z_1)\Big)^{\frac{1}{q_1}}\\
&\quad\times\sum_{i=1}^{\infty}\int_{2^{i+1}B\setminus 2^iB}\frac{|f_2(z_2)|}{\lambda(x,d(x,z_2))^{2-\frac{\alpha}{n}+\frac{\delta}{n}}}\mathrm{d}\mu(z_2)\\
&\le C\|f_1\|_{M^{p_1}_{q_1}(3,\mu)}\mu(6B)^{1-\frac{1}{p_1}}\lambda(x,d(x,y))^{\frac{\delta}{n}}\\
&\quad\times\sum_{i=1}^{\infty}\Big(\int_{2^{i+1}B}|f_2(z_2)|^{q_2}\mathrm{d}\mu(z_2)\Big)^{\frac{1}{q_2}}\frac{\mu(2^{i+1}B)^{\frac{1}{q_2}'}}{\lambda(x,2^ir)^{2-\frac{\alpha}{n}+\frac{\delta}{n}}}\\
&\le C\|f_1\|_{M^{p_1}_{q_1}(3,\mu)}\|f_2\|_{M^{p_2}_{q_2}(2,\mu)}\lambda(x,6r)^{1+\frac{\delta}{n}-\frac{1}{p_1}}\\
&\quad\times\sum_{i=1}^{\infty}\frac{\mu(2^{i+1}B)^{\frac{1}{q_2}'}\mu(2^{i+2}B)^{\frac{1}{q_2}-\frac{1}{p_2}}}{\lambda(x,2^ir)^{2-\frac{\alpha}{n}+\frac{\delta}{n}}}\\
&\le C\|f_1\|_{M^{p_1}_{q_1}(3,\mu)}\|f_2\|_{M^{p_2}_{q_2}(2,\mu)}\lambda(x,r)^{1+\frac{\delta}{n}-\frac{1}{p_1}}\sum_{i=1}^{\infty}\frac{\lambda(x,2^{i+2}r)^{1-\frac{1}{p_2}}}{\lambda(x,2^ir)^{2-\frac{\alpha}{n}+\frac{\delta}{n}}}\\
&\le C\|f_1\|_{M^{p_1}_{q_1}(3,\mu)}\|f_2\|_{M^{p_2}_{q_2}(2,\mu)}\lambda(x,r)^{1+\frac{\delta}{n}-\frac{1}{p_1}}\\
&\quad\times\sum_{i=1}^{\infty}\lambda(x,r)^{\frac{1}{p_1}-\frac{\delta}{n}-1}C(2^i)^{\frac{1}{p_1}-\frac{\delta}{n}-1}\\
&\le C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.
\end{aligned}
\tag{3.7}
$$

By the condition (1.6), Hölder's inequality and (2.9), we also obtain

$$
\begin{aligned}
E_4 &\le C\lambda(x,d(x,y))^{\frac{\delta}{n}} \int_{\mathcal{X}\setminus 2B}\int_{\mathcal{X}\setminus 2B} \frac{|f_1(z_1)f_2(z_2)|}{\Big[\sum_{i=1}^{2}\lambda(x,d(x,z_i))\Big]^{2-\frac{\alpha}{n}+\frac{\delta}{n}}}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
&\le C\lambda(x,d(x,y))^{\frac{\delta}{n}} \prod_{j=1}^{2}\int_{\mathcal{X}\setminus 2B}\frac{|f_j(z_j)|}{\lambda(x,d(x,z_j))^{1-\frac{\alpha_j}{n}+\frac{\delta_j}{n}}}\mathrm{d}\mu(z_j)\\
&\le C\lambda(x,d(x,y))^{\frac{\delta}{n}} \prod_{j=1}^{2}\sum_{i=1}^{\infty}\Big(\int_{2^{i+1}B\setminus 2^iB}|f_j(z_j)|^{q_j}\mathrm{d}\mu(z_j)\Big)^{\frac{1}{q_j}}\\
&\quad\times\Big(\int_{2^{i+1}B\setminus 2^iB}\frac{\mathrm{d}\mu(z_j)}{\lambda(x,d(x,z_j))^{(1-\frac{\alpha_j}{n}+\frac{\delta_j}{n})q_j'}}\Big)^{\frac{1}{q_j'}}\\
&\le C\lambda(x,d(x,y))^{\frac{\delta}{n}} \prod_{j=1}^{2}\sum_{i=1}^{\infty}\|f_j\|_{M^{p_j}_{q_j}(2,\mu)}\frac{(\mu(2^{i+2}B))^{\frac{1}{q_j}-\frac{1}{p_j}}(\mu(2^{i+1}B))^{\frac{1}{q_j'}}}{\lambda(x,2^{i-1}r)^{1-\frac{\alpha_j}{n}+\frac{\delta_j}{n}}}\\
&\le C\lambda(x,d(x,y))^{\frac{\delta}{n}} \prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(2,\mu)}\sum_{i=1}^{\infty}\frac{C(2^i)^{-\frac{\delta_j}{n}}\lambda(x,r)^{\frac{1}{q_j}-\frac{1}{p_j}}\lambda(x,r)^{\frac{1}{q_j'}}}{\lambda(x,r)^{1-\frac{\alpha_j}{n}+\frac{\delta_j}{n}}}\\
&\le C\lambda(x,d(x,y))^{\frac{\delta}{n}} \prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(2,\mu)}\lambda(x,r)^{-\frac{\delta_j}{n}}\sum_{i=1}^{\infty}C(2^i)^{-\frac{\delta_j}{n}}\\
&\le C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}. \qquad (3.8)
\end{aligned}
$$

Inequalities from (3.5) to (3.8) yield (3.3).

Next, we show (3.4) for chosen h_B and h_S. Denote the smallest positive integer N such that $2S\subset 6^NB$ simply by N_1. Write

$$
\begin{aligned}
&|h_B-h_S|\\
&\le |m_B(I_{\alpha,2}(f_1\chi_{6^{N_1}B\setminus 2B},f_2\chi_{2B}))|+|m_B(I_{\alpha,2}(f_1\chi_{2B},f_2\chi_{6^{N_1}B\setminus 2B}))|\\
&\quad+|m_B(I_{\alpha,2}(f_1\chi_{6^{N_1}B\setminus 2B},f_2\chi_{6^{N_1}B\setminus 2B}))|+|m_S(I_{\alpha,2}(f_1\chi_{6^{N_1}B\setminus 2S},f_2\chi_{2S}))|\\
&\quad+|m_S(I_{\alpha,2}(f_1\chi_{2S},f_2\chi_{6^{N_1}B\setminus 2S}))|+|m_S(I_{\alpha,2}(f_1\chi_{6^{N_1}B\setminus 2S},f_2\chi_{6^{N_1}B\setminus 2S}))|\\
&\quad+|m_B(I_{\alpha,2}(f_1\chi_{\mathcal{X}\setminus 6^{N_1}B},f_2\chi_{6^{N_1}B}))-m_S(I_{\alpha,2}(f_1\chi_{\mathcal{X}\setminus 6^{N_1}B},f_2\chi_{6^{N_1}B}))|\\
&\quad+|m_B(I_{\alpha,2}(f_1\chi_{6^{N_1}B},f_2\chi_{\mathcal{X}\setminus 6^{N_1}B}))-m_S(I_{\alpha,2}(f_1\chi_{6^{N_1}B},f_2\chi_{\mathcal{X}\setminus 6^{N_1}B}))|\\
&\quad+|m_B(I_{\alpha,2}(f_1\chi_{\mathcal{X}\setminus 6^{N_1}B},f_2\chi_{\mathcal{X}\setminus 6^{N_1}B}))-m_S(I_{\alpha,2}(f_1\chi_{\mathcal{X}\setminus 6^{N_1}B},f_2\chi_{\mathcal{X}\setminus 6^{N_1}B}))|\\
&:=\sum_{i=1}^{9}F_i. \qquad (3.9)
\end{aligned}
$$

By the size condition (1.5), Hölder's inequality and the fact that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{\alpha}{n}$, we have

$$
\begin{aligned}
&|I_{\alpha,2}(f_1\chi_{6^{N_1}B\setminus 2B},f_2\chi_{2B})(y)|\\
&\le C\sum_{k=1}^{N_1-1}\int_{6^{k+1}B\setminus 6^kB}\int_{2B}\frac{|f_1(z_1)f_2(z_2)|}{\Big(\sum_{i=1}^{2}\lambda(y,d(y,z_i))\Big)^{2-\frac{\alpha}{n}}}\,\mathrm{d}\mu(z_2)\,\mathrm{d}\mu(z_1)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{6B\setminus 2B}\int_{2B}\frac{|f_1(z_1)f_2(z_2)|}{\Big(\sum\limits_{i=1}^{2}\lambda(y,d(y,z_i))\Big)^{2-\frac{\alpha}{n}}}\,\mathrm{d}\mu(z_2)\,\mathrm{d}\mu(z_1)\\
&\le C\sum_{k=1}^{N_1-1}\frac{\mu(6^{k+2}B)^{1-\frac{\alpha}{2n}}}{\mu(6^{k+2}B)^{1-\frac{\alpha}{2n}}}\frac{1}{\lambda(c_B,6^k r_B)^{2-\frac{\alpha}{n}}}\prod_{i=1}^{2}\int_{6^{k+1}B}|f_i(z_i)|\,\mathrm{d}\mu(z_i)\\
&\quad+\frac{\mu(6^2B)^{1-\frac{\alpha}{2n}}}{\mu(6^2B)^{1-\frac{\alpha}{2n}}}\frac{1}{\lambda(c_B,2r_B)^{2-\frac{\alpha}{n}}}\prod_{i=1}^{2}\int_{6B}|f_i(z_i)|\,\mathrm{d}\mu(z_i)\\
&\le C\sum_{k=1}^{N_1-1}\frac{\mu(6^{k+2}B)^{1-\frac{\alpha}{2n}}}{\lambda(c_B,6^{k+2}r_B)^{1-\frac{\alpha}{2n}}}\frac{1}{\mu(6^{k+2}B)^{1-\frac{\alpha}{2n}}\lambda(c_B,6^{k+2}r_B)^{1-\frac{\alpha}{2n}}}\\
&\quad\times\prod_{i=1}^{2}\mu(6^{k+1}B)^{\frac{1}{q_i'}}\Big(\int_{6^{k+1}B}|f_i(z_i)|^{q_i}\,\mathrm{d}\mu(z_i)\Big)^{\frac{1}{q_i}}+\frac{\mu(6^2B)^{1-\frac{\alpha}{2n}}}{\lambda(c_B,6^2r_B)^{1-\frac{\alpha}{2n}}}\\
&\quad\times\frac{1}{\mu(6^2B)^{1-\frac{\alpha}{2n}}\lambda(c_B,6^2r_B)^{1-\frac{\alpha}{2n}}}\prod_{i=1}^{2}\Big(\int_{6B}|f_i(z_i)|^{q_i}\,\mathrm{d}\mu(z_i)\Big)^{\frac{1}{q_i}}\mu(6B)^{\frac{1}{q_i'}}\\
&\le C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}\Big(1+\sum_{k=1}^{N_1-1}\Big(\frac{\mu(6^{k+2}B)}{\lambda(c_B,6^{k+2}r_B)}\Big)^{1-\frac{\alpha}{2n}}\Big)\\
&\le C\widetilde{K}_{B,S}^{\frac{\alpha}{2n}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}. \qquad (3.10)
\end{aligned}
$$

Hence

$$F_1\le C\widetilde{K}_{B,S}^{\frac{\alpha}{2n}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.$$

Similarly, we see that

$$F_2\le C\widetilde{K}_{B,S}^{\frac{\alpha}{2n}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}$$

and

$$F_4+F_5\le C\widetilde{K}_{B,S}^{\frac{\alpha}{2n}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.$$

On the other hand, it follows from (1.5), (1.3) and (1.4) that for all $y\in B$,

$$
\begin{aligned}
&|I_{\alpha,2}(f_1\chi_{6^{N_1}B\setminus 2B},f_2\chi_{6^{N_1}B\setminus 2B})(y)|\\
&\le C\sum_{k=1}^{N_1-1}\int_{(6^{k+1}B\setminus 6^kB)^2}\frac{|f_1(z_1)f_2(z_2)|}{\Big(\sum\limits_{i=1}^{2}\lambda(y,d(y,z_i))\Big)^{2-\frac{\alpha}{n}}}\,\mathrm{d}\mu(z_2)\,\mathrm{d}\mu(z_1)\\
&\quad+\int_{(6B\setminus 2B)^2}\frac{|f_1(z_1)f_2(z_2)|}{\Big(\sum\limits_{i=1}^{2}\lambda(y,d(y,z_i))\Big)^{2-\frac{\alpha}{n}}}\,\mathrm{d}\mu(z_2)\,\mathrm{d}\mu(z_1)\\
&\le C\sum_{k=1}^{N_1-1}\frac{\mu(6^{k+2}B)^{1-\frac{\alpha}{2n}}}{\mu(6^{k+2}B)^{1-\frac{\alpha}{2n}}}\frac{1}{\lambda(c_B,6^k r_B)^{2-\frac{\alpha}{n}}}\prod_{i=1}^{2}\int_{6^{k+1}B}|f_i(z_i)|\,\mathrm{d}\mu(z_i)
\end{aligned}
$$

$$+\frac{\mu(6^2B)^{1-\frac{\alpha}{2n}}}{\mu(6^2B)^{1-\frac{\alpha}{2n}}}\frac{1}{\lambda(c_B,2r_B)^{2-\frac{\alpha}{n}}}\prod_{i=1}^{2}\int_{6B}|f_i(z_i)|\,\mathrm{d}\mu(z_i)$$
$$\leq C\widetilde{K}_{B,S}^{\frac{\alpha}{2n}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}, \tag{3.11}$$

which implies

$$F_3\leq C\widetilde{K}_{B,S}^{\frac{\alpha}{2n}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.$$

Analogously,

$$F_6\leq C\widetilde{K}_{B,S}^{\frac{\alpha}{2n}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.$$

Now, we estimate F_8. F_7 can be done in the same way. Notice that

$$F_8\leq\frac{1}{\mu(B)}\frac{1}{\mu(S)}\int_B\int_S|I_{\alpha,2}(f_1\chi_{6^{N_1}B},f_2\chi_{\mathcal{X}\setminus 6^{N_1}B})(y)$$
$$-I_{\alpha,2}(f_1\chi_{6^{N_1}B},f_2\chi_{\mathcal{X}\setminus 6^{N_1}B})(z)|\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y),$$

while, by a familiar argument similar to that used in the estimate for E_2,

$$|I_{\alpha,2}(f_1\chi_{6^{N_1}B},f_2\chi_{\mathcal{X}\setminus 6^{N_1}B})(y)-I_{\alpha,2}(f_1\chi_{6^{N_1}B},f_2\chi_{\mathcal{X}\setminus 6^{N_1}B})(z)|\leq C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.$$

Therefore

$$F_8\leq C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.$$

Finally, using the same method that appeared in the estimate for E_4, we can get

$$F_9\leq C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.$$

Combining all the estimates for F_i with $i=1,\cdots,9$, we get (3.9), which completes the proof of Theorem 1.1.

Proof of Theorem 1.2 For any $x,y\in\mathcal{X}$, it suffices to prove

$$|I_{\alpha,2}(f_1,f_2)(x)-I_{\alpha,2}(f_1,f_2)(y)|\leq C[\lambda(x,d(x,y))]^{\frac{\alpha}{n}-\frac{1}{p}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}. \tag{3.12}$$

Write

$$|I_{\alpha,2}(f_1,f_2)(x)-I_{\alpha,2}(f_1,f_2)(y)|$$
$$\leq\int_{2B}\int_{2B}|K_\alpha(x,z_1,z_2)|\prod_{i=1}^{2}|f_i(z_i)|\mathrm{d}\mu(z_i)$$
$$+\int_{2B}\int_{2B}|K_\alpha(y,z_1,z_2)|\prod_{i=1}^{2}|f_i(z_i)|\mathrm{d}\mu(z_i)$$

$$+\int_{2B}\int_{\mathcal{X}\setminus 2B}|K_\alpha(x,z_1,z_2)-K_\alpha(y,z_1,z_2)|\prod_{i=1}^{2}|f_i(z_i)|\mathrm{d}\mu(z_i)$$
$$+\int_{\mathcal{X}\setminus 2B}\int_{2B}|K_\alpha(x,z_1,z_2)-K_\alpha(y,z_1,z_2)|\prod_{i=1}^{2}|f_i(z_i)|\mathrm{d}\mu(z_i)$$
$$+\int_{\mathcal{X}\setminus 2B}\int_{\mathcal{X}\setminus 2B}|K_\alpha(x,z_1,z_2)-K_\alpha(y,z_1,z_2)|\prod_{i=1}^{2}|f_i(z_i)|\mathrm{d}\mu(z_i)$$
$$:=G_1+G_2+G_3+G_4+G_5. \tag{3.13}$$

Suppose $\frac{\alpha_j}{n}=\frac{1}{p_j}$ for $j=1,2$, $\delta=\delta_1+\delta_2$, where $\delta_1,\delta_2>0$. We estimate G_1 firstly. Noticing the fact (1.4), G_2 can be done in the same way. Apply the size condition (1.5), Hölder's inequality, ε-weak reverse doubling condition of dominating function λ and Lemma 2.2, we can get

$$\begin{aligned}
G_1 &\le \prod_{j=1}^{2}\int_{2B}\frac{|f_j(z_j)|}{\lambda(x,d(x,z_j))^{1-\frac{\alpha_j}{n}}}\mathrm{d}\mu(z_j)\\
&\le C\prod_{j=1}^{2}\Big(\int_{2B}|f_j(z_j)|^{q_j}\mathrm{d}\mu(z_j)\Big)^{\frac{1}{q_i}}\Big(\int_{2B}\frac{1}{\lambda(x,r)^{(1-\frac{\alpha_j}{n})q_j'}}\mathrm{d}\mu(z)\Big)^{\frac{1}{q_j'}}\\
&\le C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(2,\mu)}\mu(4B)^{\frac{1}{q_j}-\frac{1}{p_j}}\Big(\sum_{j=0}^{\infty}\frac{\mu(2^{-j+1}B)}{\lambda(x,2^{-j}r)^{(1-\frac{\alpha_j}{n})q_j'}}\Big)^{\frac{1}{q_j'}}\\
&\le C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(2,\mu)}\lambda(x,4r)^{\frac{1}{q_j}-\frac{1}{p_j}}\Big(\sum_{j=0}^{\infty}\lambda(x,2^{-j}r)^{1-(1-\frac{\alpha_j}{n})q_j'}\Big)^{\frac{1}{q_j'}}\\
&\le C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(2,\mu)}\lambda(x,r)^{\frac{\alpha_j}{n}-\frac{1}{p_j}}\Big(\sum_{j=0}^{\infty}\frac{1}{C(2^j)^{(\frac{\alpha_j}{n}-\frac{1}{q_j})q_j'}}\Big)^{\frac{1}{q_j'}}\\
&\le C[\lambda(x,d(x,y))]^{\frac{\alpha}{n}-\frac{1}{p}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.
\end{aligned}\tag{3.14}$$

Secondly, we estimate G_3 and G_4. Using the fact that $0<\frac{\alpha}{n}<\frac{\delta}{n}+\frac{1}{p}<\frac{\delta}{n}+\frac{1}{p_2}+1$, an argument similar to that used in the estimate for E_2, we have

$$\begin{aligned}
G_3+G_4 &\le C\|f_1\|_{M^{p_1}_{q_1}(3,\mu)}\|f_2\|_{M^{p_2}_{q_2}(2,\mu)}\lambda(x,r)^{1+\frac{\delta}{n}-\frac{1}{p_1}}\\
&\quad\times\sum_{i=1}^{\infty}\lambda(x,r)^{\frac{\alpha}{n}-\frac{1}{p_2}-\frac{\delta}{n}-1}C(2^i)^{\frac{\alpha}{n}-\frac{1}{p_2}-\frac{\delta}{n}-1}\\
&\le C\lambda(x,d(x,y))^{\frac{\alpha}{n}-\frac{1}{p}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.
\end{aligned}\tag{3.15}$$

Finally, we estimate G_5. Applying the same method to estimate E_4, we can get

$$\begin{aligned}
G_5 &\le C\lambda(x,d(x,y))^{\frac{\delta}{n}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(2,\mu)}\\
&\quad\times\sum_{i=1}^{\infty}C(2^i)^{\frac{\alpha_j}{n}-\frac{1}{p_j}-\frac{\delta_j}{n}}\frac{\lambda(x,r)^{\frac{1}{q_j}-\frac{1}{p_j}}\lambda(x,r)^{\frac{1}{q_j'}}}{\lambda(x,r)^{1-\frac{\alpha_j}{n}+\frac{\delta_j}{n}}}
\end{aligned}$$

$$\leq C\lambda(x,d(x,y))^{\frac{\delta}{n}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(2,\mu)}\lambda(x,r)^{\frac{\alpha_j}{n}-\frac{1}{p_j}-\frac{\delta_j}{n}}\sum_{i=1}^{\infty}C(2^i)^{\frac{\alpha_j}{n}-\frac{1}{p_j}-\frac{\delta_j}{n}}$$
$$\leq C\lambda(x,d(x,y))^{\frac{\alpha}{n}-\frac{1}{p}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}. \tag{3.16}$$

Combining inequalities (3.13) to (3.16), we finish the prove of equality (3.12), so we have finished the prove of Theorem 1.2.

Proof of Theorem 1.3 To prove the inequalities (1.10) and (1.11), we need only to prove

$$\mu(3B)^{\frac{1}{p}-\frac{1}{q}}\Big(\int_B|I_{\alpha,2}(f_1,f_2)(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}\leq C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)} \tag{3.17}$$

and

$$\mu(3B)^{\frac{1}{p}-\frac{1}{q}}\sup_{\lambda>0}(\lambda^s\mu(\{x\in B:|I_{\alpha,2}(f_1,f_2)(x)|>\lambda\}))^{\frac{1}{s}}$$
$$\leq C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}, \tag{3.18}$$

respectively.

Let $f_j^0=f_j\chi_{2B},\ f_j^\infty=f_j-f_j\chi_{2B},\ j=1,2$. Firstly, we estimate (3.17). Write

$$\mu(3B)^{\frac{1}{p}-\frac{1}{q}}\Big(\int_B|I_{\alpha,2}(f_1,f_2)(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}$$
$$\leq\mu(3B)^{\frac{1}{p}-\frac{1}{q}}\Big(\int_B|I_{\alpha,2}(f_1^0,f_2^0)(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}$$
$$+\mu(3B)^{\frac{1}{p}-\frac{1}{q}}\Big(\int_B|I_{\alpha,2}(f_1^0,f_2^\infty)(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}$$
$$+\mu(3B)^{\frac{1}{p}-\frac{1}{q}}\Big(\int_B|I_{\alpha,2}(f_1^\infty,f_2^0)(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}$$
$$+\mu(3B)^{\frac{1}{p}-\frac{1}{q}}\Big(\int_B|I_{\alpha,2}(f_1^\infty,f_2^\infty)(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}$$
$$:=H_1+H_2+H_3+H_4. \tag{3.19}$$

We estimate H_1 to H_4 respectively. Using the fact that $\frac{1}{s}=\frac{1}{q_1}+\frac{1}{q_2}-\frac{\alpha}{n}$ and Lemma 2.2, we have

$$H_1\leq C\mu(3B)^{\frac{1}{p}-\frac{1}{q}}\prod_{j=1}^{2}\Big(\int_{2B}|f_j(z_j)|^{q_j}\mathrm{d}\mu(z_j)\Big)^{\frac{1}{q_j}}$$
$$\leq C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}\left(\frac{3}{2},\mu\right)}\leq C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}. \tag{3.20}$$

By the same method to that used in the estimate for E_2 in the case $\frac{\delta}{n}=0$ and noticing the fact that $\frac{1}{s}=\frac{1}{q_1}+\frac{1}{q_2}-\frac{\alpha}{n}$, $\frac{\alpha}{n}-\frac{1}{p}<0$ and $0<\alpha<2n$, we have

$$|I_{\alpha,2}(f_1^0,f_2^\infty)(x)|$$

$$\begin{aligned}
&\leq C\int_{2B}\int_{\mathcal{X}\setminus 2B}\frac{|f_1(z_1)f_2(z_2)|}{\Big[\sum\limits_{i=1}^{2}\lambda(x,d(x,z_i))\Big]^{2-\frac{\alpha}{n}}}\mathrm{d}\mu(z_1)\mathrm{d}\mu(z_2)\\
&\leq C\|f_1\|_{M^{p_1}_{q_1}(3,\mu)}\|f_2\|_{M^{p_2}_{q_2}(2,\mu)}\lambda(x,r)^{1-\frac{1}{p_1}}\sum_{i=1}^{\infty}\frac{\lambda(x,2^{i+2}r)^{1-\frac{1}{p_2}}}{\lambda(x,2^i r)^{2-\frac{\alpha}{n}}}\\
&\leq C\lambda(x,r)^{\frac{\alpha}{n}-\frac{1}{p}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}\\
&\leq C\lambda(x,3r)^{\frac{1}{q}-\frac{1}{p}-\frac{1}{s}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}\\
&\leq C\mu(3B)^{\frac{1}{q}-\frac{1}{p}-\frac{1}{s}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.
\end{aligned}\tag{3.21}$$

Thus

$$H_2\leq C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.\tag{3.22}$$

Similarly

$$H_3\leq C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.\tag{3.23}$$

By the same method to that used in the estimate for E_4 in the case that $\frac{\delta}{n}=\frac{\delta_1}{n}=\frac{\delta_2}{n}=0$, one gets

$$\begin{aligned}
&|I_{\alpha,2}(f_1^\infty,f_2^\infty)(x)|\\
&\leq C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(2,\mu)}\sum_{i=1}^{\infty}\frac{\mu(2^{i+2}B)^{\frac{1}{q_j}-\frac{1}{p_j}}\mu(2^{i+1}B)^{\frac{1}{q_j'}}}{\lambda(x,2^i r)^{1-\frac{\alpha_j}{n}}}\\
&\leq C\lambda(x,r)^{\frac{\alpha}{n}-\frac{1}{p}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}\\
&\leq C\lambda(x,3r)^{\frac{\alpha}{n}-\frac{1}{p}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}\\
&\leq C\mu(3B)^{\frac{1}{q}-\frac{1}{p}-\frac{1}{s}}\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.
\end{aligned}\tag{3.24}$$

So

$$H_4\leq C\prod_{j=1}^{2}\|f_j\|_{M^{p_j}_{q_j}(3,\mu)}.\tag{3.25}$$

Therefore (3.17) has been proved.

Next, we prove (3.18), write

$$\mu(3B)^{\frac{1}{p}-\frac{1}{q}}\sup_{\lambda>0}(\lambda^s\mu(\{x\in B:|I_{\alpha,2}(f_1,f_2)(x)|>\lambda\}))^{\frac{1}{s}}$$

$$
\begin{aligned}
&\leq \mu(3B)^{\frac{1}{p}-\frac{1}{q}} \sup_{\lambda>0} \Big(\lambda^s \mu\Big(\Big\{x \in B : |I_{\alpha,2}(f_1^0, f_2^0)(x)| > \frac{\lambda}{4}\Big\}\Big)\Big)^{\frac{1}{s}} \\
&\quad + \mu(3B)^{\frac{1}{p}-\frac{1}{q}} \sup_{\lambda>0} \Big(\lambda^s \mu\Big(\Big\{x \in B : |I_{\alpha,2}(f_1^0, f_2^\infty)(x)| > \frac{\lambda}{4}\Big\}\Big)\Big)^{\frac{1}{s}} \\
&\quad + \mu(3B)^{\frac{1}{p}-\frac{1}{q}} \sup_{\lambda>0} \Big(\lambda^s \mu\Big(\Big\{x \in B : |I_{\alpha,2}(f_1^\infty, f_2^0)(x)| > \frac{\lambda}{4}\Big\}\Big)\Big)^{\frac{1}{s}} \\
&\quad + \mu(3B)^{\frac{1}{p}-\frac{1}{q}} \sup_{\lambda>0} \Big(\lambda^s \mu\Big(\Big\{x \in B : |I_{\alpha,2}(f_1^\infty, f_2^\infty)(x)| > \frac{\lambda}{4}\Big\}\Big)\Big)^{\frac{1}{s}} \\
&:= M_1 + M_2 + M_3 + M_4.
\end{aligned}
\tag{3.26}
$$

We first estimate M_1. Due to the fact that $\frac{1}{s} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{\alpha}{n}$, Lemma 2.3 and Lemma 2.2,

$$
\begin{aligned}
&\lambda\Big(\mu\Big(\Big\{x \in B : |I_{\alpha,2}(f_1^0, f_2^0)(x)| > \frac{\lambda}{4}\Big\}\Big)\Big)^{\frac{1}{s}} \\
&\leq \lambda\Big(\int_B |4I_{\alpha,2}(f_1^0, f_2^0)(x)/\lambda|^s \mathrm{d}\mu(x)\Big)^{\frac{1}{s}} \\
&\leq C\Big(\int_{2B} |f_1(z_1)|^{q_1} \mathrm{d}\mu(z_1)\Big)^{\frac{1}{q_1}} \Big(\int_{2B} |f_2(z_2)|^{q_2} \mathrm{d}\mu(z_2)\Big)^{\frac{1}{q_2}} \\
&\leq C\mu(3B)^{\frac{1}{q}-\frac{1}{p}} \prod_{j=1}^{2} \|f_j\|_{M_{q_j}^{p_j}(\frac{3}{2},\mu)} \\
&\leq C\mu(3B)^{\frac{1}{q}-\frac{1}{p}} \prod_{j=1}^{2} \|f_j\|_{M_{q_j}^{p_j}(3,\mu)}.
\end{aligned}
\tag{3.27}
$$

So

$$
M_1 \leq C \prod_{j=1}^{2} \|f_j\|_{M_{q_j}^{p_j}(3,\mu)}. \tag{3.28}
$$

Now, we estimate M_2. M_3 can be done in the same way. By Lemma 2.3 and inequality (3.21),

$$
\begin{aligned}
&\lambda\Big(\mu\Big(\Big\{x \in B : |I_{\alpha,2}(f_1^0, f_2^\infty)(x)| > \frac{\lambda}{4}\Big\}\Big)\Big)^{\frac{1}{s}} \\
&\leq C\Big(\int_B |I_{\alpha,2}(f_1^0, f_2^\infty)(x)|^s \mathrm{d}\mu(x)\Big)^{\frac{1}{s}} \\
&\leq C\mu(B)^{\frac{1}{s}} \mu(3B)^{\frac{1}{q}-\frac{1}{p}-\frac{1}{s}} \prod_{j=1}^{2} \|f_j\|_{M_{q_j}^{p_j}(3,\mu)} \\
&\leq C\mu(3B)^{\frac{1}{q}-\frac{1}{p}} \prod_{j=1}^{2} \|f_j\|_{M_{q_j}^{p_j}(3,\mu)}.
\end{aligned}
\tag{3.29}
$$

Hence

$$
M_2 \leq C \prod_{j=1}^{2} \|f_j\|_{M_{q_j}^{p_j}(3,\mu)}. \tag{3.30}
$$

Due to inequality (3.24), we can get

$$\begin{aligned}
&\lambda\Big(\mu\Big(\Big\{x\in B: |I_{\alpha,2}(f_1^{\infty},f_2^{\infty})(x)|>\frac{\lambda}{4}\Big\}\Big)\Big)^{\frac{1}{s}}\\
&\le C\Big(\int_B |I_{\alpha,2}(f_1^{\infty},f_2^{\infty})(x)|^s \mathrm{d}\mu(x)\Big)^{\frac{1}{s}}\\
&\le C\mu(B)^{\frac{1}{s}}\mu(3B)^{\frac{1}{q}-\frac{1}{p}-\frac{1}{s}}\prod_{j=1}^{2}\|f_j\|_{M_{q_j}^{p_j}(3,\mu)}\\
&\le C\mu(3B)^{\frac{1}{q}-\frac{1}{p}}\prod_{j=1}^{2}\|f_j\|_{M_{q_j}^{p_j}(3,\mu)}. \qquad (3.31)
\end{aligned}$$

Thus

$$M_4\le C\prod_{j=1}^{2}\|f_j\|_{M_{q_j}^{p_j}(3,\mu)}. \qquad (3.32)$$

Therefore we have finished the prove of Theorem 1.3.

Acknowledgement The authors would like to thank the referee for his/her suggestions.

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